EXTREMAL PROBLEMS AND INEQUALITIES OF MARKOV-BERNSTEIN TYPE FOR POLYNOMIALS

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Abstract. The classical Markov (1889) and Bernstein (1912) inequalities and corresponding extremal problems were generalized for various domains, various norms and for various subclasses for polynomials, both algebraic and trigonometric. Beside some classical results in uniform norm, we give a short account of L^r inequalities of Markov type for algebraic polynomials, with a special attention to the case $r = 2$. We also study extremal problems of Markov's type

$$C_{n,m} = \sup_{P \in \mathcal{P}_n} \frac{\|\mathcal{D}_m P\|}{\|A^{m/2} P\|},$$

where $\mathcal{P}_n$ is the class of all algebraic polynomials of degree at most n, $d\lambda(t) = w(t)dt$ is a nonnegative measure corresponding to the classical orthogonal polynomials, $A \in \mathcal{P}_2$, $\|P\| = \left(\int_{\mathbb{R}} |P(t)|^2 d\lambda(t)\right)^{1/2}$, and $\mathcal{D}_m$ is a differential operator defined by

$$\mathcal{D}_m P = \frac{d^m}{dt^m}\left[A^m P\right] \qquad (P \in \mathcal{P}_n,\ m \geq 1).$$

1. Introduction and Notation

Let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree at most n. We take

$$\|f\|_\infty := \max_{-1 \leq t \leq 1} |f(t)| \tag{1.1}$$

and

$$\|f\|_r := \left(\int_{\mathbb{R}} |f(t)|^r \, d\lambda(t)\right)^{1/r}, \quad r \geq 1, \tag{1.2}$$

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where $d\lambda(t)$ is a given nonnegative measure on the real line $\mathbb{R}$, with compact support or otherwise, for which all moments $\mu_k = \int_{\mathbb{R}} t^k \, d\lambda(t)$, $k = 0, 1, \ldots$, exist and are finite and $\mu_0 > 0$. In a special case $r = 2$, (1.2) reduces to

$$\|f\|_2 = \Big(\int_{\mathbb{R}} |f(t)|^2 \, d\lambda(t)\Big)^{1/2}. \tag{1.3}$$

In that case we have an inner product defined by

$$(f, g) = \int_{\mathbb{R}} f(t)\overline{g(t)} \, d\lambda(t)$$

such that $\|f\|_2 = \sqrt{(f, f)}$. Then also, there exists a unique set of (monic) orthogonal polynomials $\pi_k(\cdot) = \pi_k(\,\cdot\,; d\lambda)$, $k \geq 0$, with respect to $(\cdot, \cdot)$, such that

$$\pi_k(t) = t^k + \text{lower degree terms}, \qquad (\pi_k, \pi_m) = \|\pi_k\|_2^2 \, \delta_{km},$$

where δ_{km} is Kronecker's delta.

A standard case of orthogonal polynomials is when the measure $d\lambda$ can be express as $d\lambda(t) = w(t)\,dt$, where the weight function $t \mapsto w(t)$ is a non-negative and measurable in Lebesgue's sense for which all moments exist and $\mu_0 = \int_{\mathbb{R}} w(t)\,dt > 0$. A very important class of such orthogonal polynomials on an interval of orthogonality $(a, b) \in \mathbb{R}$ is constituted by so-called the *classical orthogonal polynomials*. They are distinguished by several particular properties (cf. [31]).

Without loss of generality, we can restrict our consideration only to the following three intervals of orthogonality: $(-1, 1)$, $(0, +\infty)$, $(-\infty, +\infty)$, with the inner product

$$(f, g) = \int_a^b w(t) f(t)\overline{g(t)} \, dt. \tag{1.4}$$

The orthogonal polynomials $\{Q_n\}$ on (a, b) with respect to the inner product (1.4) are called the *classical orthogonal polynomials* if their weight functions $t \mapsto w(t)$ satisfy the differential equation

$$\frac{d}{dt}(A(t)w(t)) = B(t)w(t),$$

where

$$A(t) = \begin{cases} 1 - t^2, & \text{if } (a, b) = (-1, 1), \\ t, & \text{if } (a, b) = (0, +\infty), \\ 1, & \text{if } (a, b) = (-\infty, +\infty), \end{cases}$$

and $B(t)$ is a polynomial of the first degree. For such classical weights we will write $w \in CW$.

Based on this definition, the classical orthogonal polynomials $\{Q_n\}$ on (a, b) can be specificated as the *Jacobi polynomials* $P_n^{(\alpha,\beta)}(t)$ $(\alpha, \beta > -1)$ on $(-1, 1)$, the *generalized Laguerre polynomials* $L_n^s(t)$ $(s > -1)$ on $(0, +\infty)$, and finally as the *Hermite* *polynomials* $H_n(t)$ on $(-\infty, +\infty)$. Their weight functions and the corresponding polynomials $A(t)$ and $B(t)$ are given in Table 1.1.

TABLE 1.1: The classification of the classical orthogonal polynomials

(a, b)	$w(t)$	$A(t)$	$B(t)$	λ_n
$(-1, 1)$	$(1-t)^\alpha(1+t)^\beta$	$1-t^2$	$\beta-\alpha-(\alpha+\beta+2)t$	$n(n+\alpha+\beta+1)$
$(0, +\infty)$	$t^s e^{-t}$	t	$s+1-t$	n
$(-\infty, +\infty)$	e^{-t^2}	1	$-2t$	$2n$

The classical orthogonal polynomial $t \mapsto Q_n(t)$ is a particular solution of the second order linear differential equation of hyphergeometric type

$$L[y] = A(t)y'' + B(t)y' + \lambda_n y = 0, \tag{1.5}$$

where λ_n is given also in Table 1.1.

