Dedekind and Hardy Type Sums and Trigonometric Sums Induced by Quadrature Formulas

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Abstract The Dedekind and Hardy sums and several their generalizations, as well as the trigonometric sums obtained from the quadrature formulas with the highest (algebraic or trigonometric) degree of exactness are studied. Beside some typical trigonometric sums mentioned in the introductory section, the Lambert and Eisenstein series are introduced and some remarks and observations for Eisenstein series are given. Special attention is dedicated to Dedekind and Hardy sums, as well as to Dedekind type Daehee-Changhee (DC) sums and their trigonometric representations and connections with some special functions. Also, the reciprocity law of the previous mentioned sums is studied. Finally, the trigonometric sums obtained from Gauss-Chebyshev quadrature formulas, as well as ones obtained from the so-called trigonometric quadrature rules, are considered.

1 Introduction and Preliminaries

Trigonometric sums play very important role in many various branches of mathematics (number theory, approximation theory, numerical analysis, Fourier analysis, etc.), physics, as well as in other computational and applied sciences. Inequalities with trigonometric sums, in particular their positivity and monotonicity are also important in many subjects (for details see [63, Chap. 4] and [66]).

There are several trigonometric sums in the well-known books [70], [71], [42, pp. 36-40] and [47]. The famous Dedekind and Hardy sums and many generalized

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sums have also trigonometric representations. In this introduction we mention some typical trigonometric sums obtained lately.

In 2000 Cvijović and Klinowski [26] gave closed form of the finite cotangent sums

$$S_n(q;\xi)=\sum_{p=0}^{q-1}\cot^n\frac{(\xi+p)\pi}{q}\qquad\text{and}\qquad S_n^*(q)=\sum_{p=1}^{q-1}\cot^n\frac{p\pi}{q},$$

where n and q are positive integers $(q\geq 2)$ and ξ is a non-integer real number. They obtained $S_n(q;\xi)$ in a determinant form, as well as the following differential recurrence relation

$$S_{n+2}(q;\xi)=-S_n(q;\xi)-\frac{q}{\pi(n+1)}\frac{\mathrm{d}}{\mathrm{d}\xi}S_{n+1}(q;\xi)\quad(n\geq 1),$$

where

$$\begin{aligned}S_1(q;\xi)&=q\cot(\pi\xi),\\ S_2(q;\xi)&=q^2[\cot^2(\pi\xi)+1]-q,\\ S_3(q;\xi)&=q^3\left[\cot^3(\pi\xi)+\cot(\pi\xi)\right]-q\cot(\pi\xi),\end{aligned}$$

etc. Evidently, according to the properties of the cotangent function, $S_{2n+1}^*(q)=0$, as well as

$$\sum_{p=1}^{q-1}\cot^{2n}\frac{p\pi}{2q}=\frac{1}{2}S_{2n}^*(2q)\quad\text{an}\quad\sum_{p=1}^{q}\cot^{2n}\frac{p\pi}{2q+1}=\frac{1}{2}S_{2n}^*(2q+1).$$

For example, $S_2^*(q)=(q^2-3q+2)/3,\ S_4^*(q)=(q^4-20q^2+45q-26)/45,\ S_6^*(q)=(2q^6-42q^4+483q^2-945q+502)/945$, etc. In general, $S_{2n}^*(q)$ is a polynomial of degree $2n$ with rational coefficients [26] (see also [70, p. 646] for $n=1$ and $n=2$).

Using contour integrals and the Cauchy residue theorem, Cvijović and Srivastava [27] derived formulas for general family of secant sums

$$S_{2n}(q,r)=\sum_{\substack{p=0\\ p\neq\frac{q}{2}\ (q\text{ is even})}}^{q-1}\cos\left(\frac{2rp\pi}{q}\right)\sec^{2n}\left(\frac{p\pi}{q}\right)\qquad(r=0,1,\ldots,q-1),$$

when $n\in\mathbb{N}$ and $q\in\mathbb{N}\setminus\{1\}$, as well as for various special cases including ones for $r=0$, i.e.,

$$S_{2n}(q)=\sum_{p=0}^{q-1}\sec^{2n}\left(\frac{p\pi}{q}\right).$$

They also obtained sums which were considered earlier by Chen [17] by using the method of generating functions. In the Appendix of [17], Chen gave tables of power sums of secant, cosecant, tangent and cotangent. Among various such trigonometric summation formulae, we mention only a few of them for tangent function:

$$\sum_{k=1}^{n-1}\tan^2\left(\frac{k\pi}{n}\right) = n(n-1),$$
$$\sum_{k=1}^{n-1}\tan^4\left(\frac{k\pi}{n}\right) = \frac{1}{3}n(n-1)(n^2+n-3),$$
$$\sum_{k=1}^{n-1}\tan^6\left(\frac{k\pi}{n}\right) = \frac{1}{15}n(n-1)(2n^4+2n^3-8n^2-8n+15).$$

Using their earlier method, Cvijović and Srivastava [28] obtained closed-form summation formulas for 12 general families of trigonometric sums of the form

$$\sum_{k=1}^{n-1}(\pm 1)^{k-1} f\left(\frac{2rk\pi}{n}\right) g\left(\frac{k\pi}{n}\right)^m \qquad (n\in\mathbb{N}\setminus\{1\},\ r=1,\dots,n-1),$$

for different combinations of the functions $x\mapsto f(x)$ and $x\mapsto g(x)$ and different values (even and odd) of $m\in\mathbb{N}$. The first function can be $f(x)=\sin x$ or $f(x)=\cos x$, while the second one can be one of the functions $\cot x$, $\tan x$, $\sec x$, and $\csc(x)$. Such a family of cosecant sums. i.e.,

$$C_{2m}(n,r) = \sum_{k=1}^{n-1}\cos\left(\frac{2rk\pi}{n}\right)\csc^{2m}\left(\frac{k\pi}{n}\right),$$

where $m\in\mathbb{N}$, $n\in\mathbb{N}\setminus\{1\}$, and $r=0,1,\dots,n-1$, was previously considered by Dowker [30]. All obtained formulas in [28] involve the higher-order Bernoulli polynomials (see also [23] and [24]).

In [32] da Fonseca, Glasser, and Kowalenko have considered the trigonometric sums of the form

$$C_{2m}(n) = \sum_{k=0}^{n-1}\cos^{2m}\left(\frac{k\pi}{n}\right) \quad \text{and} \quad S_{2m}(n) = \sum_{k=0}^{n-1}\sin^{2m}\left(\frac{k\pi}{n}\right),$$

and their extensions. In [31] da Fonseca and Kowalenko studied the sums of the form

$$\sum_{k=1}^{n}(-1)^k\cos^{2m}\left(\frac{k\pi}{2n+2}\right),$$

where n and m are arbitrary positive integers.

Recently da Fonseca, Glasser, and Kowalenko [33] have presented an elegant integral approach for computing the so-called Gardner-Fisher trigonometric inverse power sum

$$S_{m,2}(n) = \left(\frac{\pi}{2n}\right)^{2m}\sum_{k=1}^{n-1}\sec^{2m}\left(\frac{k\pi}{2n}\right), \quad n,m\in\mathbb{N}.$$

For example,

$$S_{1,2}(n) = \frac{\pi^2}{6}\left(1-\frac{1}{n^2}\right) \quad \text{and} \quad S_{2,2}(n) = \frac{\pi^4}{90}\left(1+\frac{5}{2n^2}-\frac{7}{2n^4}\right).$$

By using contour integrals and residues, similar results for secant and cosecant sums were also obtained by Grabner and Prodinger [41] in terms of Bernoulli numbers and central factorial numbers.

Recently Chu [21] has used the partial fraction decomposition method to get a general reciprocal theorem on trigonometric sums. Several interesting trigonometric reciprocities and summation formulae are derived as consequences.

In this chapter, we mainly give an overview of the Dedekind and Hardy sums and several their generalizations, as well as the trigonometric sums obtained from the quadrature formulas with the highest (algebraic or trigonometric) degree of exactness.

