INEQUALITIES FOR POLYNOMIAL ZEROS

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Abstract. This survey paper is devoted to inequalities for zeros of algebraic polynomials. We consider the various bounds for the moduli of the zeros, some related inequalities, as well as the location of the zeros of a polynomial, with a special emphasis on the zeros in a strip in the complex plane.

1. Introduction

In this paper we give an account on some important inequalities for zeros of algebraic polynomials. Let

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n \qquad (a_n \neq 0) \tag{1.1}$$

be an arbitrary algebraic polynomial of degree n with complex coefficients a_k $(k = 0, 1, \ldots, n)$. According to the well-known *fundamental theorem of algebra*, the polynomial (1.1) has exactly n zeros in the complex plane, counting their multiplicities.

Suppose that $P(z)$ has m different zeros $z_1, \ldots, z_m$, with the corresponding multiplicities $k_1, \ldots, k_m$. Then we have

$$P(z) = a_n \prod_{\nu=1}^{m} (z - z_\nu)^{k_\nu}, \qquad n = \sum_{\nu=1}^{m} k_\nu. \tag{1.2}$$

Rouché's theorem (cf. [45, p. 176]) can be applied to prove the proposition that the zeros of a polynomial are continuous functions of the coefficients of the polynomial. This property can be stated in the following form.

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Theorem 1.1. *Let $P(z)$ be given by* (1.1) *and let $z_1, \ldots, z_m$ be its zeros with the multiplicities $k_1, \ldots, k_m$, respectively, such that* (1.2) *holds. Further, let*

$$Q(z) = (a_0 + \varepsilon_0) + (a_1 + \varepsilon_1)z + \cdots + (a_{n-1} + \varepsilon_{n-1})z^{n-1} + a_n z^n,$$

and let

$$0 < r_\nu < \min |z_\nu - z_j|, \qquad j = 1, \ldots, \nu - 1, \nu + 1, \ldots, m.$$

There exists a positive number ε such that, if $|\varepsilon_i| \le \varepsilon$ for $i = 0, 1, \cdots, n-1$, then $Q(z)$ has precisely k_ν zeros in the circle C_ν with center at z_ν and radius r_ν.

There are several proofs of this result. Also, this theorem may be considered as a special case of a theorem of Hurwitz [27] (see [45, p. 178] for details and references). Thus, the zeros $z_1, \ldots, z_m$, can be considered as functions of the coefficients a_0, $a_1, \ldots, a_n$, i.e.,

$$z_\nu = \varphi_\nu(a_0, a_1, \ldots, a_n) \qquad (\nu = 1, \ldots, m).$$

Our basic task in this paper is to give some bounds for the zeros as functions of all the coefficients. For example, we try to find the smallest circle which encloses all the zeros (or k of them). Instead of the interiors of circles we are also interested in other regions in the complex plane (half-planes, sectors, rings, etc.).

The paper is organized as follows. Section 2 is devoted to the bounds for the moduli of the zeros and some related inequalities. The location of the zeros of a polynomial in terms of the coefficients of an orthogonal expansion is treated in Section 3. In particular, we give some important estimates for zeros in a strip in the complex plane.

2. Bounds for the Moduli of the Zeros

In this section we mainly consider bounds for the moduli of the polynomial zeros. We begin with classical results of Cauchy [11]:

Theorem 2.1. *Let $P(z)$ be a complex polynomial given by*

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n \qquad (a_n \ne 0), \tag{2.1}$$

and let $r = r[P]$ be the unique positive root of the algebraic equation

$$f(z) = |a_n| z^n - \big(|a_{n-1}| z^{n-1} + \cdots + |a_1| z + |a_0|\big) = 0. \tag{2.2}$$

Then all the zeros of the polynomial $P(z)$ lie in the circle $|z| \le r$.

Proof. If $|z| > r$, from (2.2) it follows that $f(|z|) > 0$. Since

$$|P(z)| \ge |a_n||z|^n - \big(|a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0|\big) = f(|z|), \tag{2.3}$$

we conclude that $|P(z)| > 0$, i.e., $P(z) \ne 0$, for $|z| > r$. Thus, all the zeros of $P(z)$ must be in the circle $|z| \le r$. □

The polynomial $f(z)$, which appears on the left hand side in (2.2), is called *associated polynomial* of $P(z)$. As usual, we call $r[P]$ the *Cauchy bound* of $P(z)$.

Theorem 2.2. *Let $P(z)$ be a complex polynomial given by* (2.1) *and let*

$$M = \max_{0\le\nu\le n-1} |a_\nu| \qquad \text{and} \qquad M' = \max_{1\le\nu\le n} |a_\nu|.$$

Then all the zeros of $P(z)$ lie in the ring

$$\frac{|a_0|}{|a_0| + M'} < |z| < 1 + \frac{M}{|a_n|}.$$

Proof. Suppose that $|z| > 1$. Then from (2.3) it follows

$$\begin{aligned} |P(z)| &\ge |a_n||z|^n - M\big(|z|^{n-1} + \cdots + |z| + 1\big) \\ &= |a_n||z|^n\Big(1 - \frac{M}{|a_n|}\sum_{\nu=1}^{n}|z|^{-\nu}\Big) \\ &> |a_n||z|^n\Big(1 - \frac{M}{|a_n|}\sum_{\nu=1}^{+\infty}|z|^{-\nu}\Big) \\ &= |a_n||z|^n\frac{|a_n|\,|z| - (|a_n| + M)}{|z| - 1}. \end{aligned}$$

Hence, if $|z| \ge 1 + M/|a_n|$ we see that $|P(z)| > 0$, i.e., $P(z) \ne 0$. Therefore, the zeros of $P(z)$ can be only in the disk $|z| < 1 + M/|a_n|$. Applying this result to the polynomial $z^nP(1/z)$ we obtain the corresponding lower bound. □

The circle $|z| \le 1 + M/|a_n|$ cannot be replaced by a circle $|z| \le 1 + \theta M/|a_n|$, with a universal constant θ such that $0 < \theta < 1$ as the simple example $P_0(z) = z^n - Mz^{n-1}$ demonstrates if only M is sufficiently large.

Cohn [13] proved that at least one of zeros of $P(z)$ satisfies the following inequality

$$|z| \ge r\big(\sqrt[n]{2} - 1\big),$$

where r is the Cauchy bound of $P(z)$. His proof based on the Grace-Apolarity theorem. Using the elementary symmetric functions and AG inequality, Berwald [4] proved:

Theorem 2.3. *Let $z_1, \ldots, z_n$ be the zeros of the polynomial $P(z)$ given by* (2.1) *and let r be the unique positive root of the equation* (2.2). *Then*

$$r \ge \frac{|z_1| + \cdots + |z_n|}{n} \ge r\big(\sqrt[n]{2} - 1\big),$$

with equality in the second inequality if and only if $z_1 = \cdots = z_n$.

Similarly as in the proof of Theorem 2.2 we can use the well-known Hölder inequality (cf. Mitrinović [48, pp. 50–51])

$$\Big(\sum_{k=1}^{n} x_k^p\Big)^{1/p}\Big(\sum_{k=1}^{n} y_k^q\Big)^{1/q} \ge \sum_{k=1}^{n} x_k y_k,$$

where $x_k \geq 0$, $y_k \geq 0$ $(k = 1, \ldots, n)$ and $1/p + 1/q = 1$ with $p > 1$, to estimate the right hand side in (2.3). So we obtain

$$\begin{aligned} |P(z)| &\geq |a_n||z|^n - \Big(\sum_{\nu=0}^{n-1} |a_\nu|^p\Big)^{1/p} \Big(\sum_{\nu=0}^{n-1} |z|^{\nu q}\Big)^{1/q} \\ &= |a_n||z|^n \Big(1 - \frac{M_p}{|a_n|} A(z)^{1/q}\Big), \end{aligned}$$

where $M_p = \Big(\sum\limits_{\nu=0}^{n-1} |a_\nu|^p\Big)^{1/p}$ and

$$A(z) = \sum_{\nu=0}^{n-1} |z|^{(\nu-n)q} < \frac{1}{|z|^q - 1} \qquad \big(|z| > 1\big).$$

Thus, for $|z| > 1$ we have

$$|P(z)| > |a_n||z|^n \Big(1 - \frac{M_p}{|a_n| \left(|z|^q - 1\right)^{1/q}}\Big).$$

Therefore, if $\left(|z|^q - 1\right)^{1/q} \geq M_p/|a_n|$, i.e.,

$$|z| \geq \Big(1 + \Big(\frac{M_p}{|a_n|}\Big)^q\Big)^{1/q},$$

we conclude that $|P(z)| > 0$, i.e., $P(z) \neq 0$.

Thus, we can state the following result (Kuniyeda [31]–[32], Montel [49]–[50], Tôya [73], Dieudonné [16], Marden [39]):

Theorem 2.4. *Let $P(z)$ be a complex polynomial given by* (2.1) *and let*

$$M_p = \Big(\sum_{\nu=0}^{n-1} |a_\nu|^p\Big)^{1/p} \quad \textit{and} \quad R_{pq} = \Big(1 + \Big(\frac{M_p}{|a_n|}\Big)^q\Big)^{1/q},$$

where $p, q > 1$, $1/p + 1/q = 1$. Then all the zeros of $P(z)$ lie in the disk

$$|z| < R_{pq}.$$

Taking $p \to +\infty$ $(q \to 1)$ we obtain Theorem 2.2.

The special case $p = q = 2$ gives the bound investigated by Carmichael and Mason [10], Fujiwara [18], and Kelleher [29]:

$$R_{22} = \frac{\sqrt{|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2}}{|a_n|}.$$

We mention here also a similar result of Williams [81], who changed R_{22} by

$$R'_{22} = \frac{\sqrt{|a_0|^2 + |a_1 - a_0|^2 + \cdots + |a_n - a_{n-1}|^2 + |a_n|^2}}{|a_n|}.$$

From some Cauchy's inequalities (see Mitrinović [48, p. 204 and p. 222]) we can obtain the following inequalities

$$\min_{1 \le \nu \le n} \frac{\alpha_\nu}{\beta_\nu} \le \frac{\sum\limits_{\nu=1}^{n} \alpha_\nu \lambda_\nu}{\sum\limits_{\nu=1}^{n} \beta_\nu \lambda_\nu} \le \max_{1 \le \nu \le n} \frac{\alpha_\nu}{\beta_\nu}, \tag{2.4}$$

which hold for the real numbers α_ν, $\beta_\nu > 0$, $\lambda_\nu > 0$ ($\nu = 1, \ldots, n$), with equality if and only if the sequences $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ are proportional. Using these inequalities, Marković [40] considered

$$P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \qquad \text{and} \qquad f(z) = \sum_{\nu=0}^{+\infty} b_\nu z^\nu,$$

with $b_\nu > 0$ ($\nu = 0, 1, \ldots$) and proved the following result:

Theorem 2.5. *Let r_0 be a positive root of the equation $Mf(r) = |a_0|$, where*

$$M = \max_{1 \le \nu \le n} \left\{ \frac{|a_\nu|}{b_\nu} \right\}.$$

Then all the zeros of $P(z)$ lie in $|z| \ge r_0$.

In particular, when $b_\nu = t^{-\nu}$ ($\nu = 1, 2, \ldots$) and

$$g(t) = \max_{1 \le \nu \le n} \left(|a_\nu| t^\nu \right),$$

where t is any positive number, one has that all the zeros of $P(z)$ lie in the domain

$$|z| \ge \frac{|a_0| t}{|a_0| + g(t)}.$$

The same result was also obtained by Landau [33] in another way.