2. Classical Extremal Problems in Uniform Norm

The first result on the extremal problems of Markov type was connected with some investigations of the well-known Russian chemist Mendeleev [18]. Namely, the question was: *If* $P(t)$ *is an arbitrary quadratic polynomial defined on an interval* $[a, b]$, *with*

$$\max_{t\in[a,b]} P(t) - \min_{t\in[a,b]} P(t) = L,$$

how large can $P'(t)$ *be on* $[a, b]$*?*

Changing the horizontal scale and shifting the coordinate axis until $|P(t)| \le 1$, the problem can be reduced to a simpler one: *If* $P(t)$ *is an arbitrary quadratic polynomial and* $|P(t)| \le 1$ *on* $[-1, 1]$, *how large can* $|P'(t)|$ *be on* $[-1, 1]$? Mendeleev found that $|P'(t)| \le 4$ on $[-1, 1]$. This result is the best possible because for $P(t) = 1 - 2t^2$ we have $P(t) \le 1$ and $P'(\pm 1) = 4$.

The corresponding problem for polynomials of degree n was considered by A. A. Markov [16]. Taking the uniform norm (1.1) he solved the extremal problem

$$A_n = \sup_{P\in\mathcal{P}_n} \frac{\|P'\|_\infty}{\|P\|_\infty},$$

finding the best constant $A_n = n^2$ and the extremal polynomial $P^*(t) = cT_n(t)$, where T_n is the Chebyshev polynomial of the first kind of degree n and c is an arbitrary constant. The best constant can be expressed also as $A_n = T_n'(1)$. Thus, the classical Markov's inequality can be expressed in the form

$$\|P'\|_\infty \le n^2 \|P\|_\infty \qquad (P \in \mathcal{P}_n).$$

In 1892, younger brother V. A. Markov [17] found the best possible inequality for k-th derivative,

$$\|P^{(k)}\|_\infty \le T_n^{(k)}(1)\|P\|_\infty \qquad (P \in \mathcal{P}_n),$$

where the extremal polynomial is also T_n. The best constant can be expressed in the form

$$T_n^{(k)}(1) = \|T_n^{(k)}\|_\infty = \frac{1}{(2k-1)!!}\prod_{i=0}^{k-1}(n^2 - i^2).$$

A version of this remarkable paper in German was published in 1916.

In 1912 Bernstein [4] considered another type of these inequalities taking $\|f\| = \max\limits_{|z|\le 1}|f(z)|$. He proved the inequality

$$\|P'\| \le n\|P\| \qquad (P \in \mathcal{P}_n),$$

with equality case when $P(z) = cz^n$ (c is an arbitrary constant).

There are several different forms of this Bernstein's inequality. If we take $\mathcal{T}_n$ to be set of all trigonometric polynomials of degree at most n and

$$\|P\| = \max_{|z|=1}|P(z)| = \max_{-\pi<\theta\le\pi}|P(e^{i\theta})|,$$

then a trigonometric version can be stated in the following form: *Let $T \in \mathcal{T}_n$ and $|T(\theta)| \le M$, then $|T'(\theta)| \le nM$. The equality holds for $T(\theta) = \gamma \sin n(\theta - \theta_0)$, where $|\gamma| = 1$.*

The standard form of Bernstein's inequality can be done as:

Theorem 2.1. *Let $P \in \mathcal{P}_n$ and $|P(t)| \le 1$ $(-1 \le t \le 1)$, then*

$$|P'(t)| \le \frac{n}{\sqrt{1-t^2}}, \qquad -1 < t < 1.$$

The equality is attained at the points $t = t_\nu = \cos\frac{(2\nu-1)\pi}{2n}$, $\nu = 1,\dots,n$, if and only if $P(t) = \gamma T_n(t)$, where $|\gamma| = 1$.

Combining the inequalities of Markov and Bernstein we can state the following result:

Theorem 2.2. *If $P \in \mathcal{P}_n$ then*

$$|P'(t)| \le \min\Big\{n^2, \frac{n}{\sqrt{1-t^2}}\Big\}\|P\|_\infty, \qquad -1 \le t \le 1.$$

A general question could be stated: *How large can $|P^{(k)}(t)|$ be, for a given t, when $|P(t)| \le 1$ on $[-1,1]$?* Let this maximum be $M_{n,k}(t)$, i.e.,

$$|P^{(k)}(t)| \le M_{n,k}(t) \qquad (1 \le k < n). \tag{2.1}$$

For $k = 1$ we put $M_{n,1}(t) = M_n(t)$. It is easy to see that the function M_n is even, i.e., $M_n(-t) = M_n(t)$.

The problem of finding $M_n(t)$ was stated by A. A. Markov himself, and solved for $n = 2$ and $n = 3$. He determined that

$$M_2(t) = \begin{cases} \dfrac{1}{1-t}, & 0 \le t \le \dfrac{1}{2}, \\[2ex] 4t, & \dfrac{1}{2} \le t \le 1, \end{cases}$$

and

$$M_3(t) = \begin{cases} 3(1-4t^2), & t \in [t_0, t_1], \\[1ex] \dfrac{7\sqrt{7}+10}{9(1+t)}, & t \in [t_1, t_2], \\[2ex] \dfrac{16t^3}{(9t^2-1)(1-t^2)}, & t \in [t_2, t_3], \\[2ex] \dfrac{7\sqrt{7}-10}{9(1-t)}, & t \in [t_3, t_4], \\[1ex] 3(4t^2-1), & t \in [t_4, t_5], \end{cases}$$

where

$$t_0 = 0, \quad t_1 = \frac{1}{6}(\sqrt{7}-2) \cong 0.1076, \quad t_2 = \frac{1}{9}(2\sqrt{7}-1) \cong 0.4768,$$
$$t_3 = \frac{1}{9}(2\sqrt{7}+1) \cong 0.6991, \quad t_4 = \frac{1}{6}(\sqrt{7}+2) \cong 0.7743, \quad t_5 = 1.$$

The determination of $M_n(t)$, $n \ge 4$, is very complicated and it can be given by a technique of Voronovskaja (see [40]). Using the same method, Gusev [14] found the corresponding function $M_{n,k}(t)$ in the inequality (2.1).

