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448. AN APPLICATION OF NEWTON'S METHOD TO SIMULTANEOUS DETERMINATION OF ZEROS OF A POLYNOMIAL*

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S. B. Prešić gave in paper [1] an iterative procedure for polynomial factorisation, which can successfully be applied to simultaneous determination of all zeros of a polynomial. Prešić noted in his paper that the formulas he obtained are similar to Newton's formulas for determination of isolated zeros of a polynomial, and he proved the quadratic convergence of his iterative method.

In this paper we shall obtain Prešić's formulas by a direct application of Newton's method.

Let $I=\{1, \ldots, n\}$ and let

$$P(x)=x^n+a_1 x^{n-1}+\cdots+a_{n-1} x+a_n \tag{1}$$

be a complex polynomial whose zeros x_i $(i \in I)$ are distinct.

We shall prove that solving the following system of nonlinear equations

$$\begin{aligned} F_1 &= \sum_i x_i=-a_1, \\ F_2 &= \sum_{i<j} x_i x_j=a_2, \\ F_2 &= \sum_{i<j<k} x_i x_j x_k=-a_3, \\ &\vdots \\ F_n &= x_1 \cdots x_n=(-1)^n a_n \end{aligned} \tag{2}$$

(which is, in fact, the system of Viète's formulas for (1)) by Newton's method is equivalent to Prešić's method of simultaneous determination of all zeros of (1).

Newton's iterative procedure for obtaining approximate solutions of a system of nonlinear equations, applied to (2), yields

$$\boldsymbol{x}(k+1)=\boldsymbol{x}(k)-W^{-1}(\boldsymbol{x}(k))(\boldsymbol{f}(\boldsymbol{x}(k))+\boldsymbol{a}) \qquad (k=0, 1, \ldots), \tag{3}$$

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where

$$f(x)=\begin{Vmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Vmatrix}, \quad a=\begin{Vmatrix} a_1 \\ -a_2 \\ \vdots \\ (-1)^{n-1}a_n \end{Vmatrix}, \quad x(k)=\begin{Vmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{Vmatrix}.$$

Matrix $W^{-1}(x)$ is the inverse matrix of JACOBI's matrix

$$W(x)=\frac{\mathrm{d}f}{\mathrm{d}x}=\| a_{ij} \|,$$

where $a_{ij}=\dfrac{\partial F_i}{\partial x_j}$ $(i, j\in I)$. For elements a_{ij} we have the following recurrent relations

$$a_{ij}=1,$$

$$a_{ij}=F_{i-1}-x_j a_{i-1,j}=F_{i-1}-x_j F_{i-2}+\cdots+(-1)^{i-1}x_j^{i-1},$$

i. e.

$$a_{ij}=F_{i-1}-x_j F_{i-2}^{(j)}=F_{i-1}^{(j)},$$

where $F_i^{(j)}$ are homogeneous functions of order i, which do not involve x_j.

Introduce the polynomials Q and R_j by

$$Q(x)=\prod_{m=1}^{n}(x-x_m), \quad R_j(x)=\prod_{\substack{m=1\\ m\neq j}}^{n}(x-x_m) \qquad (j\in I),$$

which, in the expanded form, read

$$Q(x)=x^n-F_1x^{n-1}+F_2x^{n-2}-\cdots+(-1)^nF_n,$$

$$R_j(x)=x^{n-1}-F_1^{(j)}x^{n-2}+F_2^{(j)}x^{n-3}-\cdots+(-1)^{n-1}F_{n-1}^{(j)}.$$

Notice that $Q'(x_j)=R_j(x_j)$ $(j\in I)$.

We now prove the following:

Theorem. *If $x_i\neq x_j \Leftrightarrow i\neq j$ $(i, j\in I)$, then the inverse matrix of*

$$W(x)=\begin{Vmatrix} 1 & 1 & \cdots & 1 \\ F_1^{(1)} & F_1^{(2)} & & F_1^{(n)} \\ \vdots & & & \\ F_{n-1}^{(1)} & F_{n-1}^{(2)} & & F_{n-1}^{(n)} \end{Vmatrix}$$

is given by

$$W^{-1}(x)=\begin{Vmatrix} D_1x_1^{n-1} & -D_1x_1^{n-2} & \cdots & (-1)^{n-1}D_1 \\ D_2x_2^{n-1} & -D_2x_2^{n-2} & & (-1)^{n-1}D_2 \\ \vdots & & & \\ D_nx_n^{n-1} & -D_nx_n^{n-2} & & (-1)^{n-1}D_n \end{Vmatrix},$$

with $D_i=1/Q'(x_i)$ $(i\in I)$.

Proof. Let $C = \| c_{ij} \| = W^{-1}(x)\, W(x)$. Then

$$c_{ij} = D_i \left(x_i^{n-1} - F_1^{(j)} x_i^{n-2} + F_2^{(j)} x_i^{n-3} - \cdots + (-1)^{n-1} F_{n-1}^{(j)} \right)$$
$$= D_i R_j (x_i) = \delta_{ij} \qquad (\delta_{ij} \text{ is KRONECKER's delta}).$$

Hence, C is the unit matrix, and the proof is complete.

Since

$$W^{-1}(x) \left(f(x) + a \right) = \begin{Vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{Vmatrix},$$

where $\varepsilon_i = P(x_i)/Q'(x_i)$ $(i \in I)$, from (3) follows

$$x_i(k+1) = x_i(k) - \frac{P(x_i(k))}{Q'(x_i(k))} \qquad (i \in I,\ k = 0,\ 1, \ldots),$$

which is, in fact, the algorithm given in [1].

REFERENCE

1. S. B. PREŠIĆ: *Jedan iterativni postupak za faktorizaciju polinoma.* Mat. Vesnik **5** (20) (1968), 205—216.

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