The chapter is organized as follows. In Section 2 we introduce Lambert and Eisenstein series and give some remarks and observations for Eisenstein series. Sections 3 and 4 are dedicated to Dedekind and Hardy sums. In Section 5 we consider the Dedekind type Daehee-Changhee (DC) sums. Their trigonometric representations and connections with some special functions are presented in Sections 6 and 7, respectively. The Section 8 is devoted to the reciprocity law of the previous mentioned sums. Finally, in Sections 9 and 10 we consider trigonometric sums obtained from Gauss-Chebyshev quadrature formulas, as well as ones obtained from the so-called trigonometric quadrature rules.

2 Lambert and Eisenstein Series

Lambert series $G_p(x)$ is defined by

$$G_p(x) = \sum_{m=1}^{\infty} m^{-p} \frac{x^m}{1-x^m} = \sum_{m,n=1}^{\infty} m^{-p} x^{mn},$$

where $p \geq 1$. These functions are regular for $|x| < 1$. The special case $p = 1$ gives

$$G_1(x) = -\log \prod_{m=1}^{\infty} (1-x^m).$$

For odd integer values of p, Apostol [2] gave the behavior of these functions in the neighborhood of singularities, using a technique developed by Rademacher [72] in treating the case $p = 1$.

The following series

$$\sum_{(m,n)\in\mathbb{Z}^2} (m+nz)^{-s},$$

for $\operatorname{Im} z > 0$ and $\operatorname{Re} s > 2$, has an analytic continuation to all values of s. In the paper [56] by Lewittes, it is well known this series has transformation formulae for the analytic continuation of very large class of the Eisenstein series. These transformation formulae are related to large class of functions which generalized the case of the Dedekind eta-function, which is given as follows:

Let $z = x + \mathrm{i}y$ and $s = \sigma + \mathrm{i}t$ with x, y, σ, t be real. For any complex number w, branch of $\log w$ with $-\pi \leq \arg w < \pi$. Let

$$V(z) = \frac{az+b}{cz+d}$$

be an arbitrary modular transformation. Let $\mathbb{H}$ denote the upper half-plane,

$$\mathbb{H} = \{z : \mathrm{Im}\,(z) > 0\}.$$

For $z \in \mathbb{H}$ and $\sigma > 2$, the Eisenstein series, $G(z, s, r_1, r_2)$ is defined by

$$G(z,s,r,h) = \sum_{r \neq (m,n) \in \mathbb{Z}^2} \frac{\mathrm{e}^{2\pi \mathrm{i}(mh_1+nh_2)}}{((m+r_1)z+n+r_2)^s}, \tag{1}$$

where $r_1, r_2 \in \mathbb{R}$.

Substituting $r_1 = r_2 = 0$ into Eq. (1), we have

$$G(z,s) = \sum_{r \neq (m,n) \in \mathbb{Z}^2} \frac{1}{(m+nz)^s} \tag{2}$$

(for details see [55], [56]). Let r_1 and r_2 be arbitrary real numbers. For $z \in \mathbb{H}$ and arbitrary s, generalization of Dedekind's eta-function is given by

$$A(z,s,r_1,r_2) = \sum_{m > -r_1} \sum_{k=1}^{\infty} k^{s-1} \mathrm{e}^{2\pi \mathrm{i} k r_2 + 2\pi \mathrm{i} k (m+r_1) z}.$$

For a real and $\sigma > 1$, Lewittes ([55], [56]) define $\zeta(s,a)$ by

$$\zeta(s,a) = \sum_{n > -a} (n+a)^{-s}.$$

Observe that

$$\zeta(s,a) = \zeta(s, \{a\} + \chi(a)),$$

where $\{a\}$ denotes the fractional part of a, and $\chi(a)$ denotes the characteristic function of integers. Since $0 < \chi(a) + \{a\} \leq 1$, $\zeta(s, \{a\} + \chi(a))$ denotes the classical Hurwitz zeta-function. Lewittes ([56, Eq-(18)]) showed a connection between $G(z, s, r_1, r_2)$ and $A(z, s, r_1, r_2)$ as follows

$$\begin{aligned} G(z,s,r_1,r_2) = {} & \chi(r_1) \left(\zeta(s,r_2) + \mathrm{e}^{\pi \mathrm{i} s} \zeta(s,-r_2) \right) \\ & + \frac{(-2\pi \mathrm{i})^s}{\Gamma(s)} \left(A(z,s,r_1,r_2) + \mathrm{e}^{\pi \mathrm{i} s} A(z,s,-r_1,-r_2) \right). \end{aligned}$$

The above equation was proved by Berndt [10]. He proved transformation formula under modular substitutions which is derived for very large class of generalized Eisenstein series. Berndt's results easily converted into a transformation formula for a large class of functions that includes and generalizes the Dedekind eta-function and the Dedekind sums.

A transformation formula of the function $A(z,s,r_1,r_2)$ is given by Apostol [2] as follows: Let m (>0) is an even integer. Then

$$(cz+d)^m A(V(z),-m) = A(z,-m) + \frac{1}{2}\zeta(m+1)(1-(cz+d)^m) + \frac{(2\pi\mathrm{i})^{m+1}}{2(m+2)!}\sum_{j=1}^{c}\sum_{k=0}^{m+2}\frac{\binom{m+2}{k}B_k\left(\frac{j}{c}\right)\overline{B}_{m+2-k}\left(\frac{\mathrm{i}d}{c}\right)}{(-(cz+d))^{1-k}}. \quad (3)$$

However, due to a miscalculation of residue, the term $\frac{1}{2}\zeta(m+1)(1-(cz+d)^m)$ was omitted. The result was also misstated by Carlitz [16]. The proof of this transformation is also given by Lewittes [56] and after that by Berndt [10]. A special value of the function $A(z,s,r_1,r_2)$ is given by

$$\log\eta(z) = \frac{\pi\mathrm{i}z}{12} - A(z).$$

Hence, the transformation formula for $A(z)$ is given as follows (cf. [5], [**?**], [53], [54]):

Theorem 1. *For $z \in \mathbb{H}$ we have*

$$\eta\left(-\frac{1}{z}\right) = \sqrt{(-\mathrm{i}z)}\,\eta(z)$$

Eisenstein Series

2.1 Further remarks and observations for Eisenstein series

Now we give some standard results about Eisenstein series.

For $2 \le k \in \mathbb{N}$ and $z \in \mathbb{H}$,

$$\sum_{m\in\mathbb{Z}}\frac{1}{(z+m)^k} = \frac{(-2\pi\mathrm{i})^k}{(k-1)!}\sum_{n=1}^{\infty} n^{k-1}\mathrm{e}^{2\pi\mathrm{i}nz}$$

is the known Lipschitz formula.

Apostol-Eisenstein series are given as follows:

If $2 \le k \in \mathbb{N}$ and $z \in \mathbb{H}$, the Eisenstein series $G(z,2k)$ is defined by

$$G(z,2k)=\sum_{0\neq(m,n)\in\mathbb{Z}^2}\frac{1}{(mz+n)^{2k}}. \tag{4}$$

It converges absolutely and has the Fourier expansion

$$G(z,2k)=2\zeta(2k)+\frac{2(2\pi\mathrm{i})^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)\,\mathrm{e}^{2\pi\mathrm{i}nz}$$

where, as usual, $\sigma_c(n)=\sum_{d|n}d^c$ and $\zeta(z)$ denotes Riemann zeta function.

For $k=1$, the series in (4); $G(z,2)$ is no longer absolutely convergent. $G(z,2)$ is an even function,

$$G(z,2)=2\zeta(2)+2(2\pi\mathrm{i})^2\sum_{n=1}^{\infty}\sigma(n)\,\mathrm{e}^{2\pi\mathrm{i}nz} \tag{5}$$

for $z\in\mathbb{H}$.