Instead of the condition $|P(t)| \le 1$ on $[-1,1]$, Bernstein [5] used a more general condition

$$|P(t)| \le \sqrt{H(t)} \qquad (-1 \le t \le 1),$$

where H is an arbitrary positive polynomial on $[-1,1]$ of degree $s\,(\le 2n)$. We mention an interesting result of V. Videnskiĭ [39]:

Theorem 2.3. *Let $P \in \mathcal{P}_n$ and*

$$|P(t)| \le |\alpha t + i\sqrt{1-t^2}| \qquad (\alpha \ge 0,\ -1 \le t \le 1).$$

Then, for $k = 1, \dots, n$ and $-1 \le t \le 1$, we have that

$$|P^{(k)}(t)| \le Q_n^{(k)}(1;\alpha),$$

where

$$Q_n(t;\alpha) = \frac{1}{2}(\alpha+1)T_n(t) + \frac{1}{2}(\alpha-1)T_{n-2}(t).$$

The equality is attained only for $P(t) = \gamma Q_n(t)$ at the endpoints $t = \pm 1$, where $|\gamma| = 1$.

Several inequalities of this type were given by Videnskiĭ, Duffin and Schaeffer, Turán, Rahman, Pirrre and Rahman, Rahman and Schmeisser (see Chapter 6 in [24]).

3. Extremal Problems in L^r-norm

The first results on extremal problems in L^2-norm given by (1.3),

$$\|P'\|_2 \le A_n \|P\|_2 \qquad (P \in \mathcal{P}_n), \tag{3.1}$$

were given by E. Schmidt [27] and Turán [32]:

Theorem 3.1. (a) *Let $(a,b) = (-\infty, +\infty)$ and*

$$\|f\|_2^2 = \int_{-\infty}^{\infty} e^{-t^2} f(t)^2\, dt.$$

Then the best constant in the inequality (3.1) *is $A_n = \sqrt{2n}$. An extremal polynomial is Hermite's polynomial H_n.*
(b) *Let $(a,b) = (0, +\infty)$ and*

$$\|f\|_2^2 = \int_0^{\infty} e^{-t} f(t)^2\, dt.$$

Then

$$A_n = \left(2 \sin \frac{\pi}{4n+2}\right)^{-1}.$$

The extremal polynomial is

$$P(t) = \sum_{\nu=1}^{n} \sin \frac{\nu\pi}{2n+1} L_\nu(t),$$

where L_ν is Laguerre polynomial.

Theorem 3.1 (b), in this form, was formulated by Turán [32].

An important generalization of A. A. Markov's inequality for algebraic polynomials in an integral norm was given by Hille, Szegő, and Tamarkin [15]. Taking

$$\|f\|_r = \left(\int_{-1}^{1} |f(t)|^r\, dt\right)^{1/r},$$

they proved the following theorem:

Theorem 3.2. *Let $r \geq 1$ and let $P \in \mathcal{P}_n$. Then*

$$\|P'\|_r \leq An^2 \|P\|_r, \tag{3.2}$$

where the constant $A = A(n,r)$ is given by

$$A(n,r) = 2(r-1)^{1/r-1}\Big(r+\frac{1}{n}\Big)\Big(1+\frac{r}{nr-r+1}\Big)^{n-1+1/r}, \tag{3.3}$$

for $r > 1$, and

$$A(n,1) = 2\Big(1+\frac{1}{n}\Big)^{n+1}.$$

The factor n^2 in (3.2) cannot be replaced by any function tending to infinity more slowly. Namely, for each n, there exist polynomials $P(t)$ of degree n such that the left side of (3.2) is $\leq Bn^2$, where B is a constant of the same nature as $A = A(n,r)$.

The constant $A(n,r)$ in Theorem 3.2 is not the best possible. We can see that $A(n,r) \leq 6\exp(1+1/e)$, for every n and $r \geq 1$. Also,

$$A(n,r) \ \to \ \begin{cases} 2(1+1/(n-1))^{n-1} < 2e & (n \ \text{ fixed}, \ r \to +\infty), \\ 2e & (r = 1, \ n \to +\infty), \\ 2er(r-1)^{(1/r)-1} & (r > 1 \ \text{ fixed}, \ n \to +\infty). \end{cases}$$

Some improvements of the constant A have been obtained by Goetgheluck [8]. He found that

$$A = \bar{A}(n,1) = \sqrt{\frac{8}{\pi}}\Big(1+\frac{3}{4n}\Big)^2.$$

It is easy to see that for each $n \geq 1$,

$$\sqrt{8/\pi}\Big(1+\frac{3}{4n}\Big)^2 < 2e < 2\Big(1+\frac{1}{n}\Big)^{n+1}.$$

For $r > 1$ he found the following very complicated expression

$$A = \bar{A}(n,r) = \left(\frac{(2r+1)^{2+1/r}}{r(r+1)}\right)^{(r-1)/(r+1)} \left(2r\,\frac{r+1}{r-1}\right)^{1/r} \left(\frac{r-1}{2}\right)^{2/r(r+1)} \times$$
$$\times \left(1-\frac{3}{5n}\right)^{1-1/r} \left(1+\frac{1}{nr}\right)^{n+1/r}.$$

Remark 3.1. In [8, Theorem 2] there is a misprint in the last factor in $\bar{A}(n,r)$.

Numerical calculations show that this constant is less than the corresponding constant in (3.3). Typical graphics of $\bar{A}(n,r)$ and $A(n,r)$ are displyed in Figure 3.1. Also, one can see that $\bar{A}(n,r) \to 4(1-3/5n)$ as $r \to +\infty$, n being fixed.