Assuming that $\lambda_1 > \cdots > \lambda_n > 0$, Simeunović [63] improved (2.4) in following way

$$\min_{1 \le \nu \le n} \frac{\alpha_\nu}{\beta_\nu} \le \min_{1 \le \nu \le n} \left(\frac{\sum\limits_{k=1}^{\nu} \alpha_k}{\sum\limits_{k=1}^{\nu} \beta_k} \right) \le \frac{\sum\limits_{\nu=1}^{n} \alpha_\nu \lambda_\nu}{\sum\limits_{\nu=1}^{n} \beta_\nu \lambda_\nu} \le \max_{1 \le \nu \le n} \left(\frac{\sum\limits_{k=1}^{\nu} \alpha_k}{\sum\limits_{k=1}^{\nu} \beta_k} \right) \le \max_{1 \le \nu \le n} \frac{\alpha_\nu}{\beta_\nu}$$

and then proved that all the zeros of $P(z)$ lie in the domain

$$|z| \ge \frac{|a_0| t}{|a_0| + h(t)},$$

where

$$h(t) = \max_{1 \le \nu \le n} \left(\frac{1}{\nu} \sum_{k=1}^{\nu} |a_k| t^k \right) \le \max_{1 \le k \le n} \left(|a_k| t^k \right).$$

In Bourbaki [8, p. 97] the following result is mentioned as a problem:

Theorem 2.6. *Let*

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \tag{2.5}$$

be a polynomial with non-zero complex coefficients, and let z_ν $(\nu = 1, \dots, n)$ *be the zeros of this polynomial. Then*

$$\max_{1\le\nu\le n} |z_\nu| \le \max\Bigl(2|a_{n-1}|, 2\left|\frac{a_{n-2}}{a_{n-1}}\right|, \dots, 2\left|\frac{a_1}{a_2}\right|, \left|\frac{a_0}{a_1}\right|\Bigr).$$

Introducing a new variable $w = z + a_{n-1}/n$, the polynomial (2.5) can be transformed to a polynomial of the form

$$P(w - a_{n-1}/n) = w^n + c_{n-2}w^{n-2} + \cdots + c_1 w + c_0.$$

If we define a polynomial $S(w)$ by

$$S(w) = w^n - |c_{n-2}|w^{n-2} - \cdots - |c_1|w - |c_0| \tag{2.6}$$

then we can prove the following result (cf. Milovanović [43, pp. 398–399]):

Theorem 2.7. *If at least one of the coefficients* c_ν $(\nu = 0, 1, \dots, n-2)$ *is non-zero, then all the zeros of* $P(z)$ *lie in the circle*

$$\left|z + \frac{a_{n-1}}{n}\right| \le r,$$

where r *is the unique positive zero of the polynomial* (2.6).

Setting $p_k = |a_{n-k}/a_n|$ $(k = 1, \dots, n)$ the equation (2.2) reduces to

$$z^n = \sum_{k=1}^{n} p_k z^{n-k}, \tag{2.7}$$

where $p_k \ge 0$ $(k = 1, \dots, n)$ and $\sum_{k=1}^{n} a_k > 0$. Westerfield [80] found an estimate for the positive root of this equation.

Theorem 2.8. *Let* r *be the unique positive root of the equation* (2.7) *and let positive quantities* $\sqrt[k]{p_k}$ $(k = 1, \dots, n)$*, after being arranged in order of decreasing magnitudes, form a sequence* $q_1 \ge q_2 \ge \cdots \ge q_n$*. Then* r *satisfies the inequality* $r \le \sum_{k=1}^{n} q_k s_k$*, where*

$$s_1 = y_1, \qquad s_k = y_k - y_{k-1} \quad (k = 2, \dots, n),$$

and where y_k *is the unique positive root of the equation*

$$y^k = \sum_{\nu=1}^{k} y^{k-\nu} \qquad (k = 1, \dots, n).$$

A simple proof of this theorem was given by Bojanov [5] as an application of the following his theorem:

Theorem 2.9. *If* x_ν *are positive roots of the equations*

$$x^n = a_{\nu 1} x^{n-1} + a_{\nu 2}^2 x^{n-2} + \cdots + a_{\nu n}^n$$

$$(a_{\nu 1},\, a_{\nu 2}\ \ldots,\, a_{\nu n} \ge 0;\quad \nu = 1, \ldots, m),$$

then the positive root Z *of the equation*

$$x^n = \sum_{k=1}^{n} \Big(\sum_{\nu=1}^{m} a_{\nu k} \Big)^k x^{n-k}$$

satisfies the inequality $Z \le x_1 + x_2 + \cdots + x_m$.

The method of Bojanov [5] gives also a lower bound for the positive root of (2.7). Zervos [82] proved the following result:

Theorem 2.10. *Let* $I_1, \ldots, I_n$ *be index sets and* θ_{i_j} (≥ 0) *real numbers satisfying the condition*

$$\sum_{i_\nu \in I_\nu} \theta_{i_\nu} = \nu - t \qquad (\nu = 1, \ldots, n),$$

where t $(0 < t \le 1)$ *is a fixed number. Then, the positive root* r *of the equation* (2.7) *satisfies the inequality*

$$r \le \max\Big\{ M, \Big(\sum_{\nu=1}^{n} p_\nu \Big/ \prod_{i_\nu \in I_\nu} M_{i_\nu}^{\theta_{i_\nu}} \Big)^{1/t} \Big\},$$

where $M = \max\{M_{i_\nu}\}$ *and* M_{i_ν} *are any positive numbers.*

Prešić [54] proved a lemma which with certain specifications proves the previous theorem of Zervos. An extension of this lemma, which gives a lower bound of r, was proved by Tasković [70].

Let $\lambda_2, \ldots, \lambda_n$ be arbitrary positive numbers and let r be the unique positive root of the equation (2.7). Then (see Zervos [82, p. 343] and Mitrinović [48, p. 223])

$$r \le \max\Big(\lambda_2, \ldots, \lambda_n, \Big(p_1 + \frac{p_2}{\lambda_2} + \cdots + \frac{p_n}{\lambda_n^{n-1}} \Big) \Big). \tag{2.8}$$

In order to prove this we put $\lambda = \max(\lambda_2, \ldots, \lambda_n)$. If $\alpha \ge r$, then (2.8) holds. Let $\lambda < r$. Then $\lambda_2, \ldots, \lambda_n < r$, and therefore

$$\begin{aligned} p_1 + \frac{p_2}{\lambda_2} + \cdots + \frac{p_n}{\lambda_n^{n-1}} &\ge p_1 + \frac{p_2}{r} + \cdots + \frac{p_n}{r^{n-1}} \\ &= \frac{1}{r^{n-1}} \left(p_1 r^{n-1} + \cdots + p_n \right) = r \end{aligned}$$

and inequality (2.8) is true.

Introducing a mini-max principle to a totally ordered sets, Tasković [71] stated a result which in a special case gives that

$$r = \min_{\lambda_2,\dots,\lambda_n\in\mathbb{R}^+} \max\left(\lambda_2,\dots,\lambda_n, \left(p_1 + \frac{p_2}{\lambda_2} + \cdots + \frac{p_n}{\lambda_n^{n-1}}\right)\right)$$
$$= \max_{\lambda_2,\dots,\lambda_n\in\mathbb{R}^+} \min\left(\lambda_2,\dots,\lambda_n, \left(p_1 + \frac{p_2}{\lambda_2} + \cdots + \frac{p_n}{\lambda_n^{n-1}}\right)\right).$$

Walsh [77] proved the following result:

Theorem 2.11. *If all the zeros of a polynomial* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *lie in a circle* $|z| \le r$, *then all the zeros of the polynomial* $P(z) - a$ *lie in the circle*

$$|z| \le r + |a/a_n|^{1/n}.$$

Precisely, this is an useful consequence of a general result of Walsh [77], which is known as *Coincidence Theorem* (see Theorem 2.20) Walsh [78] also proved the following result:

Theorem 2.12. *Let* $P(z)$ *be a polynomial of degree* n *given by* (2.1)*. Then all its zeros lie in the circle* $|z| \le R$, *where*

$$R = \sum_{k=0}^{n-1} \left|\frac{a_k}{a_n}\right|^{1/(n-k)}.$$

Proof. Suppose that all the zeros of the polynomial

$$P_k(z) = a_n z^k + a_{n-1} z^{k-1} + \cdots + a_{n-k+1} z$$

lie in the circle $|z| \le r_{k-1}$ $(k = 1, \dots, n)$. Since $P_1(z) = a_n z$, we have $r_0 = 0$. Applying Theorem 2.11 to $P_k(z)$, with $a = -a_{n-k}$, we conclude that all the zeros of the polynomial

$$P_k(z) + a_{n-k} = a_n z^k + a_{n-1} z^{k-1} + \cdots + a_{n-k+1} z + a_{n-k}$$

lie in the circle $|z| \le r_{k-1} + |a_{n-k}/a_n|^{1/k}$. Since

$$P_k(z) = zP_{k-1}(z) + a_{n-k+1} \qquad (k = 2, \dots, n),$$

taking $r_k = r_{k-1} + |a_{n-k}/a_n|^{1/k}$, we obtain

$$R = r_n = \left|\frac{a_{n-1}}{a_n}\right| + \left|\frac{a_{n-2}}{a_n}\right|^{1/2} + \cdots + \left|\frac{a_0}{a_n}\right|^{1/n}. \quad \square$$

Some improvements of this result were given by Rudnicki [61]. Tonkov [72] gave an elementary proof of this theorem and also determined the lower bound for the zeros.

The following result was also proved by Walsh [78]:

Theorem 2.13. *All the zeros of the polynomial $P(z)$ given by (2.1), where $a_n = 1$, lie in the disk*

$$\left| z + \frac{1}{2}\, a_{n-1} \right| \le \frac{1}{2}\, |a_{n-1}| + M, \tag{2.9}$$

where $M = \sum\limits_{\nu=2}^{n} |a_{n-\nu}|^{1/\nu}$.

Another proof of this theorem was given by Bell [3].
Rahman [55] replaced the disk (2.9) by

$$\left| z + \frac{1}{2}\, a_{n-1} \right| \le \frac{1}{2}\, |a_{n-1}| + \alpha M,$$

where (i) $\alpha = 0$ if $P(z)$ is of the form $a_{n-1}z^{n-1} + z^n$, and (ii)

$$\alpha = \max_{2 \le \nu \le n} \left(M^{-1} |a_{n-\nu}|^{1/\nu} \right)^{(\nu-1)/\nu}$$

if $P(z)$ is not of the form $a_{n-1}z^{n-1} + z^n$.
The classical Cauchy's bounds were improved in various ways by many authors. As an improvement Joyal, Labelle, and Rahman [28] proved the following theorem:

Theorem 2.14. *Let $P(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ $(a_n = 1)$ be a polynomial of degree n, and let $\beta = \max\limits_{0 \le \nu < n-1} |a_\nu|$. Then all the zeros of $P(z)$ lie in the disk*

$$|z| \le \frac{1}{2} \left\{ 1 + |a_{n-1}| + \left[(1 - |a_{n-1}|)^2 + 4\beta \right]^{1/2} \right\}. \tag{2.10}$$

The expression (2.10) takes a very simple form if $a_{n-1} = 0$. If $|a_{n-1}| = 1$, it reduces to $1 + \sqrt{\beta}$, which is smaller than the bound obtained in Theorem 2.2. If $|a_{n-1}| = \beta$, Theorem 2.14 fails to give an improvement of Theorem 2.2. A ring-shaped region containing all the zeros of $P(z)$ was obtained by Datt and Govil [14]:

Theorem 2.15. *If $P(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ $(a_n = 1)$ is a polynomial of degree n and $A = \max\limits_{0 \le \nu \le n-1} |a_\nu|$, then $P(z)$ has all its zeros in the ring-shaped region*

$$\frac{|a_0|}{2(1 + A)^{n-1}(An + 1)} \le |z| \le 1 + \lambda_0 A, \tag{2.11}$$

where λ_0 is the unique root of the equation $x = 1 - 1/(1 + Ax)^n$ in the interval $(0, 1)$. The upper bound $1 + \lambda_0 A$ in (2.11) is best possible and is attained for the polynomial $z^n - A(z^{n-1} + \cdots + z + 1)$.

