

## Research Article

# Statistics for Ratios of Rayleigh, Rician, Nakagami- $m$ , and Weibull Distributed Random Variables

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The distributions of ratios of random variables are of interest in many areas of the sciences. In this brief paper, we present the joint probability density function (PDF) and PDF of maximum of ratios  $\mu_1 = R_1/r_1$  and  $\mu_2 = R_2/r_2$  for the cases where  $R_1$ ,  $R_2$ ,  $r_1$ , and  $r_2$  are Rayleigh, Rician, Nakagami- $m$ , and Weibull distributed random variables. Random variables  $R_1$  and  $R_2$ , as well as random variables  $r_1$  and  $r_2$ , are correlated. Ascertaining on the suitability of the Weibull distribution to describe fading in both indoor and outdoor environments, special attention is dedicated to the case of Weibull random variables. For this case, analytical expressions for the joint PDF, PDF of maximum, PDF of minimum, and product moments of arbitrary number of ratios  $\mu_i = R_i/r_i$ ,  $i = 1, \dots, L$  are obtained. Random variables in numerator,  $R_i$ , as well as random variables in denominator,  $r_i$ , are exponentially correlated. To the best of the authors' knowledge, analytical expressions for the PDF of minimum and product moments of  $\{\mu_i\}_{i=1}^L$  are novel in the open technical literature. The proposed mathematical analysis is complemented by various numerical results. An application of presented theoretical results is illustrated with respect to performance assessment of wireless systems.

## 1. Introduction

Radio-wave propagation through wireless channels is a complicated phenomenon characterized by fading which is the result of multipath propagation. When a received signal envelope experiences fading during transmission, it fluctuates over time. Multivariate statistics is a useful mathematical tool for modeling and analyzing wireless channels. There is a very wide range of statistical models for fading channels [1] which their accuracy and veracity depend on propagation environment and communication scenario. Rayleigh, Rician, Nakagami- $m$ , and Weibull are the most frequently applied models in the open technical literature.

Fading can seriously degrade performance of wireless communications systems. Techniques that can be used to minimize the degradation effects due to fading have received

a great deal of research interest. Diversity combining [1], which combines two or more replicas of the received signal, is a practical and powerful technique that can be used to alleviate the detrimental effects of fading and to improve the performance of wireless communications systems without increasing transmission power and bandwidth. Space diversity [2], achieved by using multiple antennas at the receiver, is the most common form of diversity. The most popular space diversity techniques are selection combining (SC), equal-gain combining (EGC), and maximal-ratio combining (MRC).

In digital contemporary communications systems, fading channels are correlated due to insufficient antenna spacing when diversity is applied in small terminals (e.g., hand-held mobile terminals and compact base stations). Several spatial correlation models have been proposed and used for the performance analysis of various wireless systems, corresponding

to specific modulation, detection, and diversity schemes [1]. Spatial exponential correlation model is one of the most frequently used in performance analysis of wireless systems with multichannel reception [3–5]. The correlation matrix of this model is described by  $\sum_{ij} \equiv \rho^{|i-j|}$ ,  $i, j = 1, \dots, L$ , where  $\rho$  ( $0 \leq \rho < 1$ ) is the correlation coefficient between adjacent channels [1, Equation (9.164)] and  $L$  is the number of channels. The exponential correlation model corresponds to the scenario of multichannel reception from equispaced diversity antennas in which the correlation among the pairs of combined signals decays as the spacing between the antennas increases.

The distribution of the ratio of random variables is of interest in statistical analysis in biological and physical sciences, econometrics, and ranking and selection [6]. It has been studied by several authors especially when random variables are independent and come from the same family [7–10]. In [11, 12], the distribution of the ratio of correlated random variables is considered.

The ratios of random variables are also of interest in analyzing wireless communications systems in fading environment [13–17]. Namely, the random variable in nominator may present desired signal envelope while the random variable in denominator may present interference signal envelope. Rayleigh, Rician, Nakagami- $m$ , and Weibull distributions are included in our analysis. Having in mind that it is well known that the assumption of independence among the input diversity channels is not accurate for compact, hand-held, mobile terminals and indoor base stations with no sufficiently separated antennas, random variables in nominator, as well as random variables in denominator, are correlated. It shows that results presented in the paper can be efficiently used in analyzing realistic correlated fading channels.

## 2. On Two Ratios of Rayleigh, Rician, and Nakagami- $m$ Random Variables

In this section, the joint probability density function (PDF) and PDF of maximum of two ratios of random variables  $\mu_1 = R_1/r_1$  and  $\mu_2 = R_2/r_2$  are presented. Random variables  $R_1$  and  $R_2$ , as well as random variables  $r_1$  and  $r_2$ , are correlated. The joint PDF of  $\mu_1$  and  $\mu_2$  can be obtained as

$$p_{\mu_1\mu_2}(\mu_1, \mu_2) = \iint_0^\infty |J| p_{R_1R_2}(\mu_1 r_1, \mu_2 r_2) p_{r_1r_2}(r_1, r_2) dr_1 dr_2, \quad (1)$$

where  $p_{R_1R_2}(R_1, R_2)$  and  $p_{r_1r_2}(r_1, r_2)$  are the joint PDFs of random variables in nominator and denominator, respectively, and  $|J|$  is the Jacobian transformation given by

$$|J| = \begin{vmatrix} \frac{dR_1}{d\mu_1} & \frac{dR_1}{d\mu_2} \\ \frac{dR_2}{d\mu_1} & \frac{dR_2}{d\mu_2} \end{vmatrix} = r_1 r_2. \quad (2)$$

The PDF expression of maximum of ratios of random variables  $\mu_{\max} = \max\{\mu_1, \mu_2\}$  can be derived as

$$p_{\mu_{\max}}(\mu) = \int_0^\mu p_{\mu_1\mu_2}(\mu, \mu_2) d\mu_2 + \int_0^\mu p_{\mu_1\mu_2}(\mu_1, \mu) d\mu_1. \quad (3)$$

In the analysis of wireless communications systems, depending on the nature of the radio propagation environment, there are different models describing the statistical behavior of the multipath fading envelope. In the rest of this section, correlative Rayleigh, Rician, and Nakagami- $m$  fading models are considered. The existence of correlation is the real scenario in practical multiantennas wireless systems due to insufficient antenna spacing. Since, this paper considers spatial correlation—the correlation between appropriate pair of receive antennas—all signals, regardless of their nature, received by these antennas explore the same correlation coefficient.