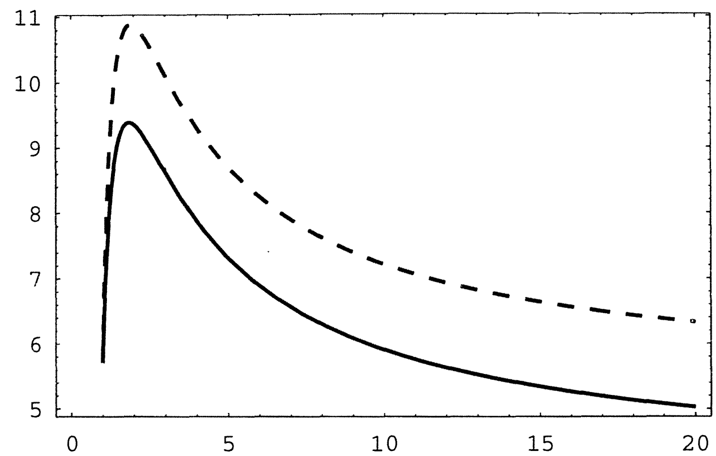

FIG. 3.1: Graphics $r \mapsto \bar{A}(n,r)$ (solid line) and $r \mapsto A(n,r)$ (broken line) for $n = 10$

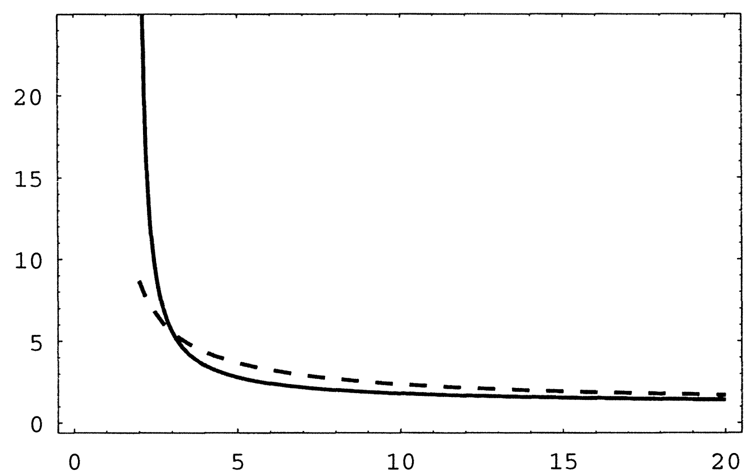

FIG. 3.2: Graphics $r \mapsto B(r)$ (solid line) and $r \mapsto C(r)$ (broken line)

Recently Baran [2] has given a new proof of the inequality (3.2) for $r > 2$, providing a better constant.

Theorem 3.3. *If* $r > 2$ *and* $P \in \mathcal{P}_n$. *Then* (3.2) *holds with*

$$A = B(r) = \left[2\nu(2/r)(r+3)^2\right]^{1/r},$$

where $\nu(t) = \pi t/\sin(\pi t)$, $t \in (0,1)$.

We note that $B(r) \to 1$ when $r \to +\infty$ and $B(r) \to +\infty$ when $r \to 2$. Since the function $\nu(t)$ is increasing on $(0, 1/2]$, it is easy to see that for $r \geq 4$, we have

$$B(r) = \pi^{1/r}(r+3)^{2/r}.$$

An improvement of the constant $B(r)$ for r near to 2 was also obtained by Baran [2]:

Theorem 3.4. *If* $r \geq 2$ *and* $P \in \mathcal{P}_n$. *Then* (3.2) *holds with*

$$A = C(r) = 2^{1/r} U_r^{1-2/r} \left(2U_r^2 + V_q U_r\right)^{1/r},$$

where

$$U_r = 2^{-1/r}(r+3)^{2/r}, \quad V_q = \left(8\nu(1/q)\right)^{1/q}, \quad \nu(t) = \pi t/\sin(\pi t), \quad 1/r + 1/q = 1.$$

This constant $C(r)$ can be expressed in the form

$$C(r) = (r+3)^{2/r}\left(2 + \frac{2^{1/r}V_q}{(r+3)^{2/r}}\right)^{1/r}.$$

Graphics of $B(r)$ and $C(r)$ are showed in Figure 3.2.

Applying the inequality $2^{1/r}V_q < 4\pi r(r+3)^{2/r}$ $\left(r \geq 2,\ q = r/(r-1)\right)$, the constant $C(r)$ can be approximated by (cf. Baran [2, Corollary 2.10])

$$C(r) < \tilde{C}(r) = (r+3)^{2/r}(2 + 4\pi r)^{1/r}.$$

Graphics of $C(r)$ and $\tilde{C}(r)$ are presented in Figure 3.3 as well as the graphic of the Goetgheluck's estimate $\bar{A}(n,r)$ for $n = 100$.

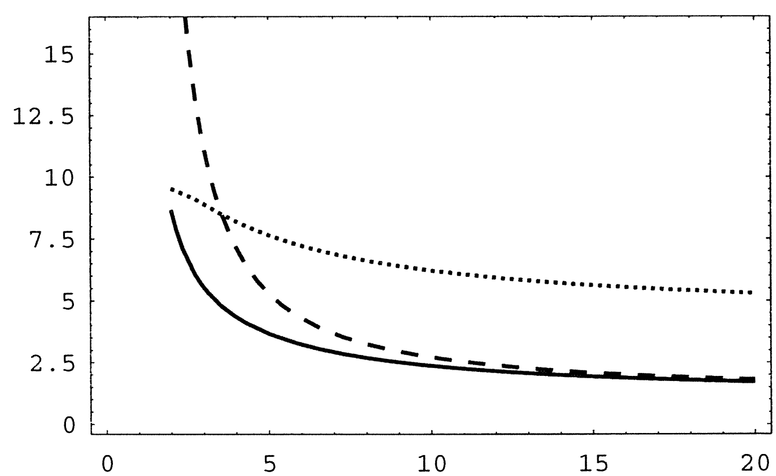

FIG. 3.3: Graphics $r \mapsto C(r)$ (solid line), $r \mapsto \tilde{C}(r)$ (broken line), and $r \mapsto \bar{A}(n,r)$ for $n = 100$ (dotted line)

The special case $r = 2$ has been investigated several times. Hille, Szegő, and Tamarkin [15] proved that $A(n, 2) \to 1/\pi$ when $n \to +\infty$. Schmidt [27] also investigated an asymptotics case of $A(n, 2)$ in (3.2). For $n \geq 5$, he obtained that

$$A(n,2) = \frac{(2+3/n)^2}{4\pi}\left(1 - \frac{\pi^2 - 3}{3(2n+3)^2} + \frac{16R}{(2n+3)^4}\right)^{-1},$$

where $-6 < R < 13$. Also, Bellman [3] proved that $A(n, 2) \leq 1/\sqrt{2}$.