For $x=\mathrm{e}^{2\pi\mathrm{i}z}$ the series in (5) is an absolutely convergent power series for $|x|<1$ so that $G(z,2)$ is analytic in $\mathbb{H}$. The behavior of $G(z,2)$ under the modular group is given by (cf. [5])

$$G\left(-\frac{1}{z},2\right)=z^2G(z,2)-2\pi\mathrm{i}z.$$

The well-known Lipschitz formula is given by the following lemma:

Lemma 1 (Lipschitz formula). *Let $2\leq k\in\mathbb{N}$ and $z\in\mathbb{H}$. Then*

$$\sum_{m\in\mathbb{Z}}\frac{1}{(z+m)^k}=\frac{(-2\pi\mathrm{i})^k}{(k-1)!}\sum_{n=1}^{\infty}n^{k-1}\mathrm{e}^{2\pi\mathrm{i}nz}.$$

By using this lemma, the Fourier expansion of the Eisenstein series is given by:

Theorem 2. *If k is an integer with $k\geq 2$ and $z\in\mathbb{H}$, then*

$$G(z,k)=2\zeta(k)+\frac{2(-2\pi\mathrm{i})^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}n^{k-1}\mathrm{e}^{2\pi\mathrm{i}nmz}.$$

Proof. We give only brief sketch of the proof since the method is well-known. Now replacing z by az, where $a>0$, substituting in Lemma 1 and summing over all $a\geq 1$, we get

$$\sum_{a=1}^{\infty}\sum_{m\in\mathbb{Z}}\frac{1}{(az+m)^k}=\frac{(-2\pi\mathrm{i})^k}{(k-1)!}\sum_{a,n=1}^{\infty}n^{k-1}\mathrm{e}^{2\pi\mathrm{i}naz}.$$

We rearrange the terms right member of the above equation, we have

$$\frac{1}{2}\sum_{0\neq a\in\mathbb{Z}}^{\infty}\sum_{m\in\mathbb{Z}}\frac{1}{(az+m)^k}-2\zeta(k)=\frac{(-2\pi\mathrm{i})^k}{(k-1)!}\sum_{a,n=1}^{\infty}n^{k-1}\mathrm{e}^{2\pi\mathrm{i}naz}.$$

After a further little rearrange and use of (2) we obtain the desired result.

Remark 1. Putting $r_1 = r_2 = 0$, we have

$$G(z,s,0,0) = \chi(0)\left(\zeta(s,0) + e^{\pi i s}\zeta(s,0)\right) + \frac{(-2\pi i)^s}{\Gamma(s)}\left(A(z,s,0,0) + e^{\pi i s}A(z,s,0,0)\right).$$

Replacing s by k (k is an integer with $k > 1$) in the above, we have

$$G(z,k) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}A(z,k).$$

After a number of straightforward calculations we arrive at the desired result.

The Fourier expansion of the function $G(z,k,r,h)$ is given by:

Corollary 1. *Let $2 \leq k \in \mathbb{N}$, r,h be rational numbers and $z \in \mathbb{H}$. Then*

$$G(z,k,r,h) = 2\zeta(k,h) + \frac{2(-2\pi i)^k}{(k-1)!}\sum_{a,n=1}^{\infty} a^{k-1}e^{2\pi i(n+r)az}.$$

3 Dedekind Sums

The history of the Dedekind sums can be traced back to famous German mathematician Julius Wilhelm Richard Dedekind (1831–1916), who did important work in abstract algebra in particularly including ring theory, algebraic and analytic number theory and the foundations of the real numbers. After Dedekind, Hans Adolph Rademacher (1892–1969), who was one of the most famous German mathematicians, worked the most deeply the Dedekind sums. Rademacher also studied important work in mathematical analysis and its applications and analytic number theory. It is well-known that, the Dedekind sums, named after Dedekind, are certain finite sums of products of a sawtooth function. The Dedekind sums are found in the functional equation that emerges from the action of the Dedekind eta function under modular groups. The Dedekind sums have occurred in analytic number theory, in some problems of topology and also in the other branches of Mathematics. Although two-dimensional Dedekind sums have been around since the 19th century and higher-dimensional Dedekind sums have been explored since the 1950s, it is only recently that such sums have figured flashily in so many different areas. The Dedekind sums have also many applications in some areas such as analytic number theory, modular forms, random numbers, the Riemann-Roch theorem, the Atiyah-Singer index theorem, and the family of zeta functions.

In many applications of elliptic modular functions to analytic number theory, and theory of elliptic curves, the Dedekind eta function plays a central role. It was introduced by Dedekind in 1877 by Dedekind. This function is defined on the upper half-plane as follows:

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{m=1}^{\infty} \left(1 - e^{2\pi i m\tau}\right),$$

The infinite product has the form $\prod_{n=1}^{\infty}(1-x^n)$, where $x = e^{2\pi i\tau}$. If $\tau \in \mathbb{H}$, then $|x| < 1$ so the product converges absolutely and it is nonzero. Furthermore, since the convergence is uniform on compact subsets of $\mathbb{H}$, $\eta(\tau)$ is analytic on $\mathbb{H}$. The function $\eta(\tau)$ is related to analysis, number theory, combinatorics, q-series, Weierstrass elliptic functions, modular forms, Kronecker limit formula, etc.

The behavior of this function under the modular group $\Gamma(1)$, defined by

$$\Gamma(1) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ ad - bc = 1, \ a,b,c,d \in \mathbb{Z} \right\},$$

we note that

$$Az = \frac{az+b}{cz+d}.$$

It is well-known that the Dedekind sums $s(h,k)$ first arose in the transformation formula of the logarithm of the Dedekind-eta function which is given by Apostol,

$$\log\eta(Az) = \log\eta(z) + \frac{\pi i(a+d)}{12c} - \pi i\left(s(d,c) - \frac{1}{4}\right) + \frac{1}{2}\log(cz+d),$$

where $z \in \mathbb{H}$ and $s(d,c)$ denotes the Dedekind sum which defined by

$$s(d,c) = \sum_{\mu \bmod c} \left(\left(\frac{\mu}{c}\right)\right)\left(\left(\frac{d\mu}{c}\right)\right),$$

where $(d,c) = 1$, $c > 0$, and

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & x \notin \mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}$$

where $[x]$ is the largest integer $\leq x$. The arithmetical function $((x))$ has a period 1 and can thus be expressed by a Fourier series as follows:

$$((x)) = -\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n}.$$

For basic properties of the Dedekind sums see monograph of Rademacher and Grosswald [76].

The most important property of Dedekind sums is the reciprocity law. Namely, if $(h,k) = 1$ and h and k are positive, then

$$s(h,k) + s(k,h) = \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} + \frac{1}{hk}\right) - \frac{1}{4}$$

(cf. [2], [100]). This will be discussed in more detail in a separate section.

Apostol [2] defined the generalized Dedekind sums $s_p(h,k)$ as

$$s_p(h,k) = \sum_{n=0}^{k-1} \frac{n}{k} \overline{B}_p\left(\frac{nh}{k}\right),$$

where $\overline{B}_p(x)$ is the p-th Bernoulli function defined by

$$\overline{B}_n(x) = B_n(x - [x]),$$

where $B_n(x)$ denotes the Bernoulli polynomial. These important polynomials are defined by the following generating function

$$\frac{t}{\mathrm{e}^t - 1}\mathrm{e}^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

For $x = 0$ these polynomials reduce to the well-known Bernoulli numbers $B_n = B_n(0)$ (cf. [72]). A few first numbers are $1, -1/2, 1/6, 0, -1/30, 0, 1/42, \ldots$.

The functions $\overline{B}_n(x)$ are 1-periodic, and they satisfy

$$\overline{B}_n(x) = B_n(x)$$

for $0 \le x < 1$, and

$$\overline{B}_n(x+1) = \overline{B}_n(x)$$

for other real x. The Bernoulli function can be expressed by the following Fourier expansion

$$\overline{B}_n(x) = -\frac{n!}{(2\pi\mathrm{i})^n} \sum_{0 \neq m \in \mathbb{Z}} \frac{1}{m^p} \mathrm{e}^{2\pi\mathrm{i}mx} \tag{6}$$

Observe that $s_1(h,k) = s(h,k)$. A representation of $s_p(h,k)$ as an infinite series has also given by Apostol [2]. Namely, for odd $p \ge 1$, $(h,k) = 1$ as

$$s_p(h,k) = \frac{p!}{(2\pi\mathrm{i})^p} \sum_{\substack{m \in \mathbb{N} \\ m \not\equiv 0 \pmod{k}}} \frac{1}{m^p} \left(\frac{\mathrm{e}^{2\pi\mathrm{i}mh/k}}{1 - \mathrm{e}^{2\pi\mathrm{i}mh/k}} - \frac{\mathrm{e}^{2\pi\mathrm{i}mh/k}}{1 - \mathrm{e}^{2\pi\mathrm{i}mh/k}} \right). \tag{7}$$

The relation between Dedekind sums $s(h,k)$ and $\cot \pi x$ are given in the lemma below. This lemma is a special case of (7). The following well-known result is easily given:

$$s(h,k) = \frac{1}{4k} \sum_{m=1}^{k-1} \cot\left(\frac{mh\pi}{k}\right) \cot\left(\frac{m\pi}{k}\right).$$

Recently, many authors proved the above nice formulas by different methods ([4], [14], [11], [12], [29], [16], [90], [101]).