(a) *Rayleigh Case.* The Rayleigh distribution is frequently used to model multipath fading with no direct line-of-sight (LOS) path. The joint bivariate PDFs of correlated Rayleigh distributed random variables  $R_i$  and  $r_i$  ( $i = 1, 2$ ) are given by [1, Equation (6.2)] as follows:

$$p_{R_1R_2}(R_1, R_2) = \frac{4R_1R_2}{\Omega_{d1}\Omega_{d2}(1-\rho)} \times \exp\left[-\frac{1}{1-\rho}\left(\frac{R_1^2}{\Omega_{d1}} + \frac{R_2^2}{\Omega_{d2}}\right)\right] \times I_0\left[\frac{2\sqrt{\rho}R_1R_2}{(1-\rho)\sqrt{\Omega_{d1}\Omega_{d2}}}\right], \quad (4)$$

$$p_{r_1r_2}(r_1, r_2) = \frac{4r_1r_2}{\Omega_{c1}\Omega_{c2}(1-\rho)} \times \exp\left[-\frac{1}{1-\rho}\left(\frac{r_1^2}{\Omega_{c1}} + \frac{r_2^2}{\Omega_{c2}}\right)\right] \times I_0\left[\frac{2\sqrt{\rho}r_1r_2}{(1-\rho)\sqrt{\Omega_{c1}\Omega_{c2}}}\right],$$

where  $I_0(\cdot)$  is the modified Bessel function of the first kind and zero order,  $\rho$  is the correlation coefficient, and  $\Omega_{di}$  and  $\Omega_{ci}$  are the mean-square values of  $R_i$  and  $r_i$ , respectively. Substituting (2) and (4) in (1) and using the infinite-series representation of the modified Bessel function [18, Equation (8.447/1)], integrals can be solved with the aid of [18, Equation (3.478/1)]. Analytical expression for the joint PDF of  $\mu_1$  and  $\mu_2$  can be finally written as

$$p_{\mu_1\mu_2}(\mu_1, \mu_2) = 4(1-\rho)^2 \sum_{k,\ell=0}^{\infty} \frac{\rho^{k+\ell} (\mu_1\mu_2)^{2k+1} \Gamma^2(2+k+\ell)}{(\Omega_{d1}\Omega_{d2})^{1+k} (\Omega_{c1}\Omega_{c2})^{1+\ell} (k!\ell!)^2} \times \left(\left(\frac{\mu_1^2}{\Omega_{d1}} + \frac{1}{\Omega_{c1}}\right)\left(\frac{\mu_2^2}{\Omega_{d2}} + \frac{1}{\Omega_{c2}}\right)\right)^{-(2+k+\ell)}. \quad (5)$$

Substituting (5) in (3) and using [18, Equation (3.194/1)], the PDF of maximum of two ratios of Rayleigh random variables can be written as

$$\begin{aligned}
 P_{\mu_{\max}}(\mu) &= (1-\rho)^2 \sum_{k,\ell=0}^{\infty} \frac{\rho^{k+\ell} \Gamma^2(2+k+\ell) \mu^{4k+3}}{(k!\ell!)^2 (1+k)} \left( \frac{\Omega_{c1}\Omega_{c2}}{\Omega_{d1}\Omega_{d2}} \right)^{1+k} \\
 &\times \left\{ \frac{2}{(1+((\Omega_{c1}/\Omega_{d1})\mu^2))^{2+k+\ell}} \right. \\
 &\times {}_2F_1\left(2+k+\ell, 1+k; 2+k; -\frac{\Omega_{c2}}{\Omega_{d2}}\mu^2\right) \\
 &+ \frac{2}{(1+((\Omega_{c2}/\Omega_{d2})\mu^2))^{2+k+\ell}} \\
 &\times {}_2F_1\left(2+k+\ell, 1+k; 2+k; -\frac{\Omega_{c1}}{\Omega_{d1}}\mu^2\right) \left. \right\}, \quad (6)
 \end{aligned}$$

where  ${}_2F_1(a, b; c; d)$  is the Gauss hypergeometric function.

(b) *Rician Case.* The Rician distribution is often used to model propagation paths consisting of one strong direct LOS component and many random weaker components. In this case,  $R_i$  and  $r_i$  ( $i = 1, 2$ ) are distributed according to [15, Equation (1)] as follows:

$$\begin{aligned}
 P_{R_1, R_2}(R_1, R_2) &= \frac{4R_1R_2(K+1)^2}{\Omega_d^2(1-\rho^2)} \\
 &\times \exp\left(-\frac{(R_1^2+R_2^2)(K+1)+2K\Omega_d(1-\rho)}{\Omega_d(1-\rho^2)}\right) \\
 &\times \sum_{k=0}^{\infty} \varepsilon_k I_k\left(\frac{2R_1R_2\rho(K+1)}{\Omega_d(1-\rho^2)}\right) \\
 &\times I_k\left(\frac{2R_1}{1+\rho}\sqrt{\frac{K(K+1)}{\Omega_d}}\right) I_k\left(\frac{2R_2}{1+\rho}\sqrt{\frac{K(K+1)}{\Omega_d}}\right), \\
 P_{r_1, r_2}(r_1, r_2) &= \frac{4r_1r_2(K+1)^2}{\Omega_c^2(1-\rho^2)} \\
 &\times \exp\left(-\frac{(r_1^2+r_2^2)(K+1)+2K\Omega_c(1-\rho)}{\Omega_c(1-\rho^2)}\right) \\
 &\times \sum_{\ell=0}^{\infty} \varepsilon_\ell I_\ell\left(\frac{2r_1r_2\rho(K+1)}{\Omega_c(1-\rho^2)}\right) \\
 &\times I_\ell\left(\frac{2r_1}{1+\rho}\sqrt{\frac{K(K+1)}{\Omega_c}}\right) I_\ell\left(\frac{2r_2}{1+\rho}\sqrt{\frac{K(K+1)}{\Omega_c}}\right), \quad (7)
 \end{aligned}$$

where  $I_n(\cdot)$  is the modified Bessel function of the first kind and  $n$ th order,  $\varepsilon_0 = 1$ ,  $\varepsilon_k = 2$ ,  $k \geq 1$ ,  $\Omega_d = \overline{R_i^2}$ ,  $\Omega_c = \overline{r_i^2}$ , and  $K$  is the Rician factor. The Rician distribution spans the range

from Rayleigh fading ( $K \rightarrow 0$ ) to no fading ( $K \rightarrow \infty$ ). Following the same procedure as for the Rayleigh case, the joint PDF of  $\mu_1$  and  $\mu_2$  becomes [17, Equation (6)] as follows:

$$\begin{aligned}
 P_{\mu_1, \mu_2}(\mu_1, \mu_2) &= \frac{\exp(-4K/(1+\rho))}{(K+1)^2} \\
 &\times \sum_{k,\ell,m,n,p,q,s,w=0}^{\infty} \frac{\mu_1^{2(m+n+k)} \Omega_d^{2(q+\ell+1)+s+w} \Omega_c^{2(m+k+1)+n+p}}{(1+\rho)^{2(n+p+k+s+w+\ell)} m!n!p!q!s!w!} \\
 &\times \left( \rho^{2(m+q)+k+\ell} \mu_2^{2(m+p+k)} (1-\rho^2)^{n+p+k+s+w+\ell+2} \right. \\
 &\times \varepsilon_k \varepsilon_\ell K^{n+p+k+s+w+\ell} E(k, \ell, m, n, p, q, s, w) \\
 &\times \left( (\Omega_d + \Omega_c \mu_1^2)^{m+n+k+q+s+\ell+2} \right. \\
 &\times \left. \left. (\Omega_d + \Omega_c \mu_2^2)^{m+p+k+q+w+\ell+2} \right)^{-1} \right), \quad (8)
 \end{aligned}$$