In 1987 Dörfler [6] considered the analogous problem for derivatives of higher order and gave a method for computing the best possible constant in the inequality of Markov type

$$\|P^{(k)}\| \leq C_{n,k}\|P\|, \tag{3.4}$$

where the norm $\|f\| = \|f\|_2$ is given by (1.4) and $d\lambda(t) = w(t)\,dt$. Here $w : (a, b) \to \mathbb{R}_+$ $(-\infty \leq a < b \leq +\infty)$ is an arbitrary weight function for which all moments are finite.

Theorem 3.5. *Let $P \in \mathcal{P}_n$. Then the best possible constant $C_{n,k}$ in* (3.4) *is equal to the largest singular value of the matrix $A_n^{(k)}$, where*

$$A_n^{(k)} = \begin{bmatrix} e_{0,0}^{(k)} & \cdots & e_{n,0}^{(k)} \\ \vdots & & \\ & & \\ e_{0,n-k}^{(k)} & & e_{n,n-k}^{(k)} \end{bmatrix}, \qquad e_{\nu,j}^{(k)} = \int_a^b \pi_\nu^{(k)}(t)\pi_j(t)w(t)\,dt,$$

and $\{\pi_\nu\}$ is a system of polynomials orthonormal with respect to the weight function w. Moreover,

$$\max_{0\leq\nu\leq n} \|\pi_\nu^{(k)}\| \leq C_{n,k} \leq \left(\sum_{\nu=0}^{n} \|\pi_\nu^{(k)}\|^2\right)^{1/2}$$

holds.

Using Dörfler's method, Goetgheluck [8] calculated $C_{n,1} = A_n n^2$ for $n \leq 65$ and showed that A_n is a decreasing function in n and $1/\pi < A_n < 1/3$ for $n > 64$.

An alternative method for computing the best constant

$$C_{n,k} = C_{n,k}(d\lambda) = \sup_{P\in\mathcal{P}_n} \frac{\|P^{(k)}\|}{\|P\|} \qquad (1 \leq k \leq n) \tag{3.5}$$

was also given in 1987 by Milovanović [20] (see also [24]):

Theorem 3.6. *The best constant $C_{n,k}$ defined in* (3.5) *is equal to the spectral norm of one triangular matrix $Q_{n,k}^T$, where*

$$Q_{n,k} = \left[q_{ij}^{(k)}\right]_{k\leq i,j\leq n} \qquad \left(q_{i,j}^{(k)} = 0 \Leftarrow i > j\right),$$

i.e.,

$$C_{n,k} = \sigma(Q_{n,k}^T) = \left(\lambda_{\max}(Q_{n,k}Q_{n,k}^T)\right)^{1/2}. \tag{3.6}$$

The elements $q_{ij}^{(k)}$ *are given by the following inner product*

$$q_{ij}^{(k)} = (\pi_j^{(k)}, \pi_{i-k}) \qquad (k \le i, j \le n).$$

Alternatively, (3.6) *can be expressed in the form*

$$C_{n,k} = \left(\lambda_{\min}(M_{n,k})\right)^{-1/2}, \tag{3.7}$$

where $M_{n,k} = \left(Q_{n,k}Q_{n,k}^T\right)^{-1}$.

We mention now a case with a special even weight function which was also considered in [20]. Let $d\lambda(t) = w(t)dt$ on $(-a, a)$, $0 < a < \infty$, where $w(-t) = w(t)$. Then we have

$$\pi_i'(t) = \frac{1}{r_i} \sum_{j=1}^{[(i+1)/2]} q_{i,j}\pi_{i-2j+1}(t), \qquad r_i \ne 0.$$

We use a class of such weight functions for which $q_{i,j} = q_{i+2,j+1}$. For example, this property holds for Gegenbauer weight. In this case, for $P \in \mathcal{P}_n$, we have

$$P'(t) = \sum_{i=1}^{n} c_i\pi_i'(t) = \sum_{i=1}^{n} q_{i,1}\Big(\sum_{j\ge 0} c_{i+2j}r_{i+2j}^{-1}\Big)\pi_{i-1}(t)$$

and

$$\|P'\|^2 = \sum_{i=1}^{n} y_i^2,$$

where

$$y_i = q_{i,1}\sum_{j\ge 0} c_{i+2j}r_{i+2j}^{-1}, \qquad i = 1, \ldots, n. \tag{3.8}$$

Putting $q_{i,1} = p_i$ and $y_{n+1} = y_{n+2} = 0$, from (3.8) follows

$$c_i = r_i\Big(\frac{y_i}{p_i} - \frac{y_{i+2}}{p_{i+2}}\Big), \qquad i = 1, \ldots, n.$$

Then

$$\|P\|^2 = \sum_{i=1}^{n} c_i^2 = \frac{r_1^2}{p_1^2}y_1^2 + \frac{r_2^2}{p_2^2}y_2^2 + \sum_{i=3}^{n} \frac{r_i^2 + r_{i-2}^2}{p_i^2}y_i^2 - 2\sum_{i=1}^{n-2} \frac{r_i^2}{p_ip_{i+2}}y_iy_{i+2}.$$

The corresponding matrix $M_{n,1}$ (see (3.7) in Theorem 3.6) is given by

$$M_{n,1} = \begin{bmatrix} \alpha_1 & 0 & \beta_1 & & & & & O \\ 0 & \alpha_2 & 0 & \beta_2 & & & & \\ \beta_1 & 0 & \alpha_3 & 0 & \beta_3 & & & \\ & \beta_2 & 0 & \alpha_4 & 0 & \beta_4 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & \beta_{n-4} & 0 & \alpha_{n-2} & 0 & \beta_{n-2} \\ & & & & \beta_{n-3} & 0 & \alpha_{n-1} & 0 \\ O & & & & & \beta_{n-2} & 0 & \alpha_n \end{bmatrix},$$

where

$$\alpha_i = \frac{r_i^2 + r_{i-2}^2}{p_i^2}, \quad \beta_i = -\frac{r_i^2}{p_i p_{i+2}} \quad (r_{-1} = r_0 = 0).$$

We define now two sequences of polynomials $\{R_i\}$ and $\{S_i\}$ by the following three-term recurrence relations:

$$\begin{aligned} tR_{i-1}(t) &= \beta_{2i-1}R_i(t) + \alpha_{2i-1}R_{i-1}(t) + \beta_{2i-3}R_{i-2}(t), \\ R_{-1}(t) &= 0, \quad R_0(t) = R_0 = \text{const}, \end{aligned}$$

where $i = 1, \ldots, [(n+1)/2]$ and

$$\begin{aligned} tS_{i-1}(t) &= \beta_{2i}S_i(t) + \alpha_{2i}S_{i-1}(t) + \beta_{2i-2}S_{i-2}(t), \\ S_{-1}(t) &= 0, \quad S_0(t) = S_0 = \text{const}, \end{aligned}$$

where $i = 1, \ldots, [n/2]$.