If one does not wish to look for the roots of the equation $x = 1 - 1/(1 + Ax)^n$, one can still obtain a result which is an improvement of Theorem 2.2, even in the case $|a_{n-1}| = \beta$:

Theorem 2.16. *Under the conditions of Theorem* 2.15, $P(z)$ *has all its zeros in the ring-shaped region*

$$\frac{|a_0|}{2(1+A)^{n-1}(An+1)} \le |z| \le 1 + \Big(1 - \frac{1}{(1+A)^n}\Big)A.$$

Some refinements of Theorems 2.14 and 2.15 were obtained by Dewan [15].

We mention here also a result of Abian [1]:

Theorem 2.17. *Let* $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ *with* $a_0 \ne 0$ *be a polynomial and let* $A(z)$ *and* $B(w)$ *be given by*

$$A(z) = \frac{1}{a_0 + \cdots + a_n z^n} \qquad \text{and} \qquad B(w) = \frac{1}{a_0 w^n + \cdots + a_n},$$

respectively. Then the precise annulus which contains all the zeros of $P(z)$ *is given by*

$$\frac{1}{\overline{\lim\limits_{k\to+\infty}} \sqrt[k]{\Big|\dfrac{1}{k!}A^{(k)}(0)\Big|}} \le |z| \le \overline{\lim_{k\to+\infty}} \sqrt[k]{\Big|\frac{1}{k!}B^{(k)}(0)\Big|}\,. \tag{2.12}$$

From the well-known Stirling's formula it follows that $\lim\limits_{k\to+\infty} k^{-1}(k!)^{k-1} = e^{-1}$. Then (2.12) reduces to

$$\frac{1}{e\,\overline{\lim\limits_{k\to+\infty}} \dfrac{\sqrt[k]{|A^{(k)}(0)|}}{k}} \le |z| \le e\,\overline{\lim_{k\to+\infty}} \frac{\sqrt[k]{|B^{(k)}(0)|}}{k}\,.$$

In 1881, Pellet [53] published the following result:

Theorem 2.18. *If the equation* $F_k(z) = 0$, *where*

$$F_k(z) = |a_0| + |a_1|z + |a_2|z^2 + \cdots + |a_{k-1}|z^{k-1} - |a_k|z^k$$
$$+|a_{k+1}|z^{k+1} + \cdots + |a_n|z^n \qquad (0 < k < n,\ a_0 a_n \ne 0),$$

has two positive roots r_k *and* ϱ_k $(0 < r_k < \varrho_k)$, *then the polynomial*

$$z \mapsto P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

has no zeros in the annulus $r_k < |z| < \varrho_k$ *and precisely* k *zeros in the disk* $|z| \le r_k$.

Pellet's proof uses Rouché's theorem [60] (see also [45, p. 176]). Walsh [79] published in 1924 another more direct proof and established a sort of converse of Pellet's theorem. Walsh allows in his proof the zeros of $P(z)$, which are in absolute value less than a, to vary continuously and monotonically (in absolute value) and to approach 0. Walsh [79] remarked that his proof of Pellet's theorem remains valid also in the case of a power-series and of its zeros inside the circle of convergence. Ostrowski [51] remarked that his proof of Walsh's theorem also applies *mutatis mutandis* to a power series when one considers its zeros within the circle of convergence.

Precisely, Walsh's converse of Pellet's theorem can be stated in the following form (cf. Marden [39, p. 129]):

Theorem 2.19. *Let F_k be defined as in Theorem 2.18, where a_0, a_1, ..., a_n are fixed coefficients, and let ε_0, ε_1, ..., ε_n be arbitrary complex numbers with* $|\varepsilon_0| = |\varepsilon_1| = \cdots = |\varepsilon_n| = 1$.

If r is a positive number such that

(1) *r is not a zero of any polynomial*

$$P(z) = \varepsilon_0 a_0 + \varepsilon_1 a_1 z + \cdots + \varepsilon_n a_n z^n,$$

(2) *every polynomial $P(z)$ has k $(0 < k < n)$ zeros in the circle $|z| = r$.*

Then $F_k(z)$ has two positive zeros r_k and ϱ_k $(0 < r_k < \varrho_k)$ and $r_k < r < \varrho_k$.

Steckhin [68] considered a generalized case

$$G(z) = \varepsilon_0\varphi_0(z) + \varepsilon_1\varphi_1(z) + \cdots + \varepsilon_n\varphi_n(z),$$

where $\varphi_k(z)$ are arbitrary complex functions and $|\varepsilon_k| = 1$ $(k = 0, 1, \ldots, n)$. Let $\mathfrak{M}_G$ be the set of all the zeros of $G(z)$ when $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n$ vary independently, but such that $|\varepsilon_k| = 1$ $(k = 0, 1, \ldots, n)$. Steckhin [68] gave an elementary proof of the following result:

Theorem 2.20. *In order that $z \in \mathfrak{M}_G$, the necessary and sufficient conditions are given by*

$$G_k(z) = |\varphi_0(z)| + \cdots + |\varphi_{k-1}(z)| - |\varphi_k(z)| + \cdots + |\varphi_n(z)| \geq 0, \tag{2.13}$$

where $k = 0, 1, \ldots, n$.

Proof. Suppose that $G_\nu(z) < 0$ for some ν $(0 \leq \nu \leq n)$. Then

$$|G(z)| \geq |\varphi_\nu(z)| - \sum_{k \neq \nu} |\varphi_k(z)| = -G_\nu(z) > 0,$$

i.e., $z \notin \mathfrak{M}$.

Conversely, let $A_k = |\varphi_k(z)|$ and $A_q = \max\limits_{0 \leq k \leq n} A_k$. Inequalities (2.13) show that $A_q \leq \sum\limits_{k \neq q} A_k$. Since every term of the right sum does not exceed A_q, this sum can be split on $B_1 = \sum_1 A_k$ and $B_2 = \sum_2 A_k$ so that $|B_1 - B_2| \leq A_q$. Then, we have

$$B_1 \leq B_2 + A_q, \quad B_2 \leq B_1 + A_q, \quad A_q \leq B_1 + B_2,$$

i.e., it is possible to build a triangle by the line segments having the lengths B_1, B_2, and A_q. This ensures existence of the numbers η_k $(k = 0, 1, \ldots, n)$, $|\eta_k| = 1$, so that $\sum\limits_{k=0}^{n} \eta_k A_k = 0$. But, $A_k = |\varphi_k(z)| = \delta_k \varphi_k(z)$, where $|\delta_k| = 1$. Thus,

$\sum\limits_{k=0}^{n} \varepsilon_k \varphi_k(z) = 0$, where $\varepsilon_k = \eta_k \delta_k$, $|\varepsilon_k| = 1$ $(k = 0, 1, \dots, n)$. In this way, $z \in \mathfrak{M}_G$ and the proof is completed. □

The conditions of this theorem can be stated in a compact form

$$\sum_{k=1}^{n} |\varphi_k(z)| \geq 2 \max_{0 \leq k \leq n} |\varphi_k(z)|.$$

Let $P(z)$ be a polynomial, defined by

$$P(z) = \varepsilon_0 a_0 + \varepsilon_1 a_1 z + \cdots + \varepsilon_n a_n z^n, \tag{2.14}$$

where

$$z = xe^{i\theta} \quad (x \geq 0,\ 0 \leq \theta < 2\pi), \quad a_k \geq 0, \quad |\varepsilon_k| = 1 \quad (k = 0, 1, \dots, n),$$

and let $\mathfrak{M}$ be the corresponding set of all the zeros of $P(z)$ when a_k are fixed and $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$ vary independently, but $|\varepsilon_k| = 1$ $(k = 0, 1, \dots, n)$.

Applying the previous theorem to (2.14) Steckhin obtained the following result:

Corollary 2.21. *In order that $z \in \mathfrak{M}$, the necessary and sufficient conditions are given by*

$$P_k(x) = a_0 + \cdots + a_{k-1}x^{k-1} - a_k x^k + \cdots + a_n x^n \geq 0,$$

where $k = 0, 1, \dots, n$.

Let $a_0 a_n \neq 0$. Then

$$P_0(x) < 0 \quad (0 \leq x < \varrho_0), \qquad P_0(x) \geq 0 \quad (\varrho_0 \leq x < +\infty),$$

$$P_n(x) \geq 0 \quad (0 \leq x \leq r_n), \qquad P_n(x) < 0 \quad (r_n < x < +\infty),$$

and for $k = 1, \dots, n-1$,

$$P_k(x) \geq 0 \quad (0 \leq x \leq r_k,\ \varrho_k \leq x < +\infty), \qquad P_k(x) < 0 \quad (r_k < x < \varrho_k),$$

which yields the result of Walsh [79] and Ostrowski [51].

Select now some subset $S \subset \{0, 1, \dots, n\}$ and denote by $\mathfrak{M}_S$ the set of zeros of all polynomials $P(z)$ for fixed values of a_k $(k = 0, 1, \dots, n)$ and ε_p $(p \in S)$, when other ε_k independently take values so that $|\varepsilon_k| = 1$. Steckhin [68] also proved:

Theorem 2.22. *Set*

$$Q(x, \varphi) = \sum_{p \in S} \varepsilon_p a_p x^p e^{ip\varphi}, \quad R(x) = \sum_{k \notin S} a_k x^k,$$

$$r(x) = \max \left\{ 0,\ 2 \max_{k \notin S} a_k x^k - R(x) \right\}.$$

For $z \in \mathfrak{M}_S$ it is necessary and sufficient that inequalities

$$r(x) \le |Q(x,\varphi)| \le R(x)$$

hold.

The special case when $S = \{p,q\}$ $(0 \le p < q \le n)$ was considered by Lipka [37] and Marden [38] (see also Marden [39, pp. 130–133]). A complete description of the set $\mathfrak{M}_{pq} \equiv \mathfrak{M}_S$ can be given by Theorem 2.22 (see Steckhin [68]).

Riddell [59] considered the problem of the zeros of the complex polynomial

$$P(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n$$

under the assumption that some $|\alpha_k|$ is large in comparison with the other $|\alpha_i|$, and he proved that then $P(z)$ has $n-k$ zeros near 0 and one zero near each of the k values of $(-\alpha_k)^{1/k}$. He established certain conditions under which precise estimates can be given. The results obtained rest on the following observation. Let k denote an integer in the range $1 \le k \le n$, chosen and fixed in the sequel. Given a polynomial $P(z)$ as above, suppose that $P(z) = 0$ and $z \ne 0$. It follows that

$$|z^k + \alpha_k| \le \sum_{i \ne k} |\alpha_i||z|^{k-i}.$$

Define

$$a = |\alpha_k|, \quad b = |\alpha_1| + \cdots + |\alpha_{k-1}|, \quad c = |\alpha_{k+1}| + \cdots + |\alpha_n|$$

and

$$g(r) = \begin{cases} br + cr^{k-n} & (0 < r \le 1), \\ br^{k-1} + cr^{-1} & (r \ge 1), \end{cases}$$

where it is understood that $b = 0$ in case $k = 1$, and $c = 0$ in case $k = n$.

It is an immediate consequence that if $P(z) = 0$ and $|z| > 0$, then

$$|z^k + \alpha_k| \le g(|z|).$$

In the following, we will consider two cases, where the quantities $P(z)$, k, a, b, c, and $g(r)$ will continue to have the meanings given above.

