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470. A METHOD TO ACCELERATE ITERATIVE PROCESSES IN BANACH SPACE*

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1. Let X be a BANACH space, $T: X \to X$ and let the sequence

$$u_{n+1} = Tu_n \qquad (n=0, 1, 2, \ldots) \tag{1}$$

converge to $a \in X$. Let U be a convex neighbourhood of the limit point a. The iterative method (1) is of order k if

$$\| Tu - a \| = O(\| u - a \|^k) \qquad (u \in U). \tag{2}$$

If the operator T is k-times differentiable in FRÉCHET's sense on U, then the iterative method (1) is of the order k if and only if the following conditions are satisfied:

1° $Ta = a$;

(3) 2° $T'_{(a)},\ T''_{(a)}, \ldots, T^{(k-1)}_{(a)}$ are zero operators;

3° $T^{(k)}_{(u)}$ is non-zero operator, with a norm limited on U (see [1]).

In [2] the following result is given.

Let (1) be an iterative method of the order k. Let the operator $T: X \to X$ be $k+1$-times differentiable in the sense of FRÉCHET in the convex neighbourhood U of the limit point a, and let an inverse operator $\left[I - \frac{1}{k} T'_{(u)}\right]^{-1}$ exist when $u \in U$. Then

$$u_{n+1} = \Psi u_n = u_n - \left[I - \frac{1}{k} T'_{(u_n)}\right]^{-1} (u_n - Tu_n), \tag{4}$$

is at least the iterative method of the order $k+1$.

A fundamental disadvantage of this procedure lies in finding an inverse operator $\left[I - \frac{1}{k} T'_{(u)}\right]^{-1}$, which is in majority of cases very complicated.

For a similar purpose an iterative process, not needing the existence of an inverse operator, is given in the present paper.

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Theorem. *Let* (1) *be an iterative method of the order* $k(\geqq 2)$ *and let the operator* $T: X \to X$ *be* $k+1$*-differentiable on* U *in Fréchet's sense. Then*

$$u_{n+1} = \Phi u_n = Tu_n - \frac{1}{k} T'_{(u_n)} (u_n - Tu_n) \qquad (n = 0, 1, 2, \dots) \tag{5}$$

is at least the iterative method of the order $k+1$.

Proof. Let $u \in U$. Combining Taylor's formula and conditions (3), we have

$$\begin{gathered} T'_{(u)} = \frac{1}{(k-1)!} T^{(k)}_{(a)} (\underbrace{u-a, \dots, u-a}_{k-1 \text{ times}}) + W(a, u-a), \\ Tu = a + \frac{1}{k!} T^{(k)}_{(a)} (\underbrace{u-a, \dots, u-a}_{k \text{ times}}) + w(a, u-a), \end{gathered} \tag{6}$$

where

$$\| W(a, u-a) \| = O(\| u-a \|^k), \quad \| w(a, u-a) \| = O(\| u-a \|^{k+1}).$$

By using $T'_{(u)}$ and Tu from (6) we may rewrite the formula (5) in the form

$$\begin{aligned} \Phi u - a &= Tu - a - \frac{1}{k} T'_{(u)} (u-a) + \frac{1}{k} T'_{(u)} (Tu - a) \\ &= w(a, u-a) - \frac{1}{k} W(a, u-a)(u-a) + \frac{1}{k} T'_{(u)} (Tu - a), \end{aligned}$$

from which if follows

$$\| \Phi u - a \| \leqq \| w(a, u-a) \| + \frac{1}{k} \| W(a, u-a) \| \cdot \| u-a \| + \frac{1}{k} \| T'_{(u)} \| \cdot \| Tu - a \|.$$

Finally, from the relation (2) and equality $\| T'_{(u)} \| = O(\| u-a \|^{k-1})$, at $k \geqq 2$ we obtain

$$\| \Phi u - a \| = O(\| u-a \|^{k+1}),$$

so that the Theorem is proved.

