UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 634 — № 677 (1979), 97—100.

650. THE STEFFENSEN INEQUALITY FOR CONVEX FUNCTION OF ORDER *n*

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0. J. F. STEFFENSEN has proved the following results (see [1], [2], [3]):

Theorem A. *Assume that two integrable functions f and g are defined on the interval (a, b), that f never increases and $0 \leqq g(t) \leqq 1$ in (a, b). Then*

$$\int_{b-\lambda}^{b} f(t)\,\mathrm{d}t \leqq \int_{a}^{b} f(t)\,g(t)\,\mathrm{d}t \leqq \int_{a}^{a+\lambda} f(t)\,\mathrm{d}t, \tag{0.1}$$

where

$$\lambda = \int_{a}^{b} g(t)\,\mathrm{d}t.$$

Theorem B. *Let g_1 and g_2 be functions defined on $[a, b]$ such that*

$$\int_{a}^{x} g_1(t)\,\mathrm{d}t \geqq \int_{a}^{x} g_2(t)\,\mathrm{d}t \quad (\forall\, x \in [a, b]) \quad \textit{and} \quad \int_{a}^{b} g_1(t)\,\mathrm{d}t = \int_{a}^{b} g_2(t)\,\mathrm{d}t. \tag{0.2}$$

Let f be an nondecreasing function on $[a, b]$, then

$$\int_{a}^{b} f(x)\,g_1(x)\,\mathrm{d}x \leqq \int_{a}^{b} f(x)\,g_2(x)\,\mathrm{d}x. \tag{0.3}$$

If f is a nonincreasing function on $[a, b]$, the reverse inequality holds.

In paper [4], M. MARJANOVIĆ gave the elegant proof of Theorem A: Let in Theorem B be $g_2(x) = g(x)$, $\lambda = \int_a^b g(x)\,\mathrm{d}x$ and $g_1(x) = 1$ $(x \in [a, a+\lambda])$ and $g_1(x) = 0$ $(x \in (a+\lambda, b])$. Then, we have

$$\int_{a}^{a+\lambda} f(x)\,\mathrm{d}x = \int_{a}^{b} f(x)\,g_1(x)\,\mathrm{d}x \geqq \int_{a}^{b} f(x)\,g(x)\,\mathrm{d}x,$$

which gives the second inequality in (0.1). First inequality in (0.1) is obtained similarly.

Let us notice that the quoted proof holds even with the weaker condition for function g, i. e. if

$$(0.4)\qquad \int_a^x g(x)\,\mathrm{d}x \leqq x-a \ (\forall\, x\in[a,\, a+\lambda]) \ \text{ and } \ \int_x^b g(x)\,\mathrm{d}x \geqq 0 \ \ (\forall\, x\in[a+\lambda,\ b]).$$

From (0.4) it follows

$$(0.5)\qquad \int_x^b g(x)\,\mathrm{d}x=\int_a^b g(x)\,\mathrm{d}x-\int_a^x g(x)\,\mathrm{d}x \geqq \lambda-(x-a)\geqq 0 \ \text{ for } x\in[a,\ a+\lambda],$$

and

$$(0.6)\qquad \int_a^x g(x)\,\mathrm{d}x=\int_a^b g(x)\,\mathrm{d}x-\int_x^b g(x)\,\mathrm{d}x \leqq \lambda \leqq x-a \ \text{ for } \ x\in[a+\lambda,\ b].$$

Combining (0.4), (0.5), (0.6), we obtain that implication (0.4) $\Rightarrow$ (0.7) holds, where

$$(0.7)\qquad \int_a^x g(x)\,\mathrm{d}x \leqq x-a \ \text{ and } \int_x^b g(x)\,\mathrm{d}x \geqq 0 \ \ (\forall\, x\in[a,\ b]).$$

Since, evidently, (0.7) $\Rightarrow$ (0.4), we conclude that (0.4) $\Leftrightarrow$ (0.7) is valid.