where  $E(k, \ell, m, n, p, q, s, w) \equiv E$  is given by

$$\begin{aligned}
 E &= \Gamma(m+n+k+q+s+\ell+2) \\
 &\times \Gamma(m+p+k+q+w+\ell+2) \\
 &\times (\Gamma(m+k+1)\Gamma(n+k+1)\Gamma(p+k+1) \\
 &\times \Gamma(q+\ell+1)\Gamma(s+\ell+1)\Gamma(w+\ell+1))^{-1}, \quad (9)
 \end{aligned}$$

while the PDF of  $\mu_{\max}$  can be written as in [17, Eq. (11)] as

$$\begin{aligned}
 P_{\mu_{\max}}(\mu) &= \exp\left(-\frac{4K}{1+\rho}\right) \\
 &\times \sum_{k,\ell,m,n,p,q,s,w=0}^{\infty} \left( \varepsilon_k \varepsilon_\ell K^{n+p+k+s+w+\ell} \rho^{2(m+q)+k+\ell} \right. \\
 &\times (1-\rho^2)^{n+p+k+s+w+\ell+2} \\
 &\times \left. \left. ((1+\rho)^{2(n+p+k+s+w+\ell)} m!n!p!q!s!w!)^{-1} \right) \\
 &\times \frac{\mu^{4(m+k)+2(p+n+1)} S^{q+\ell+1}}{(S+\mu^2)^{2(k+m)+n+p+\ell+q+3}} E(k, \ell, m, n, p, q, s, w) \\
 &\times \left[ \left( \frac{S}{S+\mu^2} \right)^w \left( {}_2F_1\left(m+n+k+1, -\ell-q-s; \right. \right. \right. \\
 &\quad \left. \left. \left. m+n+k+2; \frac{\mu^2}{\mu^2+S} \right) \right) \right. \\
 &\quad \left. \times (m+n+k+1)^{-1} \right)
 \end{aligned}$$

$$\begin{aligned}
& + \left( \frac{S}{S + \mu^2} \right)^s \left( {}_2F_1 \left( m + p + k + 1, -\ell - q - w; \right. \right. \\
& \quad \left. \left. m + p + k + 2; \frac{\mu^2}{\mu^2 + S} \right) \right. \\
& \quad \left. \times (m + p + k + 1)^{-1} \right) \Big], \tag{10}
\end{aligned}$$

where  $S = \Omega_d/\Omega_c$  and  $E$  is defined before.

(c) *Nakagami- $m$  Case.* The Nakagami- $m$  distribution has gained widespread application in the modeling of physical fading radio channels [19]. The primary justification of the use of Nakagami- $m$  fading model is its good fit to empirical fading data. It is versatile and through its parameter  $m$ , we can model signal fading conditions that range from severe to moderate, to light fading, or no fading. It includes the one-sided Gaussian distribution ( $m = 0.5$ ) and the Rayleigh distribution ( $m = 1$ ) as special cases. In Nakagami- $m$  fading environment,  $R_i$  and  $r_i$  ( $i = 1, 2$ ) follow the distributions

$$\begin{aligned}
p_{R_1, R_2}(R_1, R_2) &= \frac{4m_1^{m_1+1}(R_1 R_2)^{m_1}}{\Gamma(m_1) \Omega_{d1} \Omega_{d2} (1-\rho) (\sqrt{\rho} \Omega_{d1} \Omega_{d2})^{m_1-1}} \\
&\times \exp\left(-\frac{m_1(R_1^2 + R_2^2)}{\sqrt{\Omega_{d1} \Omega_{d2}}(1-\rho)}\right) \\
&\times I_{m_1-1}\left(\frac{2\sqrt{\rho} m_1 R_1 R_2}{\sqrt{\Omega_{d1} \Omega_{d2}}(1-\rho)}\right), \\
p_{r_1, r_2}(r_1, r_2) &= \frac{4m_2^{m_2+1}(r_1 r_2)^{m_2}}{\Gamma(m_2) \Omega_{c1} \Omega_{c2} (1-\rho) (\sqrt{\rho} \Omega_{c1} \Omega_{c2})^{m_2-1}} \\
&\times \exp\left(-\frac{m_2(r_1^2 + r_2^2)}{\sqrt{\Omega_{c1} \Omega_{c2}}(1-\rho)}\right) \\
&\times I_{m_2-1}\left(\frac{2\sqrt{\rho} m_2 r_1 r_2}{\sqrt{\Omega_{c1} \Omega_{c2}}(1-\rho)}\right). \tag{11}
\end{aligned}$$

Setting

$$\begin{aligned}
W &= \left( \frac{m_1 \mu_1^2}{\sqrt{\Omega_{d1} \Omega_{d2}}} + \frac{m_2}{\sqrt{\Omega_{c1} \Omega_{c2}}} \right) \\
&\times \left( \frac{m_1 \mu_2^2}{\sqrt{\Omega_{d1} \Omega_{d2}}} + \frac{m_2}{\sqrt{\Omega_{c1} \Omega_{c2}}} \right), \tag{12}
\end{aligned}$$

the joint PDF of ratios of the Nakagami- $m$  distributed random variables can be obtained as

$$\begin{aligned}
p_{\mu_1, \mu_2}(\mu_1, \mu_2) &= \frac{4(1-\rho)^{m_1+m_2}}{\Gamma(m_1) \Gamma(m_2)} \\
&\times \sum_{k,l=0}^{\infty} \left( \rho^{k+l} \Gamma^2(k+l+m_1+m_2) \right. \\
&\quad \times (\mu_1 \mu_2)^{2(k+m_1)-1} m_1^{2(m_1+k)} m_1^{2(m_2+l)} \\
&\quad \times (k!l!\Gamma(k+m_1)\Gamma(l+m_2)(\Omega_{c1}\Omega_{c2})^{l+m_2} \\
&\quad \left. \times (\Omega_{d1}\Omega_{d2})^{k+m_1} W^{k+l+m_1+m_2} \right)^{-1}, \tag{13}
\end{aligned}$$

while the PDF of maximum of ratios of Nakagami- $m$  distributed random variables is

$$\begin{aligned}
p_{\mu_{\max}}(\mu) &= \frac{4(1-\rho)^{m_1+m_2}}{\Gamma(m_1) \Gamma(m_2)} \\
&\times \sum_{k,l=0}^{\infty} \left( \rho^{k+l} \Gamma^2(k+l+m_1+m_2) \right. \\
&\quad \times \mu^{4(k+m_1)-1} (\Omega_{c1}\Omega_{c2})^{(m_1+k-m_2-l)/2} \\
&\quad \times m_1^{2(m_1+k)} m_2^{(m_2+l-m_1-k)} \\
&\quad \times \left( k!l!\Gamma(k+m_1)\Gamma(l+m_2) \right. \\
&\quad \times (k+m_1)(\Omega_{d1}\Omega_{d2})^{(m_1+k)} \\
&\quad \left. \left. \times \left( \frac{\mu^2 m_1}{\sqrt{\Omega_{d1} \Omega_{d2}}} + \frac{m_2}{\sqrt{\Omega_{c1} \Omega_{c2}}} \right)^{k+l+m_1+m_2} \right)^{-1} \right) \\
&\times {}_2F_1 \left( k+l+m_1+m_2, m_1+k; k+m_1+1; \right. \\
&\quad \left. \frac{-\mu^2 m_1 \sqrt{\Omega_{c1} \Omega_{c2}} / \Omega_{d1} \Omega_{d2}}{m_2} \right). \tag{14}
\end{aligned}$$