Using contour integration and Cauchy Residue Theorem, Berndt [11] proved the following result:

Lemma 2. *Let $h,k \in \mathbb{N}$ with $(h,k) = 1$. Then*

$$s(h,k) = \frac{1}{2\pi} \sum_{\substack{m \in \mathbb{N} \\ m \not\equiv 0 \,(\mathrm{mod}\, k)}} \frac{1}{m} \cot\left(\frac{mh\pi}{k}\right).$$

The sums $s_p(h,k)$ are related to the Lambert series $G_p(x)$ in the same way that $s(h,k)$ is related to $\eta(z)$, $\log \eta(z)$ being the same as $(\pi \mathrm{i} z/12) - G_1\left(e^{2\pi \mathrm{i} z}\right)$, respectively. The sums $s_p(h,k)$ are expressible as infinite series related to certain Lambert series and, for odd $p \geq 1$, $s_p(h,k)$ is also seen to be the Abel sum of a divergent series. This relation is given as follows:

Theorem 3 ([85]). *For $(h,k) = 1$, the Abel sum of the divergent series*

$$\sum_{n=1}^{\infty} \sigma_p(n) n^{-p} \sin\left(\frac{2\pi nh}{k}\right)$$

for odd p is given by

$$(-1 \mid p)(2\pi)^p (2p!)^{-1} s_p(h,k),$$

where $\sigma_p(n) = \sum_{d|n} d^p$.

Apostol [2] gave a proof of this theorem using a contour integral representation of the Lambert series $G_p(x)$, but his proof is very different from that given below (see [85]).

Brief sketch of the proof of Theorem 3: Starting with (6), replacing x by nx $(n \in \mathbb{N})$ and summing over n we get (cf. [85])

$$\sum_{n=1}^{\infty} \overline{B}_p(nx) = -\frac{p!}{(2\pi \mathrm{i})^p} \sum_{0 \neq a \in \mathbb{Z}} \sum_{n=1}^{\infty} \frac{1}{m^p} \mathrm{e}^{2\pi \mathrm{i} mnx}.$$

If we rearrange the above equation, we get

$$\sum_{n=1}^{\infty} \overline{B}_p(nx) = -\frac{p!}{(2\pi \mathrm{i})^p} \left(\sum_{n=1}^{\infty} \sum_{m=-\infty}^{-1} \frac{1}{m^p} \mathrm{e}^{2\pi \mathrm{i} mnx} + \sum_{m,n=1}^{\infty} \frac{1}{m^p} \mathrm{e}^{2\pi \mathrm{i} mnx} \right).$$

After a little calculation, we easily obtain

$$\sum_{n=1}^{\infty} \overline{B}_p(nx) = -\frac{p!}{(2\pi \mathrm{i})^p} \left(\sum_{m,n=1}^{\infty} \frac{1}{m^p} \mathrm{e}^{2\pi \mathrm{i} mnx} - \sum_{m,n=1}^{\infty} \frac{1}{m^p} \mathrm{e}^{-2\pi \mathrm{i} mnx} \right). \tag{8}$$

Because of the identity $2\mathrm{i} \sin z = \mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}$ and putting $x = a/b$ in (8), where $a,b \in \mathbb{Z}$ with $(a,b) = 1$, and writing the Lambert series as a power series $G_p(x) = \sum_{n=1}^{\infty} \sigma_p(n) n^{-p} x^n$, we get (cf. [85])

$$\sum_{n=1}^{\infty} \overline{B}_p\left(\frac{na}{b}\right) = -\frac{p!}{(2\pi \mathrm{i})^p} \sum_{m=1}^{\infty} m^{-p} \sigma_p(m) \sin\left(\frac{2\pi ma}{b}\right). \tag{9}$$

Using a definition of the Lambert series $G_p(x)=\sum_{n=1}^{\infty} n^{-p}x^n/(1-x^n)$ and replacing x by a/b, with $(a,b)=1$, in (8), we get

$$\sum_{n=1}^{\infty} \overline{B}_p\left(\frac{na}{b}\right) = -\frac{p!}{(2\pi \mathrm{i})^p} \sum_{\substack{m\,\in\,\mathbb{N}\\ m\not\equiv 0\,(\mathrm{mod}\,k)}} \frac{1}{m^p}\left(\frac{\mathrm{e}^{2\pi \mathrm{i}mh/k}}{1-\mathrm{e}^{2\pi \mathrm{i}mh/k}} - \frac{\mathrm{e}^{2\pi \mathrm{i}mh/k}}{1-\mathrm{e}^{2\pi \mathrm{i}mh/k}}\right).$$

By substituting (7) into the above, we obtain

$$\sum_{n=1}^{\infty} \overline{B}_p\left(\frac{na}{b}\right) = -s_p(a,b). \tag{10}$$

For odd p we get

$$(\mathrm{i})^{1-p} = (-1)^{(1-p)/2} = (-1 \mid p), \tag{11}$$

which is known as Jacobi (Legendre) symbol. Finally, combining (9)–(11), we find the desired result.

Zagier [101] defined the following multiple Dedekind sums

$$d(p;a_1,a_2,\ldots,a_j) = (-1)^{j/2}\sum_{m=1}^{p-1} \cot\left(\frac{\pi m a_1}{p}\right)\cot\left(\frac{\pi m a_2}{p}\right)\cdots\cot\left(\frac{\pi m a_j}{p}\right).$$

The sum $d(p;a_1,a_2,\ldots,a_j)$ vanishes identically when j is odd. In [90], Simsek, Kim and Koo gave various formulas for the above sums and finite trigonometric sums.

3.1 Some others formulas for the Dedekind sums

Theorem 4 ([85]). *Let $a,b\in\mathbb{Z}$ with $(a,b)=1$ and let p be odd integers. Then*

$$s_p(a,b) = \frac{2p!}{(2\pi b)^p}(-1\mid p)\sum_{n=1}^{\infty}\sum_{m=1}^{b-1}\sin\left(\frac{2\pi mna}{b}\right)\zeta\left(p,\frac{m}{b}\right),$$

where $\zeta(p,m/b)$ is the Hurwitz zeta function.

Proof. By substituting $x=a/b$ into (8), we easily calculate

$$\sum_{n=1}^{\infty} \overline{B}_p\left(\frac{na}{b}\right) = \frac{2p!}{(2\pi b)^p}\sum_{m,n=1}^{\infty}\frac{1}{m^p}\sin\left(\frac{2\pi nma}{b}\right).$$

Writing $m=ub+c$, with $u=0,1,3,\ldots$ and $c=1,2,\ldots,b-1$ in the above, we obtain

$$\sum_{n=1}^{\infty} \overline{B}_p\left(\frac{na}{b}\right) = -\frac{2p!}{(2\pi b)^p}(-1\mid p)\sum_{n=1}^{\infty}\sum_{c=1}^{b-1}\zeta\left(p,\frac{c}{b}\right)\sin\left(\frac{2\pi na}{b}\right), \tag{12}$$

where we assume $p>1$ in order to insure that the series involved should be absolutely convergent and the rearragements valid. Now, after combining Eqs. (10) and (12), the proof is completed.