Case 1. Annuli which contain no zero. Riddell [59] proved the following estimate which asserts the existence of a zero-free annulus

$$m_- < |z| < m_+.$$

Estimate A. *The polynomial $P(z)$ has $n-k$ zeros in the disk $|z| \le m_-$ and k zeros in the region $|z| \ge m_+$, where $m_- < m_+ \le a^{1/k}$ (zeros are being counted with their multiplicities).*

The proof of this result depends on the following lemma:

Lemma 2.23. *Estimate* A *holds if* $r = m_-$ *and* $r = m_+$ *are two solutions of an equation of the form* $a = h(r)$, *where*

$$h(r) \geq r^k + g(r).$$

Lemma 2.23 is essentially the result of Pellet [53, p. 393] (see also Dieudonnè [16, p. 10]) in our context.

The first application of Riddell's lemma is to the existence of $m_\pm$ to the right of $r = 1$. For this, Riddell [59] proved the following result:

Theorem 2.24. *Let* $1 < a \leq 1 + b + c$, *and* $D > 0$, *where*

$$D = \frac{1}{4}\,(a^{1/k} + b)^2 - \frac{a^{1/k} - 1}{a - 1}\,(ab + c).$$

Then Estimate A *holds with*

$$m_\pm = \frac{1}{2}(a^{1/k} - b) \pm D^{1/2}. \tag{2.15}$$

If m_- and m_+ lie on opposite sides of $r = 1$, Riddell [59] proved:

Theorem 2.25. *Let* $1 + b + c < a$. *Then Estimate* A *holds with*

$$m_- = \left(\frac{c}{a - b - 1}\right)^{1/(n-k)}, \qquad m_+ = \left(\frac{a - c}{b + 1}\right)^{1/k}, \tag{2.16}$$

and also with m_- *given by* (2.16) *and* m_+ *by* (2.15).

The case $b = 0$ of Theorem 2.25 strengthens a result of Parodi [52, pp. 139–140].

When both $m_\pm$ are to the left of $r = 1$, Riddell [59] obtained:

Theorem 2.26. *Let* $c < a \leq 1 + b + c$ *and* $b + 2d^{1/2} < a$, *where*

$$d \geq \begin{cases} a(c/a)^{k/(n-k)}, & \text{if } \; k < n/2, \\ \min\{1, a^{2-n/k}\}c, & \text{if } \; k \geq n/2. \end{cases}$$

Then Estimate A *holds with*

$$m_\pm = \left\{\frac{1}{2}(a - b) \pm \left[\frac{1}{4}(a - b)^2 - d\right]^{1/2}\right\}^{1/k}.$$

Case 2. Disks which contain a single zero. In the following we will present Riddell's Estimate B, which under stronger conditions implies the existence of k disks

$$|z - (-\alpha_k)^{1/k}| \leq R,$$

each one of which isolates a single zero of $P(z)$.

Estimate B. *The polynomial $P(z)$ has $n-k$ zeros in the disk $|z| \leq m_-$ and one zero in each of the k disjoint disks*

$$|z - (-\alpha_k)^{1/k}| \leq R,$$

where $m_- + R < a^{1/k}$ $(a = |\alpha_k|)$.

For the proof of this estimate the following lemma is essential.

Lemma 2.27. *Suppose that Estimate* A *holds with $m_\pm$ given as in Lemma* 2.23. *Suppose that, for some upper bound M on the moduli of the zeros of $P(z)$,*

$$g(M) \leq a - m_+^k.$$

Then Estimate B *holds with the given m_- and with $R = a^{1/k} - m_+$, provided also that, in case $k \geq 3$, $R < a^{1/k}\sin(\pi/k)$.*

The first result derived from this lemma applies, in case $k \geq 2$, only to lacunary polynomials $P(z)$.

Theorem 2.28. *Let $a > 1$, $b = 0$, and $D > 0$, where*

$$D = \frac{1}{4}a^{2/k} - \frac{a^{1/k}-1}{a-1}c.$$

Then Estimate B *holds, with*

$$R = \frac{1}{2}a^{1/k} - D^{1/2}, \qquad m_- = \begin{cases} \frac{1}{2}a^{1/k} - D^{1/2}, & \text{if } a \leq 1+c, \\ \big(c/(a-1)\big)^{1/(n-k)}, & \text{if } a > 1+c, \end{cases}$$

provided also that, in case $k \geq 3$, $R < a^{1/k}\sin(\pi/k)$.

The case $k = 1$ of the above theorem simplifies an estimate due to Parodi [52, pp. 76–77].

With some slight loss in precision for small values of k, the next theorem does not require the restriction $P(z)$ to be a lacunary polynomial. Riddell [59] proved the following results:

Theorem 2.29. *Let $1 + 2b < \min\{a^{1/k}, a+b-c\}$. Then Estimate* B *holds, with*

$$m_- = \left(\frac{c}{a-b-1}\right)^{1/(n-k)}, \qquad R = a^{1/k} - \left(a - \frac{ab+c}{1+b}\right)^{1/k},$$

provided also that, in case $k \geq 3$, $R < a^{1/k}\sin(\pi/k)$.

Theorem 2.30. *Let $c < a^{n/k} \leq 1$ and $b + 2d^{1/2} < a$, and suppose $b^2 \leq d$ if $a+b+c > 1$, where*

$$d = \begin{cases} a(c/a)^{k/(n-k)}, & \text{if } k < n/2, \\ c, & \text{if } k \geq n/2. \end{cases}$$

Then Estimate B *holds, with*

$$R = a^{1/k} - \left(a - b - \frac{2d}{a-b}\right)^{1/k}, \quad m_- = \left\{\frac{1}{2}(a-b) - \left[\frac{1}{4}(a-b)^2 - d\right]^{1/2}\right\}^{1/k},$$

provided also that, in case $k \geq 3$, $R < a^{1/k}\sin(\pi/k)$.

If $k = n$ and $a + b > 1$, the previous theorem does not apply, because of the fact $d = c = 0$ and the additional hypothesis $b^2 \leq d$ is not satisfied. If $b < a/2$, then Estimate A is satisfied with

$$m_+ = a - ab/(a-b) > 0.$$

Therefore in the case $k = n$, $a + b > 1$, we can replace the hypotheses of the previous theorem by $2b < a \leq 1$ and preserve the result with

$$R = a^{1/n} - \left(a - ab/(a-b)\right)^{1/n}.$$

At the end of this section we consider numerical radii of some companion matrices and bounds for the zeros of polynomials. Let $P(z)$ be a monic polynomial of degree $n \geq 3$ given by

$$P(z) = z^n - a_1 z^{n-1} - \cdots - a_{n-1}z - a_n \qquad (a_n \neq 0;\ a_k \in \mathbb{C}). \tag{2.17}$$

Some bounds for the zeros of $P(z)$ can be obtained using results on the numerical range and the numerical radius of the Frobenius companion matrix of $P(z)$. Some other companion matrices of $P(z)$ can be obtained by a similarity transformation of the Frobenius companion matrix of $P(z)$. Recently Linden [36] (see also [35]) used some types of generalized companion matrices, which are based on special multiplicative decompositions of the coefficients of the polynomial, in order to obtain estimates for the zeros of $P(z)$ mainly by the application of Gersgorin's theorem to the companion matrices or by computing the singular values of the companion matrices and using majorizations relations of H. Weyl between the eigenvalues and singular values of a matrix.

Proposition 2.31. *Let $P(z)$ is given by* (2.17) *and let there exist complex numbers $c_1, c_2, \ldots, c_n \in \mathbb{C}$, $0 \neq b_1, b_2, \ldots, b_{n-1} \in \mathbb{C}$ such that*

$$a_1 = c_1,\ a_2 = c_2 b_1,\ \ldots,\ a_n = c_n b_{n-1} \cdots b_2 b_1. \tag{2.18}$$

If the matrix $A \in \mathbb{C}^{n\times n}$ is given by

$$
(2.19) \qquad A = \begin{bmatrix} 0 & b_{n-1} & 0 & \dots & 0 \\ 0 & 0 & b_{n-2} & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & & 0 & b_1 \\ c_n & c_{n-1} & \dots & c_2 & c_1 \end{bmatrix},
$$

then

$$P(z) = \det(zI_n - A),$$

where I_n is the n-by-n identity matrix.

Thus, the eigenvalues of A are equal to the zeros of $P(z)$. A discussion of the normality condition for A is given in [35].

Decompositions of type (2.18) of the polynomial coefficients are always possible. The simplest one is $c_k = a_k$ $(k = 1, \dots, n)$ and $b_1 = \cdots = b_{n-1} = 1$, when we get the Frobenius companion matrix.

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$, $(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}^*\boldsymbol{x}$, and $\|\boldsymbol{x}\| = \sqrt{(\boldsymbol{x}, \boldsymbol{x})}$. For a given $M \in \mathbb{C}^{n\times n}$, we define the *numerical range* $F(M)$ by

$$F(M) = \{(M\boldsymbol{x}, \boldsymbol{y}) \,:\, \boldsymbol{x} \in \mathbb{C}^n,\ \|\boldsymbol{x}\| = 1\}$$

and the *numerical radius* $r(M)$ by

$$r(M) = \max\{|z| \,:\, z \in F(M)\}.$$

Let $\sigma(M)$ denotes the *spectrum* of M. Since $\sigma(M) \subset F(M)$, we see that estimates for $r(M)$ give estimates for the eigenvalues of M.

Theorem 2.32. *Let $M = [a_{jk}] \in \mathbb{C}^{n\times n}$ and $m \in \{1, \dots, n\}$. If the matrix $M_m \in \mathbb{C}^{(n-1)\times(n-1)}$ is obtained from M by omitting the m-th row and the m-th column and $d_m \geq r(M_m)$ is an arbitrary constant, then*

$$
\begin{aligned}
r(M) \leq{} & \frac{1}{2}\left(|a_{mm}| + d_m\right) \\
& + \frac{1}{2}\left[\left(|a_{mm}| - d_m\right)^2 + \left(\left(\sum_{k\neq m} |a_{mk}|^2\right)^{1/2} + \left(\sum_{k\neq m} |a_{km}|^2\right)^{1/2}\right)^2\right]^{1/2}.
\end{aligned}
$$

The following propositions were proved in [36].

Proposition 2.33. *Let $A \in \mathbb{C}^{n\times n}$ be given by* (2.19) *and*

$$\beta_1 = \min\left\{\cos\frac{\pi}{n+1}\max_{1\le k\le n-1}|b_k|,\ \frac{1}{2}\max_{1\le k\le n-2}(|b_k|+|b_{k+1}|)\right\},$$

$$\beta_2 = \min\left\{\cos\frac{\pi}{n}\max_{2\le k\le n-1}|b_k|,\ \frac{1}{2}\max_{2\le k\le n-2}(|b_k|+|b_{k+1}|)\right\}.$$

Then

$$r(A) \le \min(U_1, U_2),$$

where

$$U_1 = \beta_1 + \frac{1}{2}\left(|c_1| + \sqrt{\sum_{k=1}^{n}|c_k|^2}\right)$$

and

$$U_2 = \frac{1}{2}(|c_1|+\beta_2) + \frac{1}{2}\left((|c_1|-\beta_2)^2 + \left(|b_1| + \sqrt{\sum_{k=2}^{n}|c_k|^2}\right)^2\right)^{1/2}.$$

Proposition 2.34. *Let $A \in \mathbb{C}^{n\times n}$ be given by* (2.19) *and*

$$\tilde\beta_1 = \min\left\{\cos\frac{\pi}{n+1}\max_{1\le k\le n-1}\frac{1}{|b_k|},\ \frac{1}{2}\max_{1\le k\le n-2}\left(\frac{1}{|b_k|}+\frac{1}{|b_{k+1}|}\right)\right\},$$

$$\tilde\beta_2 = \min\left\{\cos\frac{\pi}{n}\max_{1\le k\le n-2}\frac{1}{|b_k|},\ \frac{1}{2}\max_{1\le k\le n-3}\left(\frac{1}{|b_k|}+\frac{1}{|b_{k+1}|}\right)\right\}.$$

Then

$$r(A^{-1}) \le \min(V_1, V_2),$$

where

$$V_1 = \tilde\beta_1 + \frac{1}{2|c_n|}\left(\left|\frac{c_{n-1}}{b_{n-1}}\right| + \sqrt{1+\sum_{k=1}^{n-1}\left|\frac{c_k}{b_k}\right|^2}\right)$$

and

$$V_2 = \frac{1}{2}\left(\frac{|c_{n-1}|}{|c_n b_{n-1}|} + \tilde\beta_2\right)$$
$$+\frac{1}{2}\left(\left(\frac{|c_{n-1}|}{|c_n b_{n-1}|} - \tilde\beta_2\right)^2 + \left(\frac{1}{|b_{n-1}|} + \frac{1}{|c_n|}\sqrt{1+\sum_{k=1}^{n-2}\left|\frac{c_k}{b_k}\right|^2}\right)^2\right)^{1/2}.$$

For $b_1 = b_2 = \cdots = b_{n-1} = b$, the constants β_i and $\tilde\beta_i$ $(i = 1, 2)$ reduce to

$$\beta_1 = |b|\cos\frac{\pi}{n+1},\quad \beta_2 = |b|\cos\frac{\pi}{n},\quad \tilde\beta_1 = \frac{1}{|b|}\cos\frac{\pi}{n+1},\quad \tilde\beta_2 = \frac{1}{|b|}\cos\frac{\pi}{n}.$$

From Propositions 2.33 and 2.34 Linden [36] determined annuli for the zeros of $P(z)$.