The above presented results can be efficiently used for analyzing wireless communications systems in fading environment. Namely, the joint bivariate PDF of ratios of random variables is necessary for performance analysis of dual diversity combining schemes. In that case,  $\mu_i$  denotes signal-to-interference ratio,  $\Omega_{di}$  and  $\Omega_{ci}$  are the average powers of desired and interference signals at  $i$ th diversity branch, respectively, and  $\rho$  is the correlation between branches. Varying the Rician and Nakagami- $m$  parameter, it is possible to simulate fading with different severity degrees. Having in mind that in interference-limited environment SC diversity receiver selects and outputs the branch with the highest signal-to-interference ratio, the PDF of  $\mu_{\max}$  presents the PDF of signal-to-interference ratio at the SC output. This expression can be used to study important performance measures such as the average bit error probability and channel capacity.

### 3. On the Ratios of Weibull Distributed Random Variables

The Weibull distribution exhibits an excellent fit to experimental fading channel measurements, for both indoor [20] and outdoor [21, 22] environments. It is the reason why Weibull distribution paved its way to wireless communications applications. The fact that the diversity receiver with larger number of branches shows better performance gives an idea to investigate the statistics of arbitrary number of ratios. In this section, the joint PDF, product moments, and PDF of maximum and minimum of arbitrary number of ratios of Weibull distributed random variables are presented. To the best of the authors' knowledge, analytical expressions for product moments and PDF of minimum of ratios of Weibull random variables are novel in the open technical literature.

(a) *The Joint PDF.* The joint bivariate PDFs of correlated Weibull distributed random variables  $R_i$  and  $r_i$  ( $i = 1, 2$ ) are given by [23, Equation (11)] as follows:

$$\begin{aligned}
 p_{R_1, R_2}(R_1, R_2) &= \frac{\beta_1 \beta_2 R_1^{\beta_1 - 1} R_2^{\beta_2 - 1}}{\Omega_{d1} \Omega_{d2} (1 - \rho)} \\
 &\times \exp \left[ -\frac{1}{1 - \rho} \left( \frac{R_1^{\beta_1}}{\Omega_{d1}} + \frac{R_2^{\beta_2}}{\Omega_{d2}} \right) \right] \\
 &\times I_0 \left[ \frac{2\sqrt{\rho} R_1^{\beta_1/2} R_2^{\beta_2/2}}{(1 - \rho) \sqrt{\Omega_{d1} \Omega_{d2}}} \right], \\
 p_{r_1, r_2}(r_1, r_2) &= \frac{\beta_1 \beta_2 r_1^{\beta_1 - 1} r_2^{\beta_2 - 1}}{\Omega_{c1} \Omega_{c2} (1 - \rho)} \\
 &\times \exp \left[ -\frac{1}{1 - \rho} \left( \frac{r_1^{\beta_1}}{\Omega_{c1}} + \frac{r_2^{\beta_2}}{\Omega_{c2}} \right) \right] \\
 &\times I_0 \left[ \frac{2\sqrt{\rho} r_1^{\beta_1/2} r_2^{\beta_2/2}}{(1 - \rho) \sqrt{\Omega_{c1} \Omega_{c2}}} \right],
 \end{aligned} \tag{15}$$

while the joint multivariate PDFs of Weibull distributed random variables  $R_i$  and  $r_i$ ,  $i = 1, \dots, L$ , with exponential correlation can be expressed as [23, Equation (22)] as follows:

$$\begin{aligned}
 p_{R_1, \dots, R_L}(R_1, \dots, R_L) &= \frac{1}{(1 - \rho)^{L-1}} \prod_{i=1}^L \frac{\beta_i R_i^{\beta_i - 1}}{\Omega_{di}} \\
 &\times \exp \left\{ -\frac{1}{1 - \rho} \left[ \frac{R_1^{\beta_1}}{\Omega_{d1}} + \frac{R_L^{\beta_L}}{\Omega_{dL}} + (1 + \rho) \sum_{i=2}^{L-1} \frac{R_i^{\beta_i}}{\Omega_{di}} \right] \right\} \\
 &\times \prod_{i=1}^{L-1} I_0 \left[ \frac{2\sqrt{\rho} R_i^{\beta_i/2} R_{i+1}^{\beta_{i+1}/2}}{(1 - \rho) \sqrt{\Omega_{d1} \dots \Omega_{dL}}} \right], \\
 p_{r_1, \dots, r_L}(r_1, \dots, r_L) &= \frac{1}{(1 - \rho)^{L-1}} \prod_{i=1}^L \frac{\beta_i r_i^{\beta_i - 1}}{\Omega_{ci}} \\
 &\times \exp \left\{ -\frac{1}{1 - \rho} \left[ \frac{r_1^{\beta_1}}{\Omega_{c1}} + \frac{r_L^{\beta_L}}{\Omega_{cL}} + (1 + \rho) \sum_{i=2}^{L-1} \frac{r_i^{\beta_i}}{\Omega_{ci}} \right] \right\} \\
 &\times \prod_{i=1}^{L-1} I_0 \left[ \frac{2\sqrt{\rho} r_i^{\beta_i/2} r_{i+1}^{\beta_{i+1}/2}}{(1 - \rho) \sqrt{\Omega_{c1} \dots \Omega_{cL}}} \right],
 \end{aligned} \tag{16}$$

where  $\beta_i$  is the Weibull parameter. Weibull parameter expresses the fading severity ( $\beta_i > 0$ ). As it increases, the severity of fading decreases, while for  $\beta_i = 2$ , the Weibull distribution reduces to the Rayleigh distribution. The joint PDF of ratios of Weibull random variables,  $\mu_i = R_i/r_i$ ,  $i = 1, \dots, L$ , can be obtained as

$$\begin{aligned}
 p_{\mu_1, \dots, \mu_L}(\mu_1, \dots, \mu_L) &= \int_0^\infty \dots \int_0^\infty |J| p_{R_1, \dots, R_L}(\mu_1 r_1, \dots, \mu_L r_L) \\
 &\times p_{r_1, \dots, r_L}(r_1, \dots, r_L) dr_1 \dots dr_L,
 \end{aligned} \tag{17}$$

where  $|J|$  is the Jacobian transformation given by

$$|J| = \begin{vmatrix} \frac{dR_1}{d\mu_1} & \dots & \frac{dR_1}{d\mu_L} \\ \vdots & & \vdots \\ \frac{dR_L}{d\mu_1} & & \frac{dR_L}{d\mu_L} \end{vmatrix} = \prod_{i=1}^L r_i. \tag{18}$$