Theorem 3.7. *The eigenvalues of the matrix $M_{n,1}$ are the zeros of polynomials*

(a) S_{m-1} *and* R_m, *when* $n = 2m - 1$,

(b) S_m *and* R_m, *when* $n = 2m$,

so that

$$C_{2m-1,1} = \left(\min\left(s_1^{(m-1)}, r_1^{(m)}\right)\right)^{-1/2} \quad \textit{and} \quad C_{2m,1} = \left(\min\left(s_1^{(m)}, r_1^{(m)}\right)\right)^{-1/2},$$

where $s_1^{(k)}$ and $r_1^{(k)}$ are the minimal zeros of the polynomials S_k and R_k respectively.

The conditions $q_{i,j} = q_{i+2,j+1}$ are satisfied for Gegenbauer measure

$$d\lambda(t) = (1 - t^2)^{\lambda - 1/2}dt, \qquad -1 < t < 1.$$

In fact, we have

$$\frac{d}{dt}\hat{C}_i^\lambda(t) = \frac{2}{h_i^{1/2}} \sum_{j=1}^{[(i+1)/2]} (i + \lambda - 2j + 1)h_{i-2j+1}^{1/2}\hat{C}_{i-2j+1}^\lambda(t),$$

where $\hat{C}_\nu^\lambda$ is the normalized Gegenbauer polynomial of degree ν, with

$$h_i = \|C_i^\lambda\|^2 = \sqrt{\pi}\frac{(2\lambda)_i\Gamma\left(\lambda+\frac{1}{2}\right)}{(i+\lambda)i!\Gamma(\lambda)}, \qquad (p)_i = p(p+1)\cdots(p+i-1).$$

Thus,

$$r_i = \frac{1}{2}\sqrt{h_i}, \qquad p_i = q_{i,1} = (i+\lambda-1)\sqrt{h_{i-1}}.$$

For $n=1$ and $n=2$, we have

$$C_{1,1} = \sqrt{2(\lambda+1)} \qquad \text{and} \qquad C_{2,1} = \sqrt{\frac{8(\lambda+1)(\lambda+2)}{2\lambda+1}},$$

respectively.

In a special case, when $\lambda = 1/2$ (Legendre case), we obtain

$$\alpha_1 = \frac{1}{3}, \quad \alpha_2 = \frac{1}{15}, \quad \alpha_i = \frac{2}{(2i+1)(2i-3)}, \quad i = 3,\ldots,n;$$

$$\beta_i = -\frac{1}{(2i+1)\sqrt{(2i-1)(2i+3)}}, \quad i = 1,\ldots,n-2.$$

Similarly, in the Chebyshev case ($\lambda = 0$), we have

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{16}, \quad \alpha_i = \frac{1}{4}\left(\frac{1}{i^2}+\frac{1}{(i-2)^2}\right), \quad i = 3,\ldots,n;$$

$$\beta_1 = -\frac{\sqrt{2}}{4}, \quad \beta_i = -\frac{1}{4i^2}, \quad i = 2,\ldots,n-2.$$

Numerical calculations of $C_{n,1}$ for some selected values of λ were given in [20].

4. Some Weighted Polynomial Inequalities in L^2-Norm

Guessab and Milovanović [11] have considered a weighted L^2-analogues of the Bernstein's inequality (see Theorem 2.1), which can be stated in the following form:

$$\|\sqrt{1-t^2}P'(t)\|_\infty \le n\|P\|_\infty. \tag{4.1}$$

Let w be the weight of the classical orthogonal polynomials ($w \in CW$) and $A(t)$ be given as in Table 1.1. Using the norm $\|f\|_2^2 = (f,f)$, we consider the following problem connected with the Bernstein's inequality (4.1): *Determine the best constant* $C_{n,m}(w)$ $(1 \le m \le n)$ *such that the inequality*

$$\|A^{m/2}P^{(m)}\|_2 \le C_{n,m}(w)\|P\|_2 \tag{4.2}$$

holds for all $P \in \mathcal{P}_n$.

At first, we note if $w \in CW$, then the corresponding classical orthogonal polynomial $t \mapsto Q_n(t)$ is a particular solution of the differential equation of the second order (1.5), i.e.,

$$\frac{d}{dt}\Big(A(t)w(t)\frac{dy}{dt}\Big) + \lambda_n w(t)y = 0,$$

where $\lambda_n = -n\Big(\frac{1}{2}(n-1)A''(0) + B'(0)\Big)$. The k-th derivative of Q_n is also the classical orthogonal polynomial, with respect to the weight $t \mapsto w_k(t) = A(t)^k w(t)$, and satisfies the following differential equation

$$\frac{d}{dt}\Big(A(t)w_k(t)\frac{dy}{dt}\Big) + \lambda_{n,k} w_k(t)y = 0,$$

where $\lambda_{n,k} = -(n-k)\Big(\frac{1}{2}(n+k-1)A''(0) + B'(0)\Big)$. We note that $\lambda_{n,0} = \lambda_n$.

A. Guessab and G. V. Milovanović [11] proved:

Theorem 4.1. *For all $P \in \mathcal{P}_n$ the inequality* (4.2) *holds, with the best constant $C_{n,m}(w) = \sqrt{\lambda_{n,0}\lambda_{n,1}\cdots\lambda_{n,m-1}}$. The equality is attained in* (4.2) *if and only if P is a constant multiple of the classical polynomial Q_n orthogonal with respect to the weight function $w \in CW$.*

In some special cases we have:

(1) Let $w(t) = (1-t)^\alpha(1+t)^\beta$ $(\alpha, \beta > -1)$ on $(-1,1)$ (Jacobi case). Then

$$\|(1-t^2)^{m/2}P^{(m)}\|_2 \le \sqrt{\frac{n!\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\Gamma(n+\alpha+\beta+1)}}\ \|P\|_2,$$

with equality if and only if $P(t) = cP_n^{(\alpha,\beta)}(t)$.