Theorem 2.35. *Let $P(z)$ be a monic polynomial as in Proposition* 2.31 *and let U_i, V_i $(i = 1,2)$ be defined as in Propositions* 2.33 *and* 2.34, *respectively. Then, all the zeros of $P(z)$ lie in the annuli*

$$\max\left(V_1^{-1}, V_2^{-1}\right) \le |z| \le \min\left(U_1, U_2\right).$$

Another cases were also considered in [36]. Some other interesting papers in this direction are [12], [17], [30]. For example, Kittaneh [30] computed the singular values of the companion matrix of a monic polynomial, and then applying some basic eigenvalue-singular value majorization relations, he obtained several sharp estimates for the zeros of $P(z)$ in terms of its coefficients. These estimates improve some classical bounds on zeros of polynomials.

3. Zeros in a Strip and Related Inequalities

Turán [74] outlined reasons why it is important to extend some classical questions of the theory of the algebraic equations to the case of other representations, different from the standard polynomial form

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n. \tag{3.1}$$

He considered in this respect the role of the Hermite expansion

$$P(z) = \sum_{k=0}^{n} \alpha_k H_k(z), \tag{3.2}$$

where the k-th Hermite polynomial $H_k(z)$ of degree k. He showed that one can obtain results for *strips* containing all zeros using the representation (2.2) as for *circles* containing all the zeros using representation (2.1) (cf. Theorems 2.2 and 2.12). Precisely, Turán [74] proved the following analogs of Cauchy's and Walsh's estimates for complex zeros.

Theorem 3.1. *If the polynomial $P(z)$ is given by* (3.2) *and*

$$\max_{0 \le k \le n-1} |\alpha_k| = M^*, \tag{3.3}$$

then all the zeros of $P(z)$ lie in the strip

$$|\operatorname{Im} z| \le \frac{1}{2}\left(1 + \frac{M^*}{|\alpha_n|}\right). \tag{3.4}$$

Theorem 3.2. *Let $P(z)$ be a polynomial of degree n given by* (3.2). *Then all the zeros of $P(z)$ lie in the strip*

$$|\operatorname{Im} z| \le \frac{1}{2} \sum_{k=0}^{n-1} \left| \frac{\alpha_k}{\alpha_n} \right|^{1/(n-k)}. \tag{3.5}$$

For the proof of the above estimates an inequality for Hermite polynomials is needed.
Using the identity $H_k'(z) = 2kH_{k-1}(z)$, we can write

$$\frac{H_{k-1}(z)}{H_k(z)} = \frac{1}{2k} \cdot \frac{H_k'(z)}{H_k(z)} = \frac{1}{2k} \sum_{\nu=1}^{k} \frac{1}{z - z_{\nu k}},$$

where $z_{\nu k}$ denote the zeros of $H_k(z)$. Therefore we have

$$\left| \frac{H_{k-1}(z)}{H_k(z)} \right| \leq \frac{1}{2k} \sum_{\nu=1}^{k} \frac{1}{|z - z_{\nu k}|}. \tag{3.6}$$

By the fact that all the $z_{\nu k}$-zeros of H_k are real, it follows that

$$\frac{1}{|z - z_{\nu k}|} \leq \frac{1}{|y|} \qquad (z = x + iy). \tag{3.7}$$

From (3.6) and (3.7) it follows that

$$\left| \frac{H_{k-1}(z)}{H_k(z)} \right| \leq \frac{1}{2|y|}.$$

For all $k \leq n - 1$ and arbitrary non-real z it follows that

$$\left| \frac{H_k(z)}{H_n(z)} \right| = \prod_{\nu=k+1}^{n} \left| \frac{H_{\nu-1}(z)}{H_\nu(z)} \right| < \frac{1}{(2|y|)^{n-k}}. \tag{3.8}$$

This inequality is very important for the proof of Theorems 3.1 and 3.2. Using (3.8), Turán [74] provided the following proofs.
Proof of Theorem 3.1. We obtain

$$|P(z)| = \left| \sum_{k=0}^{n} \alpha_k H_k(z) \right| \geq |\alpha_n| \, |H_n(z)| \left\{ 1 - \sum_{k=0}^{n-1} \left| \frac{\alpha_k}{\alpha_n} \right| \left| \frac{H_k(z)}{H_n(z)} \right| \right\}. \tag{3.9}$$

From (3.3) and (3.8) we get for

$$|y| > \frac{1}{2} \left(1 + \frac{M^*}{|\alpha_n|} \right) \tag{3.10}$$

the inequality

$$\begin{aligned} |P(z)| &\geq |\alpha_n| \, |H_n(z)| \left\{ 1 - \frac{M^*}{|\alpha_n|} \sum_{\nu=1}^{n} \left(\frac{1}{2|y|} \right)^{\nu} \right\} \\ &> |\alpha_n| \, |H_n(z)| \left\{ 1 - \frac{M^*}{|\alpha_n|} \cdot \frac{1}{2|y| - 1} \right\}. \end{aligned}$$

Because of the fact that $H_n(z)$ does not vanish in the domain given in (3.10), it follows that for such z-values, $P(z) \neq 0$. $\square$

To prove that the strip in Theorem 3.1 cannot be replaced by a strip of the form

$$|\operatorname{Im} z| < \frac{\theta}{2}\left(1 + \frac{M^*}{|\alpha_n|}\right) \tag{3.11}$$

with a fixed $0 < \theta < 1$, we consider the polynomial

$$z \mapsto P_1(z) = H_n(z) - iaH_{n-1}(z), \tag{3.12}$$

where a denotes a sufficiently large positive number. Therefore $M^* = a$. The equation $P_1(z) = 0$ can take the form

$$\frac{H_{n-1}(z)}{H_n(z)} = \frac{1}{ia} \tag{3.13}$$

From (3.12) and (3.13) we obtain

$$\sum_{\nu=1}^{n} \frac{1}{z - z_{\nu n}} = \frac{2n}{ia}\,.$$

Assume for example that n is even, i.e., $n = 2m$, then (3.13) reads as follows

$$z \sum_{\nu=1}^{m} \frac{1}{z^2 - z_{\nu n}^2} = \frac{2m}{ia} \qquad (z_{\nu n} > 0).$$

If $z = iy$ where y is a real number, then

$$\sum_{\nu=1}^{m} \frac{1}{1 + (z_{\nu n}/y)^2} = \frac{2my}{a}\,. \tag{3.14}$$

However m is fixed, therefore we can choose a sufficiently large, such that

$$\sum_{\nu=1}^{m} \frac{1}{1 + \left[z_{\nu n}/\left(\frac{1+\theta}{4}\,a\right)\right]^2} > \frac{1+\theta}{2}\,m$$

or

$$\frac{1+\theta}{4}\,a > \frac{\theta}{2}(1 + a).$$

Therefore the equation (3.14) has a real root with $y > \frac{1}{4}(1 + \theta)a$. This implies that the polynomial $z \mapsto P_1(z)$, defined by (3.12), has a zero iy_0 such that

$$|y_0| > \frac{1}{4}(1 + \theta)a > \frac{\theta}{2}(1 + a) = \frac{\theta}{2}(1 + M^*),$$

implying that inequality (3.11) is not valid.

Proof of Theorem 3.2. From (3.8) and (3.9) we have

$$\begin{aligned}|P(z)| &\geq |\alpha_n|\,|H_n(z)|\left\{1-\sum_{k=0}^{n-1}\left|\frac{\alpha_k}{\alpha_n}\right|\left(\frac{1}{2|y|}\right)^{n-k}\right\}\\ &> |\alpha_n|\,|H_n(z)|\left\{1-\sum_{k=0}^{n-1}\left(\frac{1}{2|y|}\sqrt[n-k]{\left|\frac{\alpha_k}{\alpha_n}\right|}\right)^{n-k}\right\}.\end{aligned}$$

Therefore if z is not a point in the strip (3.5), then all the following inequalities hold

$$\left|\frac{\alpha_k}{\alpha_n}\right|^{1/(n-k)}\frac{1}{2|y|}\leq 1,$$

i.e.,

$$|P(z)|\geq|\alpha_n|\,|H_n(z)|\left\{1-\sum_{k=0}^{n-1}\left|\frac{\alpha_k}{\alpha_n}\right|^{1/(n-k)}\frac{1}{2|y|}\right\}>0. \quad \square$$

Let ε be a small positive number and define

$$P(z)=H_n(z)+\varepsilon H_{n-1}(z)+\varepsilon^2H_{n-2}(z)+\cdots+\varepsilon^nH_0(z).$$

Then by (3.5) all the zeros lie in the strip $|\operatorname{Im} z|\leq(n/2)\varepsilon$, which shrinks to the real axis if $\varepsilon\to 0$. Therefore the strip (3.5) is best possible in that sense.

Turán [75] also considered the case of even polynomials and proved the following results:

Theorem 3.3. *If* $P(z)=\sum\limits_{k=0}^{n}c_{2k}H_{2k}(z)$ *with arbitrary coefficients and*

$$\max_{0\leq k\leq n-1}|c_{2k}|=M,$$

then all zeros of $P(z)$ *lie in the strip*

$$|\operatorname{Im} z|\leq\frac{1}{2}\left(1+\frac{5}{\sqrt{2n-1}}\cdot\frac{M}{|c_{2n}|}\right).$$

Theorem 3.4. *If* $z=x+iy$ *and* $P(z)=\sum\limits_{k=0}^{n}c_{2k}H_{2k}(z)$ *with* $\max\limits_{0\leq k\leq n-1}|c_{2k}|=M$, *then all zeros of* $P(z)$ *lie in the hyperbole*

$$|xy|\leq\frac{5}{4}\left(1+\frac{M}{|c_{2n}|}\right).$$

Denoting by $x_1>x_2>\cdots>x_{n-1}$ the zeros of $H_{n-1}(z)$ and using the well-known Christoffel-Darboux formula, Turán [75] obtained the formula

$$\sum_{\nu=0}^{n-2}\frac{H_\nu(x_k)^2}{2^\nu\nu!}=\frac{1}{2^n(n-1)!}H_n(x_k)^2.$$

An application of this simple formula gives the following result:

Theorem 3.5. *If the coefficients of*

$$P(z) = \sum_{k=0}^{n} c_k H_k(z)$$

are real and

$$\sum_{k=0}^{n-2} 2^\nu \nu! c_\nu^2 < 2^n (n-1)! c_n^2 \tag{3.15}$$

is fulfilled, then all zeros of $P(z)$ are real and simple.