Remark 1. The formula (5) can be obtained from (4) if in expansion

$$\left[I - \frac{1}{k} T'_{(u)} \right]^{-1} = \sum_{i=0}^{+\infty} \left(\frac{1}{k} T'_{(u)} \right)^i,$$

which is convergent for $\| T'_{(u)} \| < k$, we keep only two first members, i.e. $I + \frac{1}{k} T'_{(u)}$. The other members do not influence the order of convergence at $k \geqq 2$.

Example 1. Let X and Y be Banach spaces $F: X \to Y$ an θ be a zero-vector of a space Y. If we use Newton's method of the second order

$$u_{n+1} = Tu_n = u_n - F'^{-1}_{(u_n)} Fu_n \tag{7}$$

to solve the equation $Fu = \theta$, applying the formula (5), we obtain an iterative method of the third order, given by $u_{n+1} = \Phi u_n$ $(n = 0, 1, 2, \dots)$, where

$$\Phi u = u - F'^{-1}_{(u)} Fu - \frac{1}{2} F'^{-1}_{(u)} F''_{(u)} \left(F'^{-1}_{(u)} Fu, F'^{-1}_{(u)} Fu \right).$$

Using the same method many times we easily obtain the iterative formulas of the forth or higher orders.

2. Let us now consider the polynomial

$$P(z)=z^m+a_1 z^{m-1}+\cdots+a_{m-1}z+a_m \qquad (a_i\in C), \tag{8}$$

the zeros of which z_i $(i=1, 2, \ldots, m)$ are mutually different.

PREŠIĆ's method [3] for simultaneous finding of all zeros of polynomial (8) is given by

$$z(n+1)=z(n)-e\big(z(n)\big) \qquad (n=0, 1, 2, \ldots), \tag{9}$$

where the following notations are introduced

$$z=\begin{Vmatrix} z_1 \\ \cdot \\ \cdot \\ z_m \end{Vmatrix}, \quad e(z)=\begin{Vmatrix} e_1 \\ \cdot \\ \cdot \\ e_m \end{Vmatrix}, \quad e_i=\frac{P(z_i)}{Q'(z_i)}, \quad Q(z)=\prod_{j=1}^{m}(z-z_j).$$

With regard to the fact that the iterative process (9) is of the second order, we may come to the iterative process of the third order using the given method.

By differentiating $T(z)=z-e(z)$ we obtain

$$T'(z)=\|t_{ij}\|=\left\|\delta_{ij}-\frac{\partial e_i}{\partial z_j}\right\|,$$

where

$$\frac{\partial e_i}{\partial z_i}=\frac{P'(z_i)}{Q'(z_i)}-\frac{P(z_i)}{Q'(z_i)}\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{1}{z_i-z_j}, \qquad \frac{\partial e_i}{\partial z_j}=\frac{P(z_i)}{Q'(z_i)}\frac{1}{z_i-z_j} \qquad (i\neq j),$$

and δ_{ij} is KRONECKER's delta.

Since $P(z)-Q(z)$ is the polynomial of lower order than m and $Q(z_i)=0$ $(i=1, 2, \ldots, m)$, then on the basis of LAGRANGE interpolation formula

$$P(z)-Q(z)=Q(z)\sum_{j=1}^{m}\frac{P(z_j)}{(z-z_j)\,Q'(z_j)}.$$

From

$$\frac{d}{dz}\{P(z)-Q(z)\}_{z=z_i}=\frac{d}{dz}\left\{Q(z)\sum_{j=1}^{m}\frac{P(z_j)}{(z-z_j)\,Q'(z_j)}\right\}_{z=z_i}$$

we obtained

$$1-\frac{P'(z_i)}{Q'(z_i)}+\frac{P(z_i)}{Q'(z_i)}\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{1}{z_i-z_j}=-\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{P(z_j)}{(z_i-z_j)\,Q'(z_j)},$$

hence we may conclude that the elements of matrix $T'(z)$ are determined by

$$t_{ii}=-\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{e_j}{z_i-z_j}, \quad t_{ij}=-\frac{e_i}{z_i-z_j} \qquad (i\neq j).$$