Substituting the adequate joint PDFs of  $R_i$  and  $r_i$  in (17) and using the infinite-series representation of the modified Bessel function, after integrations, the joint bivariate PDF of

$\mu_1$  and  $\mu_2$  can be written in the following form [14, Equation (7)]:

$$\begin{aligned}
 P_{\mu_1\mu_2}(\mu_1, \mu_2) &= (1-\rho)^2 \beta_1 \beta_2 \\
 &\times \sum_{k,\ell=0}^{\infty} \frac{\rho^{k+\ell} \mu_1^{(1+k)\beta_1-1} \mu_2^{(1+\ell)\beta_2-1} \Gamma^2(2+k+\ell)}{(\Omega_{d1}\Omega_{d2})^{1+k} (\Omega_{c1}\Omega_{c2})^{1+\ell} (k!\ell!)^2} \\
 &\times \left( \left( \frac{\mu_1^{\beta_1}}{\Omega_{d1}} + \frac{1}{\Omega_{c1}} \right) \left( \frac{\mu_2^{\beta_2}}{\Omega_{d2}} + \frac{1}{\Omega_{c2}} \right) \right)^{-(2+k+\ell)}, \quad (19)
 \end{aligned}$$

while the joint multivariate PDF of  $\{\mu_i\}_{i=1}^L$  can be written as in [16, Equation (4)] as

$$\begin{aligned}
 p_{\mu_1, \dots, \mu_L}(\mu_1, \dots, \mu_L) &= (1-\rho)^2 \\
 &\times \prod_{i=1}^L \beta_i \sum_{k_1, \dots, k_{L-1}, \ell_1, \dots, \ell_{L-1}=0}^{\infty} \frac{\rho^{\sum_{i=1}^{L-1} (k_i + \ell_i)}}{(1+\rho)^{\sum_{i=2}^{L-1} (2+k_i + k_{i-1} + \ell_i + \ell_{i-1})}} \\
 &\times \left( \mu_1^{(1+k_1)\beta_1-1} \mu_L^{(1+k_{L-1})\beta_L-1} \Gamma(2+k_1+\ell_1) \right. \\
 &\quad \times \Gamma(2+k_{L-1}+\ell_{L-1}) \\
 &\quad \times \left( \prod_{i=1}^L \left( \Omega_{di}^{1+(2/L)\sum_{j=1}^{L-1} k_j} \Omega_{ci}^{1+(2/L)\sum_{j=1}^{L-1} \ell_j} \right) \right. \\
 &\quad \times \left( \frac{\mu_1^{\beta_1}}{\Omega_{d1}} + \frac{1}{\Omega_{c1}} \right)^{2+k_1+\ell_1} \\
 &\quad \times \left. \left. \left( \frac{\mu_L^{\beta_L}}{\Omega_{dL}} + \frac{1}{\Omega_{cL}} \right)^{2+k_{L-1}+\ell_{L-1}} \prod_{i=1}^{L-1} (k_i! \ell_i!)^2 \right)^{-1} \right) \\
 &\times \prod_{i=2}^{L-1} \left[ \mu_i^{\beta_i(1+k_i+k_{i-1})-1} \left( \frac{\mu_i^{\beta_i}}{\Omega_{di}} + \frac{1}{\Omega_{ci}} \right)^{-(2+k_i+k_{i-1}+\ell_i+\ell_{i-1})} \right. \\
 &\quad \left. \times \Gamma(2+k_i+k_{i-1}+\ell_i+\ell_{i-1}) \right]. \quad (20)
 \end{aligned}$$

(b) *Product Moments.* Product moments are also important in wireless communications systems because they are a useful

statistical tool to characterize a distribution. The  $(\sum_{i=1}^L n_i)$ th-order moment of the product of  $\{\mu_i\}_{i=1}^L$  can be derived as

$$\begin{aligned}
 \varepsilon \left\langle \prod_{i=1}^L \mu_i^{n_i} \right\rangle &= \int_0^{\infty} \cdots \int_0^{\infty} \left( \prod_{i=1}^L \mu_i^{n_i} \right) p_{\mu_1, \dots, \mu_L}(\mu_1, \dots, \mu_L) d\mu_1 \cdots d\mu_L. \quad (21)
 \end{aligned}$$

Substituting the adequate joint PDF of ratios of Weibull distributed random variables in (21) and using [18, Equation (3.241/4)], for  $n_i/\beta_i < 1, i = 1, \dots, L$ , product moments of two and multiple ratios of Weibull random variables are

$$\begin{aligned}
 \varepsilon \langle \mu_1^{n_1} \mu_2^{n_2} \rangle &= U \sum_{k,\ell=0}^{\infty} \frac{\rho^{k+\ell}}{(k!\ell!)^2} \Gamma\left(1+k+\frac{n_1}{\beta_1}\right) \Gamma\left(1+k+\frac{n_2}{\beta_2}\right) \\
 &\quad \times \Gamma\left(1+\ell-\frac{n_1}{\beta_1}\right) \Gamma\left(1+\ell-\frac{n_2}{\beta_2}\right), \quad (22)
 \end{aligned}$$

where  $U = (1-\rho)^2 (\Omega_{d1}/\Omega_{c1})^{n_1/\beta_1} (\Omega_{d2}/\Omega_{c2})^{n_2/\beta_2}$  and

$$\begin{aligned}
 \varepsilon \left\langle \prod_{i=1}^L \mu_i^{n_i} \right\rangle &= (1-\rho)^2 \\
 &\times \sum_{k_1, \dots, k_{L-1}, \ell_1, \dots, \ell_{L-1}=0}^{\infty} \left( \rho^{\sum_{i=1}^{L-1} (k_i + \ell_i)} \Omega_{d1}^{k_1 - (2/L)\sum_{i=1}^{L-1} k_i + (n_1/\beta_1)} \right. \\
 &\quad \times \Omega_{c1}^{\ell_1 - (2/L)\sum_{i=1}^{L-1} \ell_i - (n_1/\beta_1)} \\
 &\quad \times \left. \left( (1-\rho)^{\sum_{i=2}^{L-1} (2+k_i+k_{i-1}+\ell_i+\ell_{i-1})} \right)^{-1} \right) \\
 &\times \frac{\Omega_{dL}^{k_{L-1} - (2/L)\sum_{i=1}^{L-1} k_i + (n_L/\beta_L)} \Omega_{cL}^{\ell_{L-1} - (2/L)\sum_{i=1}^{L-1} \ell_i - (n_L/\beta_L)}}{\prod_{i=1}^{L-1} (k_i! \ell_i!)^2} \\
 &\times \Gamma\left(1+k_1+\frac{n_1}{\beta_1}\right) \Gamma\left(1+k_{L-1}+\frac{n_L}{\beta_L}\right) \\
 &\times \Gamma\left(1+\ell_1-\frac{n_1}{\beta_1}\right) \Gamma\left(1+\ell_{L-1}-\frac{n_L}{\beta_L}\right)
 \end{aligned}$$

$$\begin{aligned} & \times \prod_{i=2}^{L-1} \left\{ \Omega_{di}^{k_i+k_{i-1}-(2/L) \sum_{j=1}^{L-1} k_j+(n_i/\beta_i)} \Omega_{ci}^{\ell_i+\ell_{i-1}-(2/L) \sum_{j=1}^{L-1} \ell_j-(n_i/\beta_i)} \right. \\ & \quad \times \Gamma \left( 1 + k_i + k_{i-1} + \frac{n_i}{\beta_i} \right) \\ & \quad \left. \times \Gamma \left( 1 + \ell_i + \ell_{i-1} - \frac{n_i}{\beta_i} \right) \right\}, \end{aligned} \tag{23}$$

respectively.