The condition (3.15) is obviously fulfilled if the coefficients c_k do not decrease "too quickly". As a counterpart of the previous theorem, Turán [75] proved that the same conclusion holds if the coefficients decrease sufficiently quickly. More precisely, he proved:

Theorem 3.6. *If $P(z)$ has the form*

$$P(z) = \sum_{k=0}^{n} (-1)^k c_{2k} H_{2k}(z)$$

with positive coefficients c_{2k} and for $k = 1, 2, \ldots, n-1$ we have

$$c_{2k}^2 > 4 c_{2k-2} c_{2k+2}, \tag{3.16}$$

then all zeros of $P(z)$ are real.

Changing (3.16) by

$$\frac{c_2}{c_0} > \frac{1}{4}, \quad \frac{c_4}{c_2} > \frac{1}{4}, \quad \ldots, \quad \frac{c_{2k}}{c_{2k-2}} > \frac{1}{4} \quad (k \le n),$$

then $P(z)$ *has at least $2k$ real zeros with odd multiplicities* (see Turán [75]).

In 1966 Vermes [76] considered the location of the zeros of a complex polynomial $P(z)$ expressed in the form

$$P(z) = \sum_{k=0}^{n} a_k q_k(z), \tag{3.17}$$

where $\{q_k(z)\}$ is a given sequence of monic polynomials ($\deg q_k(z) = k$) whose zeros lie in a prescribed region E. His principal theorem states that the zeros of $P(z)$ are in the interior of a Jordan curve $S = \{z \in \mathbb{C} : |F(z)| = \max(1, R)\}$, where F maps the complement of E onto $|z| > 1$ and R is the positive root of the equation

$$\sum_{k=0}^{n-1} \lambda_k |a_k| t^k - \lambda_n |a_n| t^n = 0,$$

with $\lambda_k > 0$ depending on E only. In a special case he obtained the following result:

Theorem 3.7. *If all the zeros of the monic polynomials* $q_k(z)$ $(k \in \mathbb{N})$ *lie in* $[-1,1]$ *and the zeros of* $q_k(z)$ *and* $q_{k+1}(z)$ *separate each other, then all the zeros of* (3.17) *are in the ellipse*

$$\frac{x^2}{(R+R^{-1})^2}+\frac{y^2}{(R-R^{-1})^2}=\frac{1}{4} \qquad (z=x+iy),$$

where $R=\max(2+\sqrt{3}\,,\varrho)$ *and* ϱ *is the only positive root of the equation*

$$|a_0|+|a_1|t+|a_2|t^2+\cdots+|a_{n-1}|t^{n-1}-|a_n|t^n=0.$$

In particular, if the sequence $\{q_k(z)\}$ in Theorem 3.7 is a sequence of monic orthogonal polynomials then the zeros of $q_k(z)$ and $q_{k+1}(z)$ separate each other and we have that *all the zeros of* $P(z)$ *are in the ellipse as given in this theorem.*

We mention now a problem from the graph theory. Namely, it has been conjectured that the β-polynomials of all graphs has only real zeros. Recently, Li, Gutman and Milovanović [34] showed that the conjecture is true for complete graphs. In fact, they obtained a more general result for polynomials given by

$$\beta(n,m,t,x)=He_n(x)+t\,He_{n-m}(x), \tag{3.18}$$

where He_n is one of the forms of the Hermite polynomials [2, p. 778]. Such (monic) Hermite polynomials are orthogonal on $(-\infty,+\infty)$ with respect to the weight function $x\mapsto e^{-x^2/2}$ and their connection with the "standard" Hermite polynomials $H_k(x)$ can be expressed by $He_k(x)=2^{-k/2}H_k(x/\sqrt{2})$. Here, $1\le m\le n$ and t is a real number. Clearly, for $n\ge 3$, $|t|=2$ and $3\le m\le n$, the previous formula represents the β-polynomial of the complete graph on n vertices, pertaining to a circuit with m vertices.

Theorem 3.8. *For all* (*positive integer*) *values of* n, *for all* $m=1,2,\ldots,n$ *and for* $|t|\le n-1$ *the polynomial* $\beta(n,m,t,x)$, *given by* (3.18), *has only real zeros.*

Proof. We use here the following facts for the Hermite polynomials $He_n(x)$:

(a) The three-term recurrence relation

$$He_n(x)=xHe_{n-1}(x)-(n-1)He_{n-2}(x);$$

(b) All zeros of $He_n(x)$ are real and distinct;

(c)
$$\frac{d}{dx}He_n(x)=nHe_{n-1}(x).$$

and conclude that $He_n(x)$ has a local extreme x_i if and only if $He_{n-1}(x_i)=0$. So, the extremes of $He_n(x)$ are distinct.

Let $x_1,x_2,\ldots,x_{n-1}$ denote the distinct zeros of $He_{n-1}(x)$. If for all $i=1,2\ldots,$ $n-1$, the sign of $\beta(n,m,t,x_i)=He_n(x_i)+tHe_{n-m}(x_i)$ is the same as that of

$He_n(x_i)$, we can prove that $\beta(n,m,t,x)$ has only real zeros. Indeed, from (c) we have that x_i $(i = 1, 2, \ldots, n-1)$ are the extremes of $He_n(x)$. Since $He_n(x)$ does not have multiple zeros, we know that $He_n(x_i) \neq 0$ for all $i = 1, 2, \ldots, n-1$, and that $He_n(x_i)$ and $He_n(x_{i+1})$ have different signs $(i = 1, 2, \ldots, n-2)$. Thus, we can deduce that $\beta(n,m,t,x)$ has at least as many real zeros as $He_n(x)$, that is at least n real zeros. On the other hand the degree of $\beta(n,m,t,x)$ is n.

Then, if $|He_n(x_i)| > (n-1)|He_{n-m}(x_i)|$ for all $i = 1, 2, \ldots, n-1$, we prove that $\beta(n,m,t,x)$ has only real zeros for $|t| \leq n-1$.

Define now the auxiliary quantities $a_{n,m}$ as

$$a_{n,m} = \max_{1 \leq i \leq n-1} \left| \frac{He_{n-m}(x_i)}{He_n(x_i)} \right| . \tag{3.19}$$

Because of the previous fact, if

$$a_{n,m} \leq \frac{1}{n-1} \tag{3.20}$$

then $\beta(n,m,t,x)$ has only real zeros for $|t| \leq n-1$. Therefore, in order to complete the proof of Theorem 3.8 we only need to verify the inequality (3.20).

Using the well-known three-term recurrence relation (a) for the Hermite polynomials $He_n(x)$, (3.19) reduces to

$$a_{n,m} = \frac{1}{n-1} \max_{1 \leq i \leq n-1} \left| \frac{He_{n-m}(x_i)}{He_{n-2}(x_i)} \right| = \frac{1}{(n-1)(n-2)} \max_{1 \leq i \leq n-1} \left| \frac{x_i \, He_{n-m}(x_i)}{He_{n-3}(x_i)} \right|$$

and we conclude immediately that

$$a_{n,1} = 0, \qquad a_{n,2} = \frac{1}{n-1} \quad (n \geq 2),$$

and

$$a_{n,3} = \frac{1}{(n-1)(n-2)} \max_{1 \leq i \leq n-1} |x_i| \begin{cases} = \dfrac{1}{n-1} & (n = 3), \\[2ex] < \dfrac{2\sqrt{n-3}}{(n-1)(n-2)} \leq \dfrac{1}{n-1} & (n \geq 4). \end{cases}$$

The upper bound for $a_{n,3}$ follows from the inequality $|x_i| < 2\sqrt{n-3}$, which holds for all $i = 1, 2, \ldots, n-1$ and $n \geq 4$ (see Godsil and Gutman [25, Theorem 7]).

Note that the relation $a_{n,1} = 0$ provides a proof that the polynomial $\beta(n,1,t,x)$ has only real zeros for $n \geq 1$ and any real value of the parameter t.

The case when $n \geq m \geq 4$ can be verified using the condition (3.15), rewritten in the form

$$\sum_{k=0}^{n-2} k! \, c_k^2 < (n-1)! \, c_n^2 .$$

Then, according to Theorem 3.5, the polynomial $P(z) = \sum_{k=0}^{n} c_k \, He_k(z)$ has n distinct real zeros. Considering the β-polynomial given by (3.18), we conclude that it has all real zeros if $|t| < \sqrt{(n-1)!/(n-m)!}$. On the other hand, it is easily verified that for $n > m \geq 4$ the expression $\sqrt{(n-1)!/(n-m)!}$ is greater than $n-1$. Notice that $a_{4,4} = 1/3$.

By this, the proof of Theorem 3.8 has been completed. □

Taking other orthogonal polynomials instead of Hermite polynomials, Specht [64] – [67] obtained several results which are analogous to results of Turán. For details on orthogonal polynomials see, for example, Szegő [69].

Let $d\mu$ be a positive Borel measure on the real line, for which all the moments $\mu_k = \int_{\mathbb{R}} t^k \, d\mu(t)$, $k = 0, 1, \ldots$, are finite. We suppose also that $\operatorname{supp}(d\mu)$ contains infinitely many points, i.e., that the distribution function $\mu\colon \mathbb{R} \to \mathbb{R}$ is a non-decreasing function with infinitely many points of increase. It is well known that then there exists an infinite sequence of orthogonal polynomials with respect to the inner product $(\,.\,,\,.\,)$ defined by

$$(f, g) = \int_{\mathbb{R}} f(t)\overline{g(t)} \, d\mu(t).$$

The corresponding orthonormal and monic orthogonal polynomials will be denoted by $p_n(t)$ and $\pi_n(t)$, respectively. Thus, we have

$$p_n(t) = \gamma_n t^n + \delta_n t^{n-1} + \text{ lower degree terms}, \quad \gamma_n > 0,$$

$$(p_n, p_m) = \delta_{nm}, \quad n, m \geq 0,$$

and

$$\pi_n(t) = \frac{p_n(t)}{\gamma_n} = t^n + \text{ lower degree terms}.$$

If μ is an absolutely continuous function, then we say that $\mu'(t) = w(t)$ is a *weight function*. In that case, the measure $d\mu$ can be express as $d\mu(t) = w(t) \, dt$, where the weight function $t \mapsto w(t)$ is a non-negative and measurable in Lebesgue's sense for which all moments exists and $\mu_0 > 0$. If $\operatorname{supp}(w) = [a, b]$, where $-\infty < a < b < +\infty$, we say that $\{p_n\}$ is a system of orthonormal polynomials in a finite interval $[a, b]$. For (a, b) we say that it is an *interval of orthogonality*.

The system of orthonormal polynomials $\{p_n(t)\}$, associated with the measure $d\mu(t)$, satisfy a three-term recurrence relation

$$tp_n(t) = u_{n+1} p_{n+1}(t) + v_n p_n(t) + u_n p_{n-1}(t) \qquad (n \geq 0),$$

where $p_{-1}(t) = 0$ and the coefficients $u_n = u_n(d\mu)$ and $v_n = v_n(d\mu)$ are given by

$$u_n = \int_{\mathbb{R}} t p_{n-1}(t) p_n(t) \, d\mu(t) = \frac{\gamma_{n-1}}{\gamma_n} \qquad \text{and} \qquad v_n = \int_{\mathbb{R}} t p_n(t)^2 \, d\mu(t).$$

Since $p_0(t) = \gamma_0 = 1/\sqrt{\mu_0}$ and $\gamma_{n-1} = u_n\gamma_n$ we have that $\gamma_n = \gamma_0/(u_1u_2\cdots u_n)$. Notice that $u_n > 0$ for each n.