Lemma 2, as well as corresponding expression for $s_p(h,k)$ in (7), can be obtained without any knowledge of the function η and the finite sum $\sum_{n=1}^{k-1} nx^n$. By using the well-known equality

$$\cot \pi x = -\mathrm{i}\left(\frac{\mathrm{e}^{2\mathrm{i}\pi x}}{1-\mathrm{e}^{2\mathrm{i}\pi x}} - \frac{\mathrm{e}^{-2\mathrm{i}\pi x}}{1-\mathrm{e}^{-2\mathrm{i}\pi x}}\right),$$

Theorems 3 and 4, a relation between $s_p(h,k)$ and $\cot(an\pi/b)$ can be obtained as follows:

Theorem 5 ([85]). *Let $(h,k)=1$. For odd $p \geq 1$ we have*

$$s_p(h,k) = \mathrm{i}\frac{p!}{(2\pi \mathrm{i})^p} \sum_{\substack{n \in \mathbb{N} \\ n \not\equiv 0 \,(\mathrm{mod}\, k)}} \frac{1}{n^p} \cot\left(\frac{\pi nh}{k}\right).$$

Proof. By substituting (6) in to definition of $s_p(h,k)$, we have

$$\begin{aligned}
s_p(h,k) &= \sum_{n=1}^{k-1} \frac{n}{k} \overline{B}_p\left(\frac{nh}{k}\right) \\
&= -\frac{p!}{k(2\pi \mathrm{i})^p} \sum_{n=1}^{k-1} n \sum_{0 \neq m \in \mathbb{Z}} \frac{1}{m^p} \mathrm{e}^{\frac{2\pi \mathrm{i} mnh}{k}} \\
&= -\frac{p!}{k(2\pi \mathrm{i})^p} \sum_{n=1}^{k-1} n \left(\sum_{m=1}^{\infty} \frac{1}{m^p} \mathrm{e}^{\frac{2\pi \mathrm{i} mnh}{k}} + \sum_{-\infty}^{-1} \frac{1}{m^p} \mathrm{e}^{\frac{2\pi \mathrm{i} mnh}{k}}\right) \\
&= -\frac{p!}{k(2\pi \mathrm{i})^p} \sum_{n=1}^{k-1} n \left(\sum_{m=1}^{\infty} \frac{1}{m^p} \left(\mathrm{e}^{\frac{2\pi \mathrm{i} mnh}{k}} - \mathrm{e}^{-\frac{2\pi \mathrm{i} mnh}{k}}\right)\right).
\end{aligned}$$

By applying the well-known identity $2\mathrm{i}\sin x = \mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}$ in the above, we obtain

$$s_p(h,k) = -2\mathrm{i}\frac{p!}{k(2\pi \mathrm{i})^p} \sum_{n=1}^{k-1} n \sum_{m=1}^{\infty} \frac{1}{m^p} \sin\left(\frac{2\pi mnh}{k}\right).$$

Now, by using the following well-known identity

$$\sum_{a \bmod k} a \sin\left(\frac{2\pi n\phi}{k}\right) = -\frac{k}{2}\cot\left(\frac{\pi\phi}{k}\right),$$

where $k \nmid \phi$, $\phi \in \mathbb{Z}$, then we have the desired result.

By using Theorem 5, we arrive at the following result:

Corollary 2 ([85]). *For odd $p > 1$ we have*

$$s_p(a,b) = \mathrm{i}\frac{p!}{(2\pi \mathrm{i})^p}\sum_{n=1}^{b-1}\cot\left(\frac{\pi na}{b}\right)\zeta\left(p,\frac{n}{b}\right). \tag{13}$$

The proof this corollary was proved by Apostol [4].

Using a technique developed by Rademacher (Theorem 2, Eq. (5) in [10]) and Lewittes (Eq. (56) in [4]), we can give the behavior of Lambert series and Dedekind eta-function. Namely, substituting $r_1 = r_2 = s = 0$ into Eq. (3) (and also Eq. (5) in [10]), we have (cf. [85])

$$A(V(z)) = A(z) + \frac{\pi \mathrm{i}}{4} - \frac{1}{2}\log(cz+d) + \pi \mathrm{i} s(d,c) - \pi \mathrm{i}\left(\frac{a+d}{12c}\right).$$

By the definition of $G_1\left(\mathrm{e}^{\pi \mathrm{i} z}\right)$ and $A(z)$, we get (cf. [85])

$$G_1\left(\mathrm{e}^{\pi \mathrm{i} V(z)}\right) = G_1\left(\mathrm{e}^{\pi \mathrm{i} z}\right) + \frac{\pi \mathrm{i}}{4} - \frac{1}{2}\log(cz+d) + \pi \mathrm{i} s(d,c) - \pi \mathrm{i}\left(\frac{a+d}{12c}\right). \tag{14}$$

This relation gives modular transformation of $G_1\left(\mathrm{e}^{\pi \mathrm{i} z}\right)$. Replacing $V(z) = \frac{az+b}{cz+d}$, by $W(z) = \frac{bz-a}{dz-c}$, where $c, d > 0$ in (14), we have

$$G_1\left(\mathrm{e}^{\pi \mathrm{i} W(z)}\right) = G_1\left(\mathrm{e}^{\pi \mathrm{i} z}\right) + \frac{\pi \mathrm{i}}{4} - \frac{1}{2}\log(dz-c) + \pi \mathrm{i} s(-c,d) - \pi \mathrm{i}\left(\frac{b-c}{12d}\right). \tag{15}$$

Comparing (14) with (15) and using reciprocity law of Dedekind sums

$$12s(d,c) + 12s(c,d) = -3 + \frac{d}{c} + \frac{c}{d} + \frac{1}{dc}, \quad (d,c) = 1,$$

we deduce that

$$G_1\left(\mathrm{e}^{\pi \mathrm{i} V(z)}\right) - G_1\left(\mathrm{e}^{\pi \mathrm{i} W(z)}\right) = \frac{1}{2}\log\left(\frac{dz-c}{cz+d}\right) + \frac{\pi \mathrm{i}(cb-da-3dc+1)}{12dc}.$$

Thus, we arrive at the following results (see [85]).

Theorem 6. *Let $V(z) = \frac{az+b}{cz+d}$ and $W(z) = \frac{bz-a}{dz-c}$ be arbitrary modular transformations, with $c > 0$, and let*

$$\mathbb{K} = \left\{z : \mathrm{Re}\,(z) > -\frac{d}{c}, \ \ \mathrm{Im}\,(z) > 0\right\}.$$

Then, for $z \in \mathbb{K}$ we have

$$\sum_{m,n=1}\frac{1}{m}\left(\mathrm{e}^{2\pi \mathrm{i} nmV(z)} - \mathrm{e}^{2\pi \mathrm{i} nmW(z)}\right) = \frac{1}{2}\log\left(\frac{dz-c}{cz+d}\right) + \frac{\pi \mathrm{i}(cb-da-3dc+1)}{12dc}.$$

Theorem 7. *Let $m > 0$ be an even integer and $(a,b) = 1$. We have*

$$s_{m+1}(a,b) = \frac{2(m+1)!}{\pi^2 (4\pi \mathrm{i})^m} \sum_{\substack{n \in \mathbb{N} \\ n \not\equiv 0 \pmod b}} \frac{\lim_{s\to 1}\left(\zeta\left(s, \frac{na}{b}\right) - \zeta\left(s, 1-\frac{na}{b}\right)\right)}{n^{1+m}}.$$

4 Hardy Sums

Hardy's sums are derived from theta function. Thus, the well known theta-functions, $\vartheta_n(0,q)(n = 2,3,4)$ related to infinite products are given by

$$\vartheta_2(0,q) = 2q^{1/2}\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n})^2,$$

$$\vartheta_3(0,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})^2,$$

$$\vartheta_4(0,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2.$$

In the sequel we denote $\vartheta_2(0,q)$, $\vartheta_3(0,q)$ and $\vartheta_4(0,q)$ as $\vartheta_2(z)$, $\vartheta_3(z)$ and $\vartheta_4(z)$, respectively, where $q = \mathrm{e}^{\pi \mathrm{i} z}$. The relations between theta-functions and Dedekind eta-function are defined by

$$\vartheta_3(z) = \frac{\eta^5(z)}{\eta^2(2z)\eta^2(z/2)}.$$

These relations, as well as others, are studied by Rademacher [72] (see also [84], as well as the books [99] and [82]).

Let h and k with $k > 0$ be relatively prime integers (i.e., $(h,k) = 1$). The Hardy sums are defined by (see [48])

$$S(h,k) = \sum_{j=1}^{k-1}(-1)^{j+1+[hj/k]},$$

$$s_1(h,k) = \sum_{j=1}^{k}(-1)^{[hj/k]}\left(\left(\frac{j}{k}\right)\right),$$

$$s_2(h,k) = \sum_{j=1}^{k}(-1)^{j}\left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{hj}{k}\right)\right),$$

$$s_3(h,k) = \sum_{j=1}^{k}(-1)^{j}\left(\left(\frac{hj}{k}\right)\right),$$

$$s_4(h,k) = \sum_{j=1}^{k-1}(-1)^{[hj/k]},$$
$$s_5(h,k) = \sum_{j=1}^{k}(-1)^{j+[hj/k]}\left(\left(\frac{j}{k}\right)\right).$$

By using the following well-known trigonometric formulas, we mention some relations including the Hardy sums (for details see [40]).