(c) *PDF of Maximum.* The PDF expression of maximum of ratios of random variables,  $\mu_{\max} = \max\{\mu_1, \mu_2, \dots, \mu_L\}$ , can be derived as

$$\begin{aligned} P_{\mu_{\max}}(\mu) &= \underbrace{\int_0^\mu \dots \int_0^\mu}_{(L-1)\text{-fold}} P_{\mu_1\mu_2\dots\mu_L}(\mu, \mu_2, \dots, \mu_L) d\mu_2 d\mu_3 \dots d\mu_L \\ &+ \underbrace{\int_0^\mu \dots \int_0^\mu}_{(L-1)\text{-fold}} P_{\mu_1\mu_2\dots\mu_L}(\mu_1, \mu, \dots, \mu_L) d\mu_1 d\mu_3 \dots d\mu_L \\ &\vdots \\ &+ \underbrace{\int_0^\mu \dots \int_0^\mu}_{(L-1)\text{-fold}} P_{\mu_1\mu_2\dots\mu_L}(\mu_1, \mu_2, \dots, \mu) d\mu_1 d\mu_2 \dots d\mu_{L-1}. \end{aligned} \tag{24}$$

Substituting the joint PDF of  $\{\mu_i\}_{i=1}^L$  in the previous expression and after integrations, the PDFs of maximum of two and multiple ratios of Weibull random variables can be written, respectively, as in [14, Equation(11)] and [16, Equation(12)] as follows:

$$\begin{aligned} P_{\mu_{\max}}(\mu) &= (1-\rho)^2 \sum_{k,\ell=0}^{\infty} \frac{\rho^{k+\ell} \Gamma^2(2+k+\ell) \mu^{(1+k)(\beta_1+\beta_2)-1}}{(1+k)(k!\ell!)^2} \left( \frac{\Omega_{c1}\Omega_{c2}}{\Omega_{d1}\Omega_{d2}} \right)^{1+k} \\ & \times \left\{ \frac{\beta_1}{(1+((\Omega_{c1}/\Omega_{d1})\mu^{\beta_1}))^{2+k+\ell}} \right. \\ & \quad \times {}_2F_1 \left( 2+k+\ell, 1+k, 2+k; -\frac{\Omega_{c2}}{\Omega_{d2}} \mu^{\beta_2} \right) \\ & \quad + \frac{\beta_2}{(1+((\Omega_{c2}/\Omega_{d2})\mu^{\beta_2}))^{2+k+\ell}} \\ & \quad \left. \times {}_2F_1 \left( 2+k+\ell, 1+k, 2+k; -\frac{\Omega_{c1}}{\Omega_{d1}} \mu^{\beta_1} \right) \right\}, \end{aligned} \tag{25}$$

$$\begin{aligned} P_{\mu_{\max}}(\mu) &= (1-\rho)^2 \\ & \times \sum_{k_1, k_2, \dots, k_{L-1}, \ell_1, \ell_2, \dots, \ell_{L-1}=0}^{\infty} \frac{\Omega_{c1}^{1+k_1+\ell_1-(2/L) \sum_{i=1}^{L-1} \ell_i} \Omega_{cL}^{1+k_{L-1}+\ell_{L-1}-(2/L) \sum_{i=1}^{L-1} \ell_i}}{(1+\rho)^{\sum_{i=2}^{L-1} (2+k_i+k_{i-1}+\ell_i+\ell_{i-1})}} \\ & \times \frac{\Gamma(2+k_1+\ell_1) \Gamma(2+k_{L-1}+\ell_{L-1})}{(\prod_{i=1}^L \Omega_{di})^{1+(2/L) \sum_{i=1}^{L-1} k_i} \prod_{i=0}^{L-1} (k_i! \ell_i!)^2} \\ & \times \rho^{\sum_{i=1}^{L-1} (k_i+\ell_i)} \mu^{(1+k_1)\beta_1+(1+k_{L-1})\beta_L+\sum_{i=2}^{L-1} (1+k_i+k_{i-1})\beta_i-1} \\ & \times \prod_{i=2}^{L-1} \left\{ \Omega_{ci}^{1+k_i+k_{i-1}+\ell_i+\ell_{i-1}-(2/L) \sum_{i=1}^{L-1} \ell_i} \right. \\ & \quad \left. \times \Gamma(2+k_i+k_{i-1}+\ell_i+\ell_{i-1}) \right\} Z(\mu), \end{aligned} \tag{26}$$

respectively, where

$$\begin{aligned} Z(\mu) &= \left( \frac{\beta_1}{(1+k_{L-1})(1+((\Omega_{c1}/\Omega_{d1})\mu^{\beta_1}))^{2+k_1+\ell_1}} \right. \\ & \quad \times {}_2F_1 \left( 2+k_{L-1}+\ell_{L-1}, 1+k_{L-1}; 2+k_{L-1}; -\frac{\Omega_{cL}}{\Omega_{dL}} \mu^{\beta_L} \right) \\ & \quad + \frac{\beta_L}{(1+k_1)(1+((\Omega_{cL}/\Omega_{dL})\mu^{\beta_L}))^{2+k_{L-1}+\ell_{L-1}}} \\ & \quad \left. \times {}_2F_1 \left( 2+k_1+\ell_1, 1+k_1; 2+k_1; -\frac{\Omega_{c1}}{\Omega_{d1}} \mu^{\beta_1} \right) \right) \\ & \times \prod_{i=2}^{L-1} \left( {}_2F_1 \left( 2+k_i+k_{i-1}+\ell_i+\ell_{i-1}, 1+k_i+k_{i-1}; \right. \right. \\ & \quad \left. \left. 2+k_i+k_{i-1}; -\frac{\Omega_{ci}}{\Omega_{di}} \mu^{\beta_i} \right) \right. \\ & \quad \left. \times (1+k_i+k_{i-1})^{-1} \right) \\ & + \left( {}_2F_1 \left( 2+k_1+\ell_1, 1+k_1; 2+k_1; -\frac{\Omega_{c1}}{\Omega_{d1}} \mu^{\beta_1} \right) \right. \\ & \quad \times {}_2F_1 \left( 2+k_{L-1}+\ell_{L-1}, 1+k_{L-1}; 2+k_{L-1}; \right. \\ & \quad \left. -\frac{\Omega_{cL}}{\Omega_{dL}} \mu^{\beta_L} \right) \\ & \quad \left. \times ((1+k_1)(1+k_{L-1}))^{-1} \right) \\ & \times \sum_{j=2}^{L-1} \frac{\beta_j}{(1+((\Omega_{cj}/\Omega_{dj})\mu^{\beta_j}))^{2+k_j+k_{j-1}+\ell_j+\ell_{j-1}}} \\ & \times \prod_{i=2, i \neq j}^{L-1} \left[ {}_2F_1 \left( 2+k_i+k_{i-1}+\ell_i+\ell_{i-1}, 1+k_i+k_{i-1}; \right. \right. \\ & \quad \left. \left. 2+k_i+k_{i-1}; -\frac{\Omega_{ci}}{\Omega_{di}} \mu^{\beta_i} \right) \right. \\ & \quad \left. \times (1+k_i+k_{i-1})^{-1} \right]. \end{aligned} \tag{27}$$