The corresponding monic orthogonal polynomials $\{\pi_n(t)\}$ satisfy the following three-term recurrence relation

$$\pi_{n+1}(t) = (t-\alpha_n)\pi_n(t) - \beta_n\pi_{n-1}(t), \qquad n = 0,1,2,\dots, \tag{3.21}$$

where $\alpha_n = v_n$ and $\beta_n = u_n^2 > 0$.

Because of orthogonality, we have that

$$\alpha_n = \frac{(t\pi_n,\pi_n)}{(\pi_n,\pi_n)} \quad (n \ge 0), \quad \beta_n = \frac{(\pi_n,\pi_n)}{(\pi_{n-1},\pi_{n-1})} \quad (n \ge 1).$$

The coefficient β_0, which multiplies $\pi_{-1}(t) = 0$ in three-term recurrence relation (3.21) may be arbitrary. Sometimes, it is convenient to define it by $\beta_0 = \mu_0 = \int_{\mathbb{R}} d\mu(t)$. Then the norm of π_k can be express in the form $\|\pi_k\| = \sqrt{h_k}$, where

$$h_k = (\pi_k,\pi_k) = \beta_0\beta_1\cdots\beta_k\,. \tag{3.22}$$

Consider now an arbitrary polynomial $P(z)$ of degree n, given by

$$P(z) = A_0 + A_1z + \cdots + A_nz^n. \tag{3.23}$$

Then, it can be expanded, for example, in terms of the orthonormal polynomials $p_k(z)$ $(k = 0,1,\dots,n)$ in the form

$$P(z) = c_0p_0(z) + c_1p_1(z) + \cdots + c_np_n(z). \tag{3.24}$$

Specht [64] proved the following result:

Theorem 3.9. *All the zeros of a complex polynomial $P(z)$ expanded in the form* (3.24) *lie in the strip*

$$|\operatorname{Im} z| \le \frac{\gamma_{n-1}}{\gamma_n}\left(\sum_{k=0}^{n-1}\left|\frac{c_k}{c_n}\right|^2\right)^{1/2},$$

where γ_k is the leading coefficient in the orthonormal polynomial $p_k(z)$.

In the case of the Legendre polynomials $P_k(x)$, i.e., when

$$P(z) = c_0P_0(z) + c_1P_1(z) + \cdots + c_nP_n(z),$$

Specht obtained the following estimate

$$|\operatorname{Im} z| \le \frac{n}{\sqrt{2n-1}}\frac{1}{|c_n|}\left(\sum_{k=0}^{n-1}\frac{|c_k|^2}{2k-1}\right)^{1/2}.$$

Giroux [24] proved a sharper result than Theorem 3.9.

Theorem 3.10. *Let $z_1, \dots, z_n$ be the zeros of the polynomial $P(z)$ given by (3.24). Then*

$$\sum_{k=1}^{n} |\operatorname{Im} z_k| \le \frac{\gamma_{n-1}}{\gamma_n} \left(\sum_{k=0}^{n-1} \left| \frac{c_k}{c_n} \right|^2 \right)^{1/2}, \tag{3.25}$$

with equality if and only if

$$c_0 = \cdots = c_{n-2} = 0 \quad \textit{and} \quad \operatorname{Re}(c_{n-1}/c_n) = 0.$$

Proof. Following Giroux [24] we start with the identity

$$\sum_{k=0}^{n} |c_k|^2 = \int_{\mathbb{R}} |P(t)|^2 \, d\mu(t) = \|P\|^2.$$

In particular, we have $|c_{n-1}|^2 + |c_n|^2 \le \|P\|^2$. Since

$$P(z) = c_n \gamma_n \prod_{k=0}^{n} (z - z_k) = c_n \gamma_n \left(z^n - \left(\sum_{k=1}^{n} z_k \right) z^{n-1} + \cdots \right)$$

and

$$P(z) = \sum_{k=0}^{n} c_k p_k(z) = c_n \gamma_n z^n + (c_n \delta_n + c_{n-1} \gamma_{n-1}) z^{n-1} + \cdots,$$

we have

$$c_n \delta_n + c_{n-1} \gamma_{n-1} = -c_n \gamma_n \sum_{k=1}^{n} z_k.$$

It is sufficient to prove the theorem when $c_n = 1$. In that case, since δ_n is real, we have $\operatorname{Im} c_{n-1} = -(\gamma_n/\gamma_{n-1}) \sum_{k=1}^{n} \operatorname{Im} z_k$. Hence

$$\frac{\gamma_n}{\gamma_{n-1}} \left| \sum_{k=1}^{n} \operatorname{Im} z_k \right| = |\operatorname{Im} c_{n-1}| \le |c_{n-1}| \le \left(\|P\|^2 - 1 \right)^{1/2},$$

so that

$$1 + \left(\frac{\gamma_n}{\gamma_{n-1}} \right)^2 \left| \sum_{k=1}^{n} \operatorname{Im} z_k \right|^2 \le \|P\|^2.$$

Applying this result to the polynomial

$$Q(z) = P(z) \prod_{\nu} (z - \bar{z}_\nu)/(z - z_\nu),$$

where the zeros z_ν appearing in the product are precisely those for which $\operatorname{Im} z_\nu < 0$, we get

$$1 + \left(\frac{\gamma_n}{\gamma_{n-1}}\right)^2 \left(\sum_{k=1}^{n} |\operatorname{Im} z_k|\right)^2 \le \|Q\|^2 = \|P\|^2.$$

This is the statement of the theorem (when $c_n = 1$). Equality in (3.25) is attained if only if $c_0 = \cdots = c_{n-2} = 0$, c_{n-1}/c_n is purely imaginary and the zeros z_k $(k = 1, \dots, n)$ are either all above or all below the real axis. □

Remark. For every real number c, the zeros of $p_n(z) + icp_{n-1}(z)$ are either all above or all below the real axis.

Using an inequality of de Bruijn [6] (see also [45, p. 114]), Giroux [24] also proved:

Corollary 3.11. *Let $P(z)$ be a polynomial of degree $n > 1$ given by* (3.24) *and let $w_1, \dots, w_{n-1}$ be the zeros of $P'(z)$. Then we have*

$$\sum_{k=1}^{n-1} |\operatorname{Im} w_k| \le \frac{n-1}{n} \frac{\gamma_{n-1}}{\gamma_n} \left(\sum_{k=0}^{n-1} \left|\frac{c_k}{c_n}\right|^2\right)^{1/2},$$

with equality if and only if $P(z)$ is a multiple of the polynomial $p_n(z) + icp_{n-1}(z)$ with c real.

Another consequence of Theorem 3.10 is the following result:

Corollary 3.12. *There is at least one zero of the polynomial* (3.24) *in the strip*

$$|\operatorname{Im} z| \le \frac{1}{n} \frac{\gamma_{n-1}}{\gamma_n} \left(\sum_{k=0}^{n-1} \left|\frac{c_k}{c_n}\right|^2\right)^{1/2}.$$

Giroux [24] also proved:

Theorem 3.13. *Let*

$$f(x) = (x - x_1)(x - x_2) \cdots (x - x_n),$$
$$g(x) = (x - y_1)(x - y_2) \cdots (x - y_{n-1}),$$

with $x_1 < y_1 < x_2 < \cdots < y_{n-1} < x_n$. Then, for any real number c, the zeros of the polynomial $h(x) = f(x) + icg(x)$ are all in the half strip $\operatorname{Im} z \ge 0$, $x_1 \le \operatorname{Re} z \le x_n$, or all are in the conjugate half strip.

Using the system of monic orthogonal polynomials $\{\pi_k(z)\}_{k=0}^{+\infty}$, defined by the three-term recurrence relation (3.21), Gol'berg and Malozemov [26] considered estimates for zeros of polynomials of the type

$$Q(z) = \pi_n(z) + b_1\pi_{n-1}(z) + \cdots + b_n\pi_0(z). \tag{3.26}$$

Setting $\beta_k = u_k^2 > 0$, $c_1 = b_1 = \alpha + i\beta$,

$$c_2 = \frac{b_2}{u_{n-1}}, \quad c_3 = \frac{b_3}{u_{n-1}u_{n-2}}, \quad \dots, \quad c_n = \frac{b_n}{u_{n-1}u_{n-2}\cdots u_1}$$

and $C = \sum_{k=2}^{n} |c_k|^2$, Gol'berg and Malozemov [26] proved:

Theorem 3.14. *Let x_1 and x_n be the minimal and the maximal zero of the polynomial $\pi_n(z)$, respectively, and let ξ be an arbitrary zero of the polynomial $Q(z)$, defined by* (3.26). *Then*

$$x_1 - \frac{1}{2}\left(\alpha + \sqrt{\alpha^2 + C}\right) \le \operatorname{Re}\xi \le x_n - \frac{1}{2}\left(\alpha - \sqrt{\alpha^2 + C}\right)$$

and

$$-\frac{1}{2}\left(\beta + \sqrt{\beta^2 + C}\right) \le \operatorname{Im}\xi \le -\frac{1}{2}\left(\beta - \sqrt{\beta^2 + C}\right).$$

Suppose now that an arbitrary polynomial $P(z)$ of degree n is given by (3.23). Let

$$P(z) = a_0\pi_0(z) + a_1\pi_1(z) + \cdots + a_n\pi_n(z) \qquad (a_n \neq 0) \tag{3.27}$$

be its representation in terms of monic orthogonal polynomials $\{\pi_k(z)\}$. Comparing (2.24) and (3.27) we see that $c_k\gamma_k = a_k$ $(k = 0, 1, \ldots, n)$, so that the Specht's estimate given in Theorem 3.9 can be expressed in the form

$$|\operatorname{Im} z| \le \left(\sum_{k=0}^{n-1} \frac{h_k}{h_{n-1}} \left|\frac{a_k}{a_n}\right|^2\right)^{1/2}, \tag{3.28}$$

where h_k is given by (3.22).

An interesting property of (3.28) is that its right hand side may be expressed in terms of a norm (see Schmeisser [62]). Namely, since for the L^2-norm of $P(z)$ we have

$$\|P\|^2 = \int_{\mathbb{R}} |P(t)|^2\, d\mu(t) = \sum_{k=0}^{n} |a_k|^2 \|\pi_k\|^2 = \sum_{k=0}^{n} h_k |a_k|^2,$$

the inequality (3.28) can be rewritten as

$$|\operatorname{Im} z| \le \frac{1}{|a_n|\sqrt{h_{n-1}}}\|P - a_n\pi_n\| = \frac{1}{\sqrt{h_{n-1}}}\left\|\frac{P}{a_n} - \pi_n\right\|. \tag{3.29}$$

This may be interpreted as a perturbation theorem. Namely, since $\pi_n(z)$ has all its zeros on the real line, (3.29) tells us that, apart from a constant, the deviation of $P(z)/a_n$ from $\pi_n(z)$, measured by the norm, is an upper bound for the distances of the zeros of $P(z)$ from the real line. Several refinements of (3.28) or its equivalent form (3.29) were derived in [62]. We mention some of them.

Theorem 3.15. *Denote by $\xi_1, \ldots, \xi_n$ the zeros of $\pi_n(z)$. Then every polynomial $P(z)$ of the form* (3.27) *has all its zeros in the union $\mathcal{U}$ of the disks*

$$\mathcal{D}_k = \{z \in \mathbb{C}\ |z - \xi_k| \le r\} \qquad (k = 1, \ldots, n),$$

where

$$r = \sqrt{\sum_{k=0}^{n-1} \frac{h_k}{h_{n-1}} \left|\frac{a_k}{a_n}\right|^2}.$$

Moreover, if m of these disks constitute a connected component of $\mathcal{U}$, then their union contains exactly m zeros of $P(z)$.