If $2m-1 \not\equiv 0\,(\mathrm{mod}\,k)$, $(h,k)=1$, then

$$\sum_{j=1}^{k-1} j\sin\left(\frac{\pi(2m-1)hj}{k}\right) = -\frac{k}{2}\cot\left(\frac{\pi h(2m-1)}{2k}\right).$$

If $m=(2n-1)h$, $2n-1 \not\equiv 0\,(\mathrm{mod}\ k)$, and h and k are of opposite parity, an elementary calculation gives

$$\sum_{j=1}^{k-1}(-1)^j\sin\left(\frac{\pi mj}{k}\right) = -\tan\left(\frac{\pi m}{2k}\right).$$

If $2m \not\equiv 0\,(\mathrm{mod}\ k)$, and h and k are of opposite parity, an elementary calculation gives

$$\sum_{j=1}^{k-1}(-1)^j j\sin\left(\frac{\pi mj}{k}\right) = \frac{k}{2}\tan\left(\frac{\pi m}{2k}\right).$$

If k is odd, then

$$\sum_{j=1}^{k-1}(-1)^j\sin\left(\frac{2h\pi mj}{k}\right) = -\tan\left(\frac{\pi hm}{k}\right);$$

if k is even, then

$$\sum_{j=1}^{k-1}\sin\left(\frac{2h\pi mj}{k}\right) = 0.$$

Also (cf. [15])

$$\sum_{j=1}^{k-1}\cot^2\left(\frac{\pi j}{k}\right) = \frac{(k-1)(k-3)}{3}.$$

The Fourier series of the function $f(x)=(-1)^{[x]}$ is given by (cf. [40])

$$f(x) = \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)x\pi)}{2n-1}.$$

Combining the above finite trigonometric sums and Fourier series of the function $f(x)=(-1)^{[x]}$ with definitions of the Hardy sums, some relations between Hardy sums and trigonometric functions can be given. Such results were obtained by Goldberg [40] and Berndt and Goldberg [14]:

Theorem 8. *Let h and k denote relatively prime integers with* $k > 0$*.*

1° *If* $h+k$ *is odd, then*

$$S(h,k) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\tan\Big(\frac{\pi h(2n-1)}{2k}\Big); \tag{16}$$

2° *If h is even and k is odd, then*

$$s_1(h,k) = -\frac{2}{\pi}\sum_{\substack{n=1\\ 2n-1\not\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{1}{2n-1}\cot\Big(\frac{\pi h(2n-1)}{2k}\Big); \tag{17}$$

3° *If h is odd and k is even, then*

$$s_2(h,k) = -\frac{1}{2\pi}\sum_{\substack{n=1\\ 2n\not\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{1}{n}\tan\Big(\frac{\pi hn}{k}\Big); \tag{18}$$

4° *If k is odd, then*

$$s_3(h,k) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\tan\big(\frac{\pi hn}{k}\big); \tag{19}$$

5° *If h is odd, then*

$$s_4(h,k) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\cot\Big(\frac{\pi h(2n-1)}{2k}\Big); \tag{20}$$

6° *If h and k are odd, then*

$$s_5(h,k) = \frac{2}{\pi}\sum_{\substack{n=1\\ 2n-1\not\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{1}{2n-1}\tan\Big(\frac{\pi h(2n-1)}{2k}\Big). \tag{21}$$

Using the well-known sum

$$\sum_{n=-\infty}^{\infty}\frac{1}{n+y} = \pi\cot\pi y$$

in (16) through (21), the relations between Hardy sums and finite trigonometric sums can be also obtained [40], [14]:

Theorem 9. *Let h and k be coprime integers with* $k > 0$*.*

1° *If* $h+k$ *is odd, then*

$$S(h,k)=\frac{1}{k}\sum_{m=1}^{k}\tan\left(\frac{\pi h(2m-1)}{2k}\right)\cot\left(\frac{\pi(2m-1)}{2k}\right);$$

2° *If h is even and k is odd, then*

$$s_1(h,k)=-\frac{1}{2k}\sum_{\substack{m=1\\ m\neq(k+1)/2}}^{k}\cot\left(\frac{\pi h(2m-1)}{2k}\right)\cot\left(\frac{\pi(2m-1)}{2k}\right);$$

3° *If h is odd and k is even, then*

$$s_2(h,k)=-\frac{1}{4k}\sum_{\substack{m=1\\ m\neq k/2}}^{k-1}\tan\left(\frac{\pi hm}{k}\right)\cot\left(\frac{\pi m}{k}\right);$$

4° *If k is odd, then*

$$s_3(h,k)=\frac{1}{2k}\sum_{m=1}^{k-1}\tan\left(\frac{\pi hm}{k}\right)\cot\left(\frac{\pi m}{k}\right);$$

5° *If h is odd, then*

$$s_4(h,k)=\frac{1}{k}\sum_{m=1}^{k}\cot\left(\frac{\pi h(2m-1)}{2k}\right)\cot\left(\frac{\pi(2m-1)}{2k}\right);$$

6° *If h and k are odd, then*

$$s_5(h,k)=\frac{1}{2k}\sum_{\substack{m=1\\ m\neq(k+1)/2}}^{k}\tan\left(\frac{\pi h(2m-1)}{2k}\right)\cot\left(\frac{\pi(2m-1)}{2k}\right).$$

Using elementary methods, the previous identities were also obtained by Sitaramachandrarao [91]. Some new higher dimensional generalizations of the Dedekind sums associated with the Bernoulli functions, as well as ones of Hardy sums, have been recently introduced by Rassias and Tóth [77]. They derived the so-called Zagier-type identities for these higher dimensional sums, as well as a sequence of corollaries with interesting particular sums.

5 Dedekind Type Daehee-Changhee (DC) Sums

The first kind n-th Euler function $\overline{E}_m(x)$ is defined by

$$\overline{E}_n(x)=E_n(x)$$

for $0 \leq x < 1$, and by

$$\overline{E}_n(x+1) = -\overline{E}_n(x)$$

for other real x. This function can be expressed by the following Fourier expansion

$$\overline{E}_m(x) = \frac{2m!}{(\pi \mathrm{i})^{m+1}} \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{(2n-1)\pi \mathrm{i} x}}{(2n-1)^{m+1}}, \tag{22}$$

where $m \in \mathbb{N}$ (for details on Euler polynomials and functions and their Fourier series see [1], [16], [49], [50], [51], [94], [89], [96]). Hoffman [49] studied the Fourier series of Euler polynomials and expressed the values of Euler polynomials at any rational argument in terms of $\tan x$ and $\sec x$. Suslov [96] considered explicit expansions of some elementary and q-functions in basic Fourier series of the q-extensions of the Bernoulli and Euler polynomials and numbers.

Observe that if $0 \leq x < 1$, then (22) reduces to the first kind n-th Euler polynomials $E_n(x)$ which are defined by means of the following generating function

$$\frac{2\mathrm{e}^{tx}}{\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi. \tag{23}$$

Observe that $E_n(0) = E_n$ denotes the first kind Euler number which is given by the following recurrence formula

$$E_0 = 1 \quad \text{and} \quad E_n = -\sum_{k=0}^{n} \binom{n}{k} E_k. \tag{24}$$

Some of them are given by $1, -1/2, 0, 1/4, \ldots, E_n = 2^n E_n(1/2)$ and $E_{2n} = 0$ ($n \in \mathbb{N}$).

In [50] and [51], by using Fourier transform for the Euler function, Kim derived some formulae related to infinite series and the first kind Euler numbers. From (22) it is easy to see that (cf. [1], [51], [49], [89])

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2m+2}} = \frac{(-1)^{m+1}\pi^{2m+2} E_{2m+1}}{4(2m+1)!}. \tag{25}$$

By using the first kind n-th Euler function and above infinite series, we can construct infinite series representation of the Dedekind type Daehee-Changhee-sum (DC-sum) and reciprocity law of this sum. We also can give relations between the Dedekind type DC-sum and some special functions.