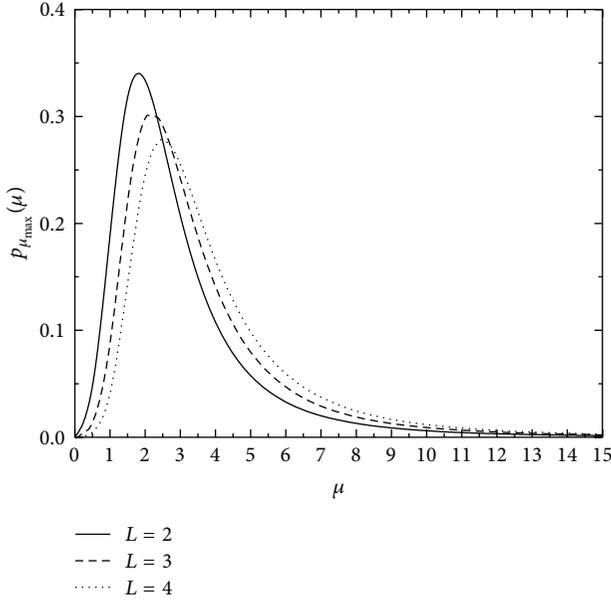


FIGURE 1: Probability density function of maximum of ratios of Weibull random variables for  $\rho = 0.2$ ,  $\Omega_{di} = 0.8$ ,  $\Omega_{ci} = 0.2$ , and  $\beta_i = 2.5$ ,  $i = 1, \dots, L$ .

Figure 1 illustrates the PDF of maximum of two, three, and four ratios of random variables.

(d) *PDF of Minimum.* The minimum of ratios of Weibull random variables,  $\mu_{\min} = \min\{\mu_1, \mu_2, \dots, \mu_L\}$ , can be obtained by

$$\begin{aligned}
 P_{\mu_{\min}}(\mu) &= \underbrace{\int_{\mu}^{\infty} \dots \int_{\mu}^{\infty} P_{\mu_1 \mu_2 \dots \mu_L}(\mu, \mu_2, \dots, \mu_L) d\mu_2 d\mu_3 \dots d\mu_L}_{(L-1)\text{-fold}} \\
 &+ \underbrace{\int_{\mu}^{\infty} \dots \int_{\mu}^{\infty} P_{\mu_1 \mu_2 \dots \mu_L}(\mu_1, \mu, \dots, \mu_L) d\mu_1 d\mu_3 \dots d\mu_L}_{(L-1)\text{-fold}} \\
 &\vdots \\
 &+ \underbrace{\int_{\mu}^{\infty} \dots \int_{\mu}^{\infty} P_{\mu_1 \mu_2 \dots \mu_L}(\mu_1, \mu_2, \dots, \mu) d\mu_1 d\mu_2 \dots d\mu_{L-1}}_{(L-1)\text{-fold}}.
 \end{aligned} \tag{28}$$

Substituting the adequate joint PDF in (28) and using [18, Equation (3.194/2)], with  $(\mu^{\beta_i}/\Omega_{di}) + (1/\Omega_{ci}) = \Phi_i$ ,  $\Omega_{di}/\Omega_{ci}\mu^{\beta_i} = \varphi_i$ ,  $i = 1, 2, \dots, L$ , the PDF of minimum of two ratios of random variables becomes

$$\begin{aligned}
 P_{\mu_{\min}}(\mu) &= (1 - \rho)^2 \\
 &\times \sum_{k, \ell=0}^{\infty} \frac{\rho^{k+\ell} \Gamma^2(2+k+\ell)}{(\Omega_{d1}\Omega_{d2})^{1+k} (\Omega_{c1}\Omega_{c2})^{1+\ell} (k!\ell!)^2 (1+\ell)}
 \end{aligned}$$

$$\begin{aligned}
 &\times \left\{ \beta_1 \Omega_{d2}^{2+k+\ell} \frac{\mu^{(1+k)\beta_1 - (1+\ell)\beta_2 - 1}}{\Phi_1^{2+k+\ell}} \right. \\
 &\quad \times {}_2F_1(2+k+\ell, 1+\ell; 2+\ell; -\varphi_2) \\
 &\quad + \beta_2 \Omega_{d1}^{2+k+\ell} \frac{\mu^{(1+k)\beta_2 - (1+\ell)\beta_1 - 1}}{\Phi_2^{2+k+\ell}} \\
 &\quad \left. \times {}_2F_1(2+k+\ell, 1+\ell; 2+\ell; -\varphi_1) \right\}, \tag{29}
 \end{aligned}$$

and finally, the PDF of minimum of multiple ratios of random variables can be expressed in a more complicated form as follows:

$$\begin{aligned}
 P_{\mu_{\min}}(\mu) &= (1 - \rho)^2 \\
 &\times \sum_{k_1, k_2, \dots, k_{L-1}, \ell_1, \ell_2, \dots, \ell_{L-1}=0}^{\infty} \left( \rho^{\sum_{i=1}^{L-1} (k_i + \ell_i)} (1 + \rho)^{-\sum_{i=2}^{L-1} (2+k_i+k_{i-1}+\ell_i+\ell_{i-1})} \right. \\
 &\quad \times \left( \left( \prod_{i=1}^L \Omega_{di} \right)^{1+(2/L)\sum_{i=1}^{L-1} k_i} \right. \\
 &\quad \left. \left. \times \left( \prod_{i=1}^L \Omega_{ci} \right)^{1+(2/L)\sum_{i=1}^{L-1} \ell_i} \right)^{-1} \right) \\
 &\times \left( \Gamma(2+k_1+\ell_1) \Gamma(2+k_{L-1}+\ell_{L-1}) \right. \\
 &\quad \times \prod_{i=2}^{L-1} \Gamma(2+k_i+k_{i-1}+\ell_i+\ell_{i-1}) \\
 &\quad \times \left( \prod_{i=1}^{L-1} (k_i! \ell_i!) \right)^{-1} \Big) \\
 &\times \left\{ \prod_{i=2}^{L-1} \left( \frac{\Omega_{di}^{2+k_i+k_{i-1}+\ell_i+\ell_{i-1}}}{1+\ell_i+\ell_{i-1}} {}_2F_1 \right. \right. \\
 &\quad \times (2+k_i+k_{i-1}+\ell_i+\ell_{i-1}, \\
 &\quad \left. \left. 1+\ell_i+\ell_{i-1}; 2+\ell_i+\ell_{i-1}; -\varphi_i) \right) \right. \\
 &\quad \times \left( \frac{\beta_1 \Omega_{dL}^{2+k_{L-1}+\ell_{L-1}} \mu^{(1+k_1)\beta_1 - \sum_{i=2}^{L-1} (1+\ell_i+\ell_{i-1})\beta_i - (1+\ell_{L-1})\beta_{L-1}}}{(1+\ell_{L-1}) \Phi_1^{2+k_1+\ell_1}} \right. \\
 &\quad \left. \times {}_2F_1(2+k_{L-1}+\ell_{L-1}, 1+\ell_{L-1}; 2+\ell_{L-1}; -\varphi_L) \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\beta_L \Omega_{d1}^{2+k_1+\ell_1} \mu^{(1+k_{L-1})\beta_L - \sum_{i=2}^{L-1} (1+\ell_i+\ell_{i-1})\beta_i - (1+\ell_1)\beta_1 - 1}}{(1+\ell_1) \Phi_L^{2+k_{L-1}+\ell_{L-1}}} \\
 & \times {}_2F_1(2+k_1+\ell_1, 1+\ell_1; 2+\ell_1; -\varphi_1) \Big) \\
 & + {}_2F_1(2+k_1+\ell_1, 1+\ell_1; 2+\ell_1; -\varphi_1) \\
 & \times {}_2F_1(2+k_{L-1}+\ell_{L-1}; 1+\ell_{L-1}; 2+\ell_{L-1}; -\varphi_L) \\
 & \times \frac{\Omega_{d1}^{2+k_1+\ell_1} \Omega_{dL}^{2+k_{L-1}+\ell_{L-1}}}{(1+\ell_1)(1+\ell_{L-1})} \\
 & \times \sum_{i=2}^{L-1} \left[ \frac{\beta_i}{\Phi_i^{2+k_i+k_{i-1}+\ell_i+\ell_{i-1}}} \right. \\
 & \times \prod_{j=2, j \neq i}^{L-1} \left( \frac{\Omega_{dj}^{2+k_j+k_{j-1}+\ell_j+\ell_{j-1}}}{1+\ell_j+\ell_{j-1}} \right. \\
 & \times {}_2F_1(2+k_j+k_{j-1}+\ell_j+\ell_{j-1}, \\
 & \left. \left. \left. 1+\ell_j+\ell_{j-1}; 2+\ell_j+\ell_{j-1}; -\varphi_j \right) \right) \right] \\
 & \times \mu^{(1+k_i+k_{i-1})\beta_i - (1+\ell_1)\beta_1 - (1+\ell_{L-1})\beta_L - \sum_{j=2, j \neq i}^{L-1} (1+\ell_j+\ell_{j-1})\beta_{j-1}} \Big\}. \tag{30}
 \end{aligned}$$

The PDF of minimum of ratios of random variables is important for analyzing multihop relayed communication systems in which the source terminal communicates with the destination terminal through a number of relay terminals [24]. In that case,  $\mu_i$  is signal-to-interference ratio,  $\Omega_{di}$  and  $\Omega_{ci}$  are the average signal desired and interference powers at  $i$ th terminal input, respectively, and  $\rho$  is the correlation between two successive terminals (in the case of independent terminals, which is the real scenario in practice, correlation coefficient tends to be zero). Figure 2 illustrates the PDF of minimum of two, three, and four ratios of random variables.

The main problem in the infinite-series expressions can be their convergence. Expressions presented in the paper converge rapidly, and thus they can be efficiently used. As an illustrative example, the number of terms needed to be summed in expressions for PDF of minimum of ratios of Weibull random variables is shown in Table 1. It is evident that the number of terms depends strongly on the correlation coefficient.

#### 4. Conclusion

The distributions of ratios of random variables,  $\mu_i = R_i/r_i$ ,  $i = 1, \dots, L$ , are of interest in many areas of science. In this paper, expressions for the joint PDF and PDF of maximum of two ratios of Rayleigh, Rician, and Nakagami- $m$  distributed random variables have been presented. Motivated by the

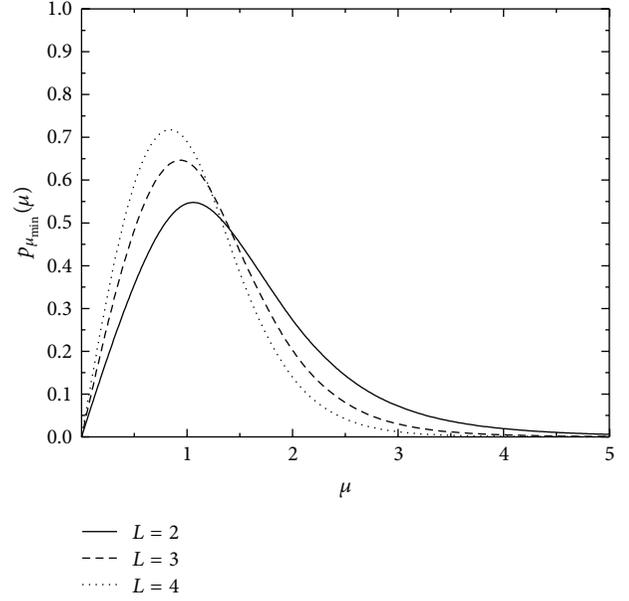


FIGURE 2: Probability density function of minimum of ratios of Weibull random variables for  $\rho = 0.2$ ,  $\Omega_{di} = 0.8$ ,  $\Omega_{ci} = 0.2$ , and  $\beta_i = 2.5$ ,  $i = 1, \dots, L$ .

TABLE 1: Number of terms needed to be summed to achieve accuracy at the 4th digit of expressions for PDF of minimum of ratios of Weibull random variables for  $\Omega_{di} = 0.8$ ,  $\Omega_{ci} = 0.2$ ,  $\beta_i = 2.5$ ,  $i = 1, \dots, L$ .

	$\mu = 0.5$	$\mu = 1$	$\mu = 2$
$L = 2, \rho = 0.2$	7	5	5
$L = 2, \rho = 0.6$	17	15	13
$L = 3, \rho = 0.2$	8	6	5
$L = 3, \rho = 0.6$	18	16	16

fact that Weibull distribution exhibits an excellent fit to experimental fading channel measurements, special attention is dedicated to the case of ratios of Weibull random variables. For this case, expressions for the joint PDF, product moments, PDF of maximum, and PDF of minimum of arbitrary number of ratios of Weibull distributed random variables have been obtained. An application of these results for the wireless communications community has also been described.

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