Theorem 3.16. *Let z_ν $(\nu = 1, \dots, n)$ be an arbitrary zero of the polynomial* (3.27). *Then*

$$\left| \operatorname{Im}\Big(z_\nu + \frac{1}{2}\frac{a_{n-1}}{a_n}\Big) \right| \le \frac{1}{2}\sqrt{\Big(\operatorname{Im}\frac{a_{n-1}}{a_n}\Big)^2 + \sum_{k=0}^{n-2} \frac{h_k}{h_{n-1}} \left|\frac{a_k}{a_n}\right|^2}\,.$$

Theorem 3.17. *Let $z_1, \dots, z_n$ be the zeros of the polynomial* (3.27). *Then*

$$\sum_{\nu=1}^{n} (\operatorname{Im} z_\nu)^2 \le \Big(\operatorname{Im}\frac{a_{n-1}}{a_n}\Big)^2 + \frac{1}{2h_{n-1}} \sum_{k=0}^{n-2} h_k \left|\frac{a_k}{a_n}\right|^2 .$$

Theorem 3.17 improves upon (3.28) but it does not imply Theorem 3.16. As a consequence, Schmeisser [62] obtained the following individual bounds.

Corollary 3.18. *Let $z_1, \dots, z_n$ be the zeros of the polynomial* (3.27) *ordered as*

$$|\operatorname{Im} z_1| \le |\operatorname{Im} z_2| \le \cdots \le |\operatorname{Im} z_n| \,.$$

Then

$$|\operatorname{Im} z_\nu| \le \sqrt{\frac{1}{n-\nu+1}\left(\Big(\operatorname{Im}\frac{a_{n-1}}{a_n}\Big)^2 + \frac{1}{2h_{n-1}}\sum_{k=0}^{n-2} h_k \left|\frac{a_k}{a_n}\right|^2\right)}$$

for $\nu = 1, 2, \dots, n$.

Notice that the estimate for z_n is not as good as that of Theorem 3.16.

In the case of real polynomials Schmeisser [62] proved the following result:

Theorem 3.19. *Let the polynomial* (3.27) *have real coefficients. Then each zero z_ν $(\nu = 1, \dots, n)$ of $P(z)$ satisfies the inequality*

$$|\operatorname{Im} z_\nu| \le \sqrt{\left(\sum_{k=0}^{n-2} \frac{h_k}{h_{n-2}} \left|\frac{a_k}{a_n}\right|^2\right)^{1/2} - \frac{h_{n-1}}{h_{n-2}}\Delta_{n-1}}$$

provided that the radicand is non-negative, else $P(z)$ has n distinct real zeros which separate those of π_{n-1}. Here, Δ_m is defined by

$$\Delta_m = \min_{1 \le \nu \le m} \sqrt{\frac{\pi_{m-1}(\xi_\nu)}{\pi'_m(\xi_\nu)}}\,,$$

where $\xi_1, \dots, \xi_m$ are zeros of $\pi_m(z)$.

Schmeisser [62] considered also some estimates involving the *distance function*

$$d_n(z) = \min_{\xi \subset J_n} |z - \xi| \qquad (z \in \mathbb{C}),$$

where J_n is the smallest compact interval that contains the zeros of the monic orthogonal polynomial $\pi_n(z)$. It is easy to see that

$$d_1(z) \geq d_2(z) \geq \cdots \geq d_n(z) \geq \cdots \geq |\operatorname{Im} z|,$$

which means that any upper bound for $d_n(z)$ is also an upper bound for $|\operatorname{Im} z|$.

Let $P(z)$ be given by (3.23) or by its equivalent form (3.27). Taking the Cauchy bound of $P(z)$ as the unique positive zero of the associated polynomial (see Theorem 2.1)

$$f(z) = \sum_{k=0}^{n-1} |A_k| z^k - |A_n| z^n,$$

Schmeisser [62] gave a short proof of the following result:

Theorem 3.20. *Let $P(z)$ be a polynomial given in the form* (3.27). *Then each zero z_ν $(\nu = 1, \ldots, n)$ of $P(z)$ satisfies the inequality $d_n(z_\nu) \leq r$, where $r = r[P]$ is the Cauchy bound of $P(z)$.*

Using this fact and upper bounds for $r[P]$, he obtained several estimates for $d_n(z_\nu)$:

$$d_n(z_\nu) \leq 1 + \max_{0 \leq k \leq n-1} \left| \frac{a_k}{a_n} \right|,$$

$$d_n(z_\nu) \leq \max \left\{ 1, \sum_{k=0}^{n-1} \left| \frac{a_k}{a_n} \right| \right\},$$

$$d_n(z_\nu) \leq \left(\sum_{k=0}^{n} \left| \frac{a_k}{a_n} \right|^2 \right)^{1/2},$$

$$d_n(z_\nu) \leq 2 \max_{0 \leq k \leq n-1} \left| \frac{a_k}{a_n} \right|^{1/(n-k)},$$

$$d_n(z_\nu) \leq \left| \frac{a_0}{a_n} \right|^{1/n} + \left| \frac{a_1}{a_n} \right|^{1/(n-1)} + \cdots + \left| \frac{a_{n-1}}{a_n} \right|,$$

$$d_n(z_\nu) \leq \max_{0 \leq k \leq n-1} \left(n \left| \frac{a_k}{a_n} \right| \right)^{1/(n-k)}.$$

Notice that in these estimates the parameters which determine the system of orthogonal polynomials, do not appear explicitly. The reason is that Theorem 3.20 holds for a much wider class of expansions.

Now we mention a few results which also were given in [62].

Theorem 3.21. *Let $z_1, \ldots, z_n$ be the zeros of the polynomial* (3.27) *in an arbitrary order. Then*

$$\sum_{\nu=1}^{n} h_{\nu-1} d_n(z_\nu)^2 \cdots d_n(z_n)^2 \le \sum_{k=0}^{n-1} h_k \left|\frac{a_k}{a_n}\right|^2 . \tag{3.30}$$

In this theorem, we can order the zeros as

$$d_n(z_1) \le d_n(z_2) \le \cdots \le d_n(z_n) . \tag{3.31}$$

The left hand side of (3.30) is a sum of non-negative terms and $h_{n-1}d_n(z_n)^2$ is one of them. Hence dividing both sides by h_{n-1}, we see that (3.30) is a refinement of (3.28). Furthermore, if (3.31) holds, then we may estimate the left hand side of (3.30) from below by

$$\sum_{\nu=k}^{n} h_{\nu-1} d_n(z_\nu)^2 \cdots d_n(z_n)^2 \ge \sum_{\nu=k}^{n} h_{\nu-1} d_n(z_k)^{2(n-\nu+1)} \ge h_{k-1} d_n(z_k)^{2(n-k+1)},$$

where $1 \le k \le n$. This allows the following individual bounds for the zeros of $P(z)$.

Corollary 3.22. *Let $z_1, \ldots, z_n$ be the zeros of the polynomial* (3.27) *ordered as in* (3.31). *Then*

$$d_n(z_\nu) \le \left(\sum_{k=0}^{n-1} \frac{h_k}{h_{\nu-1}} \left|\frac{a_k}{a_n}\right|^2 \right)^{1/(2n-2\nu+2)} \qquad (\nu = 1, 2, \ldots, n) .$$

Gautschi and Milovanović [21] considered special linear combinations of the form

$$p_n(z) = \pi_n(z) - i\theta_{n-1}\pi_{n-1}(z), \tag{3.32}$$

where $\{\pi_k(z)\}_{k=0}^{+\infty}$ is a system of monic polynomials orthogonal with respect to an even weight function $x \mapsto w(x)$ on $(-a, a)$, $0 < a < +\infty$, and θ_{n-1} is a real constant. Then these monic polynomials satisfy a three-term recurrence relation of the form (3.21) with $\alpha_k = 0$ and $\beta_k > 0$. Since $\pi_k(-z) = (-1)^k \pi_k(z)$, $k = 0, 1, \ldots$, the polynomial $p_n(z)$, defined by (3.32), can be expanded in the form

$$p_n(z) = z^n - i\theta_{n-1} z^{n-1} + \cdots ,$$

so that $\sum\limits_{k=1}^{n} z_k = i\theta_{n-1}$, hence

$$\sum_{k=1}^{n} \operatorname{Im} z_k = \theta_{n-1}, \tag{3.33}$$

where $z_1, z_2, \dots, z_n$ are the zeros of the polynomial $p_n(z)$.

By Theorem 3.13 and (3.33) all zeros of the polynomial $p_n(z)$ lie in the half strip

$$\operatorname{Im} z > 0, \quad -a < \operatorname{Re} z < a \quad \text{if} \quad \theta_{n-1} > 0, \tag{3.34}$$

or

$$\operatorname{Im} z < 0, \quad -a < \operatorname{Re} z < a \quad \text{if} \quad \theta_{n-1} < 0, \tag{3.35}$$

strict inequality holding in the imaginary part, since $p_n(z)$ for $\theta_{n-1} \neq 0$ cannot have real zeros. Of course, if $\theta_{n-1} = 0$, all zeros lie in $(-a, a)$.

Let D_a be the disk $D_a = \{z \in \mathbb{C} : |z| < a\}$ and ∂D_a its boundary. Gautschi and Milovanović [21] first proved the following auxiliary result:

Lemma 3.23. *For each $z \in \partial D_a$ one has*

$$\left| \frac{\pi_k(z)}{\pi_{k-1}(z)} \right| \geq \frac{\pi_k(a)}{\pi_{k-1}(a)} \qquad (k = 1, 2, \dots). \tag{3.36}$$

Their main result can be stated in the form:

Theorem 3.24. *If the constant θ_{n-1} satisfies*

$$0 < \theta_{n-1} < \pi_n(a)/\pi_{n-1}(a),$$

then all zeros of the polynomial $p_n(z)$ lie in the upper half disk

$$|z| < a \ \wedge \ \operatorname{Im} z > 0.$$

If $-\pi_n(a)/\pi_{n-1}(a) < \theta_{n-1} < 0$, then all zeros of the polynomial $p_n(z)$ are in the lower half disk

$$|z| < a \ \wedge \ \operatorname{Im} z < 0.$$

Proof. By (3.36) we have

$$\left| \frac{\pi_n(z)}{\pi_{n-1}(z)} \right| \geq \frac{\pi_n(a)}{\pi_{n-1}(a)} \qquad (z \in \partial D_a),$$

hence, if $\pi_n(a)/\pi_{n-1}(a) > |\theta_{n-1}|$,

$$|\pi_n(z)| > |\theta_{n-1}\pi_{n-1}(z)| \quad (z \in \partial D_a).$$

Applying Rouché's theorem to $p_n(z)$, we conclude that all zeros of $p_n(z)$ lie in the open disk D_a. Combining this with (3.34) or (3.35), we obtain the assertions of the theorem. □

Remark. A class of orthogonal polynomials on the semicircle

$$\Gamma = \{z \in \mathbb{C} : z = e^{i\theta},\ 0 \le \theta \le \pi\}$$

with respect to the complex-valued inner product

$$(f, g) = \int_\Gamma (iz)^{-1} f(z)g(z)\,dz = \int_0^\pi f(e^{i\theta})g(e^{i\theta})\,d\theta$$

was introduced and studied by Gautschi and Milovanović [22]–[23]. Such polynomials can be expressed in the form (3.32), where $\pi_k(z)$ should be replaced by the monic Legendre polynomial $\hat{P}_k(z)$. Generalizing previous work, Gautschi, Landau, and Milovanović [20] studied a more general case of complex polynomials orthogonal with respect to the complex-valued inner product

$$(f, g) = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\,d\theta,$$

under suitable assumptions on the *complex "weight function"* $w(z)$. Some further results in this direction and applications of such polynomials were obtained by Gautschi [19], Milovanović [41]–[42], [44], de Bruin [7], Milovanović and Rajković [46]–[47], and Calio', Frontini, and Milovanović [9].

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