On the basis of the above, we can formulate the following results:

Theorem A1. *Assume that two integrable functions f and g on $[a, b]$, that f is noninc-reasing and that* (0.7) *holds. Then the second inequality in* (0.1) *is valid.*

Theorem A2. *Let function f fulfils conditions as in Theorem* A1. *If*

$$\int_x^b g(x)\,\mathrm{d}x \leqq b-x \ \text{ and } \int_a^x g(x)\,\mathrm{d}x \geqq 0 \ \ (\forall\, x\in[a,\ b]),$$

then the first inequality in (0.1) *is valid.*

1. In this partion of paper we will generalize Theorems A1 and A2 in the case when function f is convex of order n. There we will use result ([5]):

Theorem C. *Let $x\mapsto f(x)$ be a convex function of order n $(n\geqq 1)$ on $[a, b]$. Then, for every $c\in[a, b]$, the function $x\mapsto \dfrac{G(x)}{(x-c)^n}$ is nondecreasing on $[a, b]$, where*

$$G(x)=f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k,$$

with $f^{(k)}(c)$ being the right derivative for $c=a$ $(f_+^{(k)}(a))$ and the left derivative for $c=b$ $(f_-^{(k)}(b))$.

Theorem 1. *Let the functions f and g satisfy the conditions:*

$1°$ *f is convex of order n ($n \in \mathbf{N}$);*

$2°$ $f^{(k)}(a)=0 \quad (k=0, 1, \ldots, n-1)$;

$3°$ $\int_a^x (x-a)^n g(x)\,dx \leqq \frac{(x-a)^{n+1}}{n+1}$ *and* $\int_x^b (x-a)^n g(x)\,dx \geqq 0$ ($\forall\ x \in [a, b]$).

Then

$$\int_a^{a+\lambda_1} f(x)\,dx \leqq \int_a^b f(x) g(x)\,dx, \tag{1.1}$$

where

$$\lambda_1 = \left[(n+1)\int_a^b (x-a)^n g(x)\,dx\right]^{\frac{1}{n+1}}. \tag{1.2}$$

Proof. According to Theorem C for $c=a$, and with regard to the assumption for function f, the function $x \mapsto \frac{f(x)}{(x-a)^n}$ is nondecreasing.

Let the functions g_1 and g_2 satisfy the conditions

$$\int_a^x (t-a)^n g_1(t)\,dt \geqq \int_a^x (t-a)^n g_2(t)\,dt \quad (\forall\, x \in [a, b]) \tag{1.3}$$

and

$$\int_a^b (t-a)^n g_1(t)\,dt = \int_a^b (t-a)^n g_2(t)\,dt. \tag{1.4}$$

If we replace $f(x)$, $g_1(x)$, $g_2(x)$ by $\frac{f(x)}{(x-a)^n}$, $(x-a)^n g_1(x)$, $(x-a)^n g_2(x)$ respectively in the Theorem B, we obtain

$$\int_a^b f(x) g_1(x)\,dx \leqq \int_a^b f(x) g_2(x)\,dx. \tag{1.5}$$

Let be now $g_1(x)=1$ ($x \in [a, a+\lambda_1]$), $g_1(x)=0$ ($x \in (a+\lambda_1, b]$), $g_2(x)=g(x)$, where λ_1 is given by (1.2). It is easy to show that the conditions (1.3) and (1.4) are satisfied.

According to (1.5), we get

$$\int_a^{a+\lambda_1} f(x)\,dx = \int_a^b f(x) g_1(x)\,dx \leqq \int_a^b f(x) g(x)\,dx,$$

which proves the theorem.

The following result can be similarly proved:

Theorem 2. *Let the functions f and g satisfy the conditions:*

1° *f is convex of order n ($n \in \mathbf{N}$);*

2° $f^{(k)}(b) = 0 \quad (k = 0, 1, \ldots, n-1)$;

3° $\int_x^b (b-x)^n g(x)\,\mathrm{d}x \leqq \frac{(b-x)^{n+1}}{n+1}$ *and* $\int_a^x (b-x)^n g(x)\,\mathrm{d}x \geqq 0$ $(\forall\, x \in [a, b])$.

If n is even number, the following inequality

$$\int_a^b f(x)\,g(x)\,\mathrm{d}x \leqq \int_{b-\lambda_2}^b f(x)\,\mathrm{d}x$$

holds, where

$$\lambda_2 = \left[(n+1)\int_a^b g(x)\,(b-x)^n\,\mathrm{d}x\right]^{\frac{1}{n+1}}.$$

If n is odd number, the reverse inequality holds.

REMARK 1. If $0 \leqq g(x) \leqq 1$, the condition 3° in Theorem 1 (also in Theorem 2) is fulfilled.

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