Applying (5) to $T(z)$ we obtain

$$\Phi(z)=T(z)-\frac{1}{2}T'(z)\big(z-T(z)\big)=z-e(z)+s(z),$$

where

$$s(z)=\begin{Vmatrix} s_1 \\ \vdots \\ s_m \end{Vmatrix}, \quad s_i=e_i\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{e_j}{z_i-z_j} \quad (i=1, 2, \ldots, m).$$

Thus, the corresponding iterative process of the third order is

$$(10)\qquad z_i(n+1)=z_i(n)-e_i\Big(1-\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{e_j}{z_i(n)-z_j(n)}\Big) \quad (i=1, 2, \ldots, m; \quad n=0, 1, \ldots).$$

Let us mention that formulas (10) are determined in [4] in another way.

By further application of method (5) to (10) it is possible to obtain an iterative process with convergence of the fourth order.

REMARK 2. Applying JOVANOVIĆ's method (4) to $T(z)$ we obtain

$$\Phi_1(z)=z-\Big[I-\frac{1}{2}T'(z)\Big]^{-1}(z-T(z)),$$

i.e.

$$\Phi_1(z)=z-H(z)^{-1}e(z),$$

where the elements of matrix $H(z)=||h_{ij}||$ are determined by

$$h_{ii}=1+\frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{e_j}{z_i-z_j}, \quad h_{ij}=\frac{1}{2}\frac{e_i}{z_i-z_j} \quad (i\neq j).$$

EXAMPLE 2. The accurate zeros of polynomial

$$P(z)=z^5-(3.2+3.9\,i)\,z^4-(13.83-1.61\,i)\,z^3+(9.83+29.99\,i)\,z^2$$
$$-(3.63+14.79\,i)\,z+(29.43+49.09\,i)$$

are

$$z_1=1.7+1.1\,i, \quad z_2=4.5+2i, \quad z_3=-3, \quad z_4=-i, \quad z_5=1.8\,i.$$

Starting with approximate values

$$z_1(0)=1+i, \quad z_2(0)=4+2.5\,i, \quad z_3(0)=-2+0.5\,i, \quad z_4(0)=0.5-1.1\,i, \quad z_5(0)=-0.2+2.2\,i,$$

and using formula (10), we have

$$\begin{aligned}
z_1(1)&=1.86594010+i\,1.16539200, & z_1(2)&=1.70313403+i\,1.09663271,\\
z_2(1)&=4.48809503+i\,1.97590059, & z_2(2)&=4.50008890+i\,1.99997626,\\
z_3(1)&=-3.13623734+i\,0.25103344, & z_3(2)&=-2.99977371-i\,0.00297130,\\
z_4(1)&=-0.09598915-i\,1.07210234, & z_4(2)&=-0.00219258-i\,1.00216835,\\
z_5(1)&=0.07819136+i\,2.08184318, & z_5(2)&=-0.00125665+i\,1.80853067,
\end{aligned}$$

$$\begin{aligned}
z_1(3)&=1.69999987+i\,1.09999987, & z_1(4)&=1.69999999+i\,1.10000000,\\
z_2(3)&=4.50000000+i\,2.00000000, & z_2(4)&=4.50000000+i\,1.99999999\\
z_3(3)&=-2.99999999-i\,0.00000000, & z_3(4)&=-3.00000000-i\,0.00000000,\\
z_4(3)&=-0.00000003-i\,1.00000000, & z_4(4)&=-0.00000000-i\,0.99999999,\\
z_5(3)&=0.00000014+i\,1.80000000, & z_5(4)&=-0.00000000+i\,1.80000000,
\end{aligned}$$

It can be concluded from the results obtained that all zeros are determined with accuracy of 10^{-8}.

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