The second kind Euler numbers, E_m^* are defined by means of the following generating functions (cf. [75], [1], [89])

$$\sec hx = \frac{1}{\cosh x} = \frac{2\mathrm{e}^x}{\mathrm{e}^{2x} + 1} = \sum_{n=0}^{\infty} E_n^* \frac{x^n}{n!}, \quad |x| < \frac{\pi}{2}. \tag{26}$$

Kim [51] studied the second kind Euler numbers and polynomials in details. By (23) and (26), it is easy to see that

$$E_m^* = \sum_{n=0}^{m} \binom{m}{n} 2^n E_n \quad \text{and} \quad E_{2m}^* = -\sum_{n=0}^{m-1} \binom{2m}{2n} E_{2n}^*.$$

From the above $E_0^* = 1$, $E_1^* = 0$, $E_2^* = -1$, $E_3^* = 0$, $E_4^* = 5$, $\ldots$, and $E_{2m+1}^* = 0$ $(m \in \mathbb{N})$.

The first and the second kind Euler numbers are also related to $\tan z$ and $\sec z$,

$$\tan z = -\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}} = \frac{\mathrm{e}^{2\mathrm{i}z}}{2\mathrm{i}} \left(\frac{2}{\mathrm{e}^{2\mathrm{i}z} + 1} \right) - \frac{\mathrm{e}^{-2\mathrm{i}z}}{2\mathrm{i}} \left(\frac{2}{\mathrm{e}^{-2\mathrm{i}z} + 1} \right).$$

By using (23) and Cauchy product, we have (cf. [89])

$$\begin{aligned}
\tan z &= \frac{1}{2\mathrm{i}} \sum_{n=0}^{\infty} E_n \frac{(2\mathrm{i}z)^n}{n!} \sum_{n=0}^{\infty} \frac{(2\mathrm{i}z)^n}{n!} - \frac{1}{2i} \sum_{n=0}^{\infty} E_n \frac{(-2\mathrm{i}z)^n}{n!} \sum_{n=0}^{\infty} \frac{(-2\mathrm{i}z)^n}{n!} \\
&= \frac{1}{2\mathrm{i}} \sum_{n=0}^{\infty} \sum_{k=0}^{n} E_k \frac{(2\mathrm{i}z)^k}{k!} \frac{(2\mathrm{i}z)^{n-k}}{(n-k)!} - \frac{1}{2\mathrm{i}} \sum_{n=0}^{\infty} \sum_{k=0}^{n} E_k \frac{(-2\mathrm{i}z)^k}{k!} \frac{(-2\mathrm{i}z)^{n-k}}{(n-k)!} \\
&= \frac{1}{2\mathrm{i}} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{E_k}{k!(n-k)!} (2\mathrm{i})^n z^n - \frac{1}{2\mathrm{i}} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{E_k}{k!(n-k)!} (-2\mathrm{i})^n z^n \\
&= \sum_{j=0}^{\infty} (-1)^n 2^{2j+1} \left(\sum_{k=0}^{2j+1} \binom{2j+1}{k} E_k \right) \frac{z^{2j+1}}{(2j+1)!}.
\end{aligned}$$

Finally, using (24) we obtain that (cf. [1], [75], [89])

$$\tan z = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^{2n+1} E_{2n+1}}{(2n+1)!} z^{2n+1}, \quad |z| < \frac{\pi}{2}. \tag{27}$$

Remark 2. There are several proofs of (27). For example, Kim [51] used

$$\mathrm{i}\tan z = \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}} = 1 - \frac{2}{\mathrm{e}^{2\mathrm{i}z} - 1} + \frac{4}{\mathrm{e}^{4\mathrm{i}z} - 1}$$

to get

$$z \tan z = \sum_{n=1}^{\infty} (-1)^n \frac{4^n (1 - 4^n) B_{2n}}{(2n)!} z^{2n},$$

i.e., (27). Similarly, Kim [51] proved the following relation for the secant function (see also [75], [1], [94], [89], [96])

$$\sec z = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}^*}{(2n)!} z^{2n}, \quad |z| < \frac{\pi}{2}.$$

Kim [52] defined *the Dedekind type Daehee-Changhee (DC) sums* as follows:

Definition 1. Let h and k be coprime integers with $k > 0$. Then

$$T_p(h,k) = 2\sum_{j=1}^{k-1}(-1)^{j-1}\frac{j}{k}\overline{E}_p\left(\frac{hj}{k}\right), \tag{28}$$

where $\overline{E}_p(x)$ denotes the p-th Euler function of the first kind.

The behavior of these sum $T_p(h,k)$ is similar to that of the Dedekind sums. Several properties and identities of the sum $T_p(h,k)$ and Euler polynomials, as well as some other interesting results, were derived in [52]. The most fundamental result in the theory of the Dedekind sums, Hardy-Berndt sums, Dedekind type DC and the other arithmetical sums is the reciprocity law and it can be used as an aid in calculating these sums (see Section 8).

6 Trigonometric Representation of the DC-sums

In this section we can give relations between trigonometric functions and the sum $T_p(h,k)$. We establish analytic properties of the sum $T_p(h,k)$ and give their trigonometric representation.

Starting from (22) in the form

$$\frac{(\pi \mathrm{i})^{m+1}}{2m!}\overline{E}_m(x) = \sum_{n=-\infty}^{0}\frac{\mathrm{e}^{(2n-1)\pi \mathrm{i}x}}{(2n-1)^{m+1}} + \sum_{n=1}^{\infty}\frac{\mathrm{e}^{(2n-1)\pi \mathrm{i}x}}{(2n-1)^{m+1}}, \tag{29}$$

we can get the following auxiliary result (see [89]):

Lemma 3. *Let $m \in \mathbb{N}$ and $0 \le x \le 1$, except for $m = 1$ when $0 < x < 1$. Then we have*

$$\overline{E}_{2m-1}(x) = \frac{(-1)^m 4(2m-1)!}{\pi^{2m}}\sum_{n=1}^{\infty}\frac{\cos((2n-1)\pi x)}{(2n-1)^{2m}}, \tag{30}$$

and

$$\overline{E}_{2m}(x) = \frac{(-1)^m 4(2m)!}{\pi^{2m+1}}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\pi x)}{(2n-1)^{2m+1}}. \tag{31}$$

For $0 \le x < 1$, $\overline{E}_{2m-1}(x)$ and $\overline{E}_{2m}(x)$ reduce to the Euler polynomials, which are related to Clausen functions (see Section 7).

We now modify the sums $T_p(h,k)$ for odd and even integer p. Thus, by (28), we define $T_{2m-1}(h,k)$ and $T_{2m}(h,k)$ sums as follows:

Definition 1′ [89]. Let h and k be coprime integers with $k > 0$. Then

$$T_{2m-1}(h,k) = 2\sum_{j=1}^{k-1}(-1)^{j-1}\frac{j}{k}\overline{E}_{2m-1}\left(\frac{hj}{k}\right) \tag{32}$$

and

$$T_{2m}(h,k) = 2\sum_{j=1}^{k-1}(-1)^{j-1}\frac{j}{k}\overline{E}_{2m}\Big(\frac{hj}{k}\Big), \tag{33}$$

where $\overline{E}_{2m-1}(x)$ and $\overline{E}_{2m}(x)$ denote the Euler functions.