(2) Let $w(t) = t^s e^{-t}$ $(s > -1)$ on $(0, +\infty)$ (generalized Laguerre case). Then

$$\|t^{m/2}P^{(m)}\|_2 \le \sqrt{n!/(n-m)!}\ \|P\|_2,$$

with equality if and only if $P(t) = cL_n^s(t)$.

(3) The Hermite case with the weight $w(t) = e^{-t^2}$ on $(-\infty, +\infty)$ is the simplest. Then the best constant is $C_{n,m}(w) = 2^{m/2}\sqrt{n!/(n-m)!}$.

In connection with the previous results is also the following characterization of the classiacal orthogonal polynomials given by Agarwal and Milovanović [1].

Theorem 4.2. *For all $P \in \mathcal{P}_n$ the inequality*

$$(2\lambda_n + B'(0))\|\sqrt{A}P'\|_2^2 \le \lambda_n^2\|P\|_2^2 + \|AP''\|_2^2 \tag{4.3}$$

holds, with equality if only if $P(t) = cQ_n(t)$, where Q_n is the classical orthogonal polynomial of degree n orthogonal to all polynomials of degree $\le n-1$ with respect *to the weight function $w(t)$ on (a,b), and c is an arbitrary real constant. λ_n, $A(t)$ and $B(t)$ are given in Table* 1.1.

The Hermite case was considered by Varma [37]. Then, the inequality (4.3) reduces to

$$\|P'\|_2^2 \le \frac{1}{2(2n-1)}\|P''\|_2^2 + \frac{2n^2}{2n-1}\|P\|_2^2.$$

In the generalized Laguerre case, the inequality (4.3) becomes

$$\|\sqrt{t}P'\|_2^2 \le \frac{n^2}{2n-1}\|P\|_2^2 + \frac{1}{2n-1}\|tP''\|_2^2,$$

where $w(t) = t^s e^{-t}$ on $(0,+\infty)$.
In the Jacobi case the inequality (4.3) reduces to the inequality

$$\big((2n-1)(\alpha+\beta) + 2(n^2+n-1)\big)\|\sqrt{1-t^2}P'\|_2^2$$
$$\le n^2(n+\alpha+\beta+1)^2\|P\|_2^2 + \|(1-t^2)P''\|_2^2.$$

In the simplest case, when $\alpha = \beta = 0$ (Legendre case), we obtain

$$\|\sqrt{1-t^2}P'\|_2^2 \le \frac{n^2(n+1)^2}{2(n^2+n-1)}\|P\|_2^2 + \frac{1}{2(n^2+n-1)}\|(1-t^2)P''\|_2^2.$$

In the Chebyshev case ($\alpha = \beta = -1/2$), we get

$$\|\sqrt{1-t^2}P'\|_2^2 \le \frac{n^4}{2n^2-1}\|P\|_2^2 + \frac{1}{2n^2-1}\|(1-t^2)P''\|_2^2,$$

where $\|f\|_2^2 = \int_{-1}^{1}(1-t^2)^{-1/2}f(t)^2\,dt$.
The corresponding result for trigonometric polynomials was obtained by Varma [38].
Recently Guessab [9] obtained sharp Markov-Bernstein inequalities in L^2 norms that are weighted with classical weights.

Theorem 4.3. *Let $P \in \mathcal{P}_n$ and $w \in CW$. Then*

$$\|w^{-1/2}(V(t)P)'\|_2^2 + \|\sqrt{A(t)C(t)}\,P\|_2^2 \le \beta_n\|P\|_2^2, \tag{4.4}$$

where $A(t)$ and λ_n are given in Table 1.1, *and $V(t) = \sqrt{A(t)w(t)}$,*

$$C(t) = \begin{cases} \dfrac{1}{4}\left(\dfrac{\alpha^2-1}{(1-t)^2} + \dfrac{\beta^2-1}{(1+t)^2}\right), & \textit{Jacobi case,} \\[2ex] \dfrac{1}{4}\left(\dfrac{s^2-1}{t^2}+1\right), & \textit{generalized Laguerre case,} \\[2ex] t^2, & \textit{Hermite case,} \end{cases}$$

and

$$\beta_n = \lambda_n + \begin{cases} \dfrac{1}{2}(\alpha+1)(\beta+1), & \textit{Jacobi case,} \\ \dfrac{1}{2}(s+1), & \textit{generalized Laguerre case,} \\ 1, & \textit{Hermite case.} \end{cases}$$

The equality is attained in (4.4) *if and only if* P *is a constant multiple of the classical polynomial* Q_n *orthogonal with respect to the weight function* $t \mapsto w(t)$.

This elegant result is established by using the second-order Sturm-Liouville type differential equations satisfied by the classical orthogonal polynomials.
Using the method from [11], Guessab [10] has investigated the extremal problem

$$\max_{P \in \mathcal{P}_n^1} \left\| \left(\sqrt{A}/w_m\right)\left(w_m P^{(m)}\right)' \right\|_{w_m},$$

where $w \in CW$, $w_m = A^m w$, $\mathcal{P}_n^1 = \{P \in \mathcal{P}_n \mid \|P\|_{w_m} \le 1\}$, and

$$\|f\|_{w_m} = \left(\int_a^b w_m(t)|f(t)|^2 w_m(t)\,dt\right)^{1/2}.$$

Theorem 4.4. *Let* $P \in \mathcal{P}_n^1$ *and* $w \in CW$. *Then*

$$\left\| \left(\sqrt{A}/w_m\right)\left(w_m P^{(m)}\right)' \right\|_{w_m} \le \sqrt{\lambda_{n,0}\lambda_{n,1}\cdots\lambda_{n,m-1}\beta_{n,m}}\,, \tag{4.5}$$

where $\lambda_{n,\nu}$ *is as in Theorem* 4.1,

$$\beta_{n,m} = \lambda_{n,m} + B'(0) + (k-1)A''(0),$$

and $A(t)$ *and* $B(t)$ *are given in Table* 1.1.
The equality is attained in (4.5) *if and only if* P *is a constant multiple of the classical polynomial* Q_n *orthogonal with respect to the weight function* $t \mapsto w(t)$.

At the end we mention a result of Guessab and Milovanović [12]. They considered the extremal problems of Markov's type

$$C_{n,m}(d\lambda) = \sup_{P \in \mathcal{P}_n} \frac{\|\mathcal{D}_m P\|_2}{\|A^{m/2}P\|_2} \qquad (m \ge 1) \tag{4.6}$$

for the differential operator $\mathcal{D}_m$ defined by

$$\mathcal{D}_m P = \frac{d^m}{dt^m}\left[A^m P\right] \qquad (P \in \mathcal{P}_n), \tag{4.7}$$

where

$$\|P\|_2 = \left(\int_{\mathbb{R}} |P(t)|^2\,d\lambda(t)\right)^{1/2},$$

and found the best constant $C_{n,m}(d\lambda)$ in three following cases:
1° *The Legendre measure* $d\lambda(t) = dt$ on $[-1, 1]$;
2° *The Laguerre measure* $d\lambda(t) = e^{-t}\,dt$ on $[0, +\infty)$.
3° *The Hermite measure* $d\lambda(t) = e^{-t^2}\,dt$ on $(-\infty, +\infty)$.
Some extremal problems for differential operators were investigated by Stein [29] and Džafarov [7].
Let $P \in \mathcal{P}_n$, $d\lambda(t) = w(t)\,dt$ on (a,b), and $\mathcal{D}_m$ be given by (4.7). An application of integration by parts gives

$$\|\mathcal{D}_m P\|_2^2 = \int_a^b \big(\mathcal{D}_m P\big)^2 w\,dt = (-1)^m \int_a^b A^m P[w\mathcal{D}_m P]^{(m)}\,dt.$$

Since

$$(-1)^m \int_a^b A^m P[w\mathcal{D}_m P]^{(m)} dt = \int_a^b (-1)^m \big(\sqrt{w} A^{m/2} P\big)\Big(\frac{A^{m/2}[w\mathcal{D}_m P]^{(m)}}{\sqrt{w}}\Big) dt,$$

using Cauchy-Schwarz-Buniakowsky inequality we obtain

$$\|\mathcal{D}_m P\|_2^2 \le \|A^{m/2} P\|_2 \Big(\int_a^b \frac{A^m}{w}\big([w\mathcal{D}_m P]^{(m)}\big)^2\,dt\Big)^{1/2},$$

with equality if and only if

$$\mathcal{F}_m P = \frac{(-1)^m}{w}[w\mathcal{D}_m P]^{(m)} = \gamma P \qquad (P \in \mathcal{P}_n),$$

where γ is an arbitrary constant.
Taking a norm with respect to the measure $d\lambda_m(t) = A^m d\lambda(t) = A^m w dt$,

$$\|f\|_* = \Big(\int_a^b |f(t)|^2\,d\lambda_m(t)\Big)^{1/2},$$

we have

$$\frac{\|\mathcal{D}_m P\|_2}{\|A^{m/2} P\|_2} \le \Big(\frac{\|\mathcal{F}_m P\|_*}{\|P\|_*}\Big)^{1/2}, \tag{4.8}$$

with equality if and only if $\mathcal{F}_m P = \gamma P$, i.e.,

$$(-1)^m \frac{d^m}{dt^m}\Big[w \frac{d^m}{dt^m}(A^m P)\Big] = \gamma w P \qquad (P \in \mathcal{P}_n). \tag{4.9}$$

We are interested only in polynomial solutions of this equation. If they exist, then from the eigenvalue problem (4.9) and the inequality (4.8), we can determine the best constant in the extremal problem (4.6). Namely,

$$C_{n,m}(d\lambda) = \sqrt{\max_{0 \le \nu \le n} |\lambda_{\nu,m}|},$$

where $\lambda_{\nu,m}$ are eigenvalues of the operator $\mathcal{F}_m$. Then, the extremal polynomial is the eigenfunction corresponding to the maximal eigenvalue. Guessab and Milovanović [12] solved the following cases:

1° In the Legendre case ($d\lambda(t) = dt$ on $(-1,1)$),

$$C_{n,m}(d\lambda) = \sqrt{\frac{(n+2m)!}{n!}},$$

with the extremal polynomial $P^*(t) = \gamma C_n^{m+1/2}(t)$, where C_n^μ is the Gegenbauer polynomial of degree n.

2° In the Laguerre case ($d\lambda(t) = e^{-t}\,dt$ on $(0,+\infty)$),

$$C_{n,m}(d\lambda) = \sqrt{\frac{(n+m)!}{n!}},$$

with the extremal polynomial $P^*(t) = \gamma L_n^m(t)$, where L_n^m is the generalized Laguerre polynomial of degree n.

3° In the Hermite case ($d\lambda(t) = e^{-t^2}\,dt$ on the real line $\mathbb{R}$),

$$C_{n,m}(d\lambda) = 2^{m/2}\sqrt{n!/(n-m)!}\,,$$

with the extremal polynomial $P^*(t) = \gamma H_n(t)$, where H_n is the Hermite polynomial of degree n. This result can be found in Ph. D. Thesis of Shampine [28] (see also, Dörfler [6] and Milovanović [20]). The case $m = 1$ was investigated by Schmidt [27] and Turán [32].

Remark 4.1. In the Jacobi case with the weight $t \mapsto (1-t)^\alpha(1+t)^\beta$ $(\alpha,\beta > -1)$ the equation (4.8) has no polynomial solution for $|\alpha| + |\beta| > 0$. Similarly, in the generalized Laguerre case with the weight function $t \mapsto t^s e^{-t}$ $(s > -1)$ the equation (4.8) has no polynomial solution for $s \neq 0$.

Remark 4.2. For extremal problems of Markov-Bernstein and Turán type on restricted polynomial classes in L^r norm see [13], [19], [21–26], [30], and [33–36].

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