By substituting (30) into (32), we get (cf. [89])

$$T_{2m-1}(h,k) = -\frac{8(-1)^m(2m-1)!}{k\pi^{2m}}\sum_{j=1}^{k-1}(-1)^j j\sum_{n=1}^{\infty}\frac{\cos(\frac{(2n-1)\pi hj}{k})}{(2n-1)^{2m}},$$

i.e.,

$$T_{2m-1}(h,k) = -\frac{8(-1)^m(2m-1)!}{k\pi^{2m}}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2m}}\sum_{j=1}^{k-1}(-1)^j j\cos\left(\frac{(2n-1)\pi hj}{k}\right).$$

Since (cf. [14] and [40])

$$\sum_{j=1}^{k-1} j\,\mathrm{e}^{(2n-1)\pi\mathrm{i}hj/k} = \begin{cases} \dfrac{k}{\mathrm{e}^{(2n-1)\pi\mathrm{i}h/k}-1}, & \text{if } 2n-1\not\equiv 0\,(\mathrm{mod}\,k),\\[2mm] \frac{1}{2}k(k-1), & \text{if } 2n-1\equiv 0\,(\mathrm{mod}\,k),\end{cases}$$

we conclude that

$$\sum_{j=1}^{k-1}(-1)^j j\,\mathrm{e}^{(2n-1)\pi\mathrm{i}hj/k} = \frac{k}{\mathrm{e}^{(k+(2n-1)h)\pi\mathrm{i}/k}-1},$$

from which, by some elementary calculations, the following sums follow

$$\sum_{j=1}^{k-1}(-1)^j j\cos\left(\frac{(2n-1)\pi hj}{k}\right) = -\frac{k}{2} \tag{34}$$

and

$$\sum_{j=1}^{k-1}(-1)^j j\sin\left(\frac{(2n-1)\pi hj}{k}\right) = \frac{k}{2}\tan\left(\frac{(2n-1)\pi h}{2k}\right), \tag{35}$$

where $2n-1\not\equiv 0\,(\mathrm{mod}\,k)$.

Using (34) and (25) we obtain the following result:

Theorem 10. *Let h and k be coprime positive integers and $m\in\mathbb{N}$. Then*

$$T_{2m-1}(h,k) = \Big(1-\frac{1}{k^{2m}}\Big)E_{2m-1}.$$

Indeed, here we have

$$T_{2m-1}(h,k)=\frac{4(-1)^m(2m-1)!}{\pi^{2m}}\sum_{\substack{n=1\\2n-1\not\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{1}{(2n-1)^{2m}},$$

i.e.,

$$\begin{aligned}T_{2m-1}(h,k)&=\frac{4(-1)^m(2m-1)!}{\pi^{2m}}\left\{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2m}}-\sum_{\substack{n=1\\2n-1\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{1}{(2n-1)^{2m}}\right\}\\&=\frac{4(-1)^m(2m-1)!}{\pi^{2m}}\left\{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2m}}-\sum_{j=1}^{\infty}\frac{1}{k^{2m}(2j-1)^{2m}}\right\},\end{aligned}$$

where we put $2n-1=(2j-1)k$ in order to calculate the second sum when $2n-1\equiv 0\,(\mathrm{mod}\,k)$. It proves the statement.

Similarly, by substituting (31) into (33) we get

$$T_{2m}(h,k)=\frac{8(-1)^m(2m)!}{k\pi^{2m+1}}\sum_{j=1}^{k-1}(-1)^j j\sum_{n=1}^{\infty}\frac{\sin\left(\frac{(2n-1)hj\pi}{k}\right)}{(2n-1)^{2m+1}},$$

and then, using (35), we arrive at the following theorem.

Theorem 11 ([89]). *Let h and k be coprime positive integers and $m\in\mathbb{N}$. Then*

$$T_{2m}(h,k)=\frac{4(-1)^m(2m)!}{\pi^{2m+1}}\sum_{\substack{n=1\\2n-1\not\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{\tan\left(\frac{(2n-1)\pi h}{2k}\right)}{(2n-1)^{2m+1}}. \tag{36}$$

7 DC-Sums Related to Special Functions

In this section, we give relations between DC-sums and some special functions.

In [94], Srivastava and Choi gave many applications of the Riemann zeta function, Hurwitz zeta function, Lerch zeta function, Dirichlet series for the polylogarithm function and Dirichlet's eta function. In [45], Guillera and Sandow obtained double integral and infinite product representations of many classical constants, as well as a generalization to Lerch's transcendent of Hadjicostas's double integral formula for the Riemann zeta function, and logarithmic series for the digamma and Euler beta functions. They also gave many applications. The *Lerch transcendent* $\Phi(z,s,a)$ (cf. [94, p. 121 et seq.], [45]) is the analytic continuation of the series

$$\Phi(z,s,a)=\frac{1}{a^s}+\frac{z}{(a+1)^s}+\frac{z}{(a+2)^s}+\cdots=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s},$$

which converges for $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$, and $\mathrm{Re}\,(s) > 1$ when $|z| = 1$, where as usual, $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$ and $\mathbb{Z}^- = \{-1, -2, \ldots\}$. Φ denotes the familiar Hurwitz-Lerch zeta function. Here, we mention some relations between this function Φ and other special functions (cf. [45].

Special cases include the analytic continuations of the Riemann zeta function

$$\Phi(1,s,1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}\,(s) > 1,$$

the Hurwitz zeta function

$$\Phi(1,s,a) = \zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \mathrm{Re}\,(s) > 1,$$

the alternating zeta function (also called Dirichlet's eta function $\eta(s)$)

$$\Phi(-1,s,1) = \zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s},$$

the Dirichlet beta function

$$2^{-s}\,\Phi\Big(-1,s,\frac{1}{2}\Big) = \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s},$$

the Legendre chi function

$$2^{-s} z\,\Phi\Big(z^2,s,\frac{1}{2}\Big) = \chi_s(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^s}, \quad |z| \le 1, \mathrm{Re}\,(s) > 1,$$

the polylogarithm

$$z\Phi(z,n,1) = \mathrm{Li}_m(z) = \sum_{n=0}^{\infty} \frac{z^k}{n^m}$$

and the Lerch zeta function (sometimes called the Hurwitz-Lerch zeta function)

$$L(\lambda, \alpha, s) = \Phi(\mathrm{e}^{2\pi\mathrm{i}\lambda}, s, \alpha),$$

which is a generalization of the Hurwitz zeta function and polylogarithm (cf. [3], [9], [19], [20], [25], [22], [45], [92], [93], [94]).

By using (29), we can give a relation between the Legendre chi function $\chi_s(z)$ and the function $\overline{E}_m(x)$:

Corollary 3 ([89]). *Let $m \in \mathbb{N}$. Then we have*

$$\overline{E}_m(x) = \frac{2(m!)}{(\pi\mathrm{i})^{m+1}} \Big((-1)^{m+1} \chi_{m+1}(\mathrm{e}^{-\pi\mathrm{i}x}) + \chi_{m+1}(\mathrm{e}^{\pi\mathrm{i}x}) \Big).$$

The graphics of the functions $x \mapsto \overline{E}_m(x)$ on $(0,1)$ for $m = 1,2,\ldots,6$ are presented in Figure 1. Most of the aforementioned functions are implemented in Mathematica and Matlab software packages, and the values of these functions can be calculated with arbitrary precision.

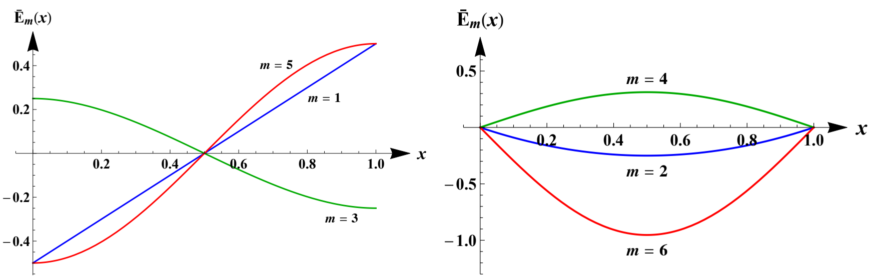

Fig. 1 Graphics of the functions $\overline{E}_m(x)$ for $m = 1,3,5$ (left) and $m = 2,4,6$ (right)

Choi et al. gave relations between the Clausen function, multiple gamma function and other functions. The higher-order *Clausen function* $\mathrm{Cl}_n(t)$ is defined for all $n \in \mathbb{N} \setminus \{1\}$ by (see [94])

$$\mathrm{Cl}_n(t) = \begin{cases} \sum_{k=1}^{\infty} \frac{\sin(kt)}{k^n}, & \text{if } n \text{ is even,} \\ \sum_{k=1}^{\infty} \frac{\cos(kt)}{k^n}, & \text{if } n \text{ is odd.} \end{cases}$$

The following functions are related to the higher-order Clausen function (cf. [92], [22, Eq. (5) and Eq. (6)])

$$\mathsf{S}(s,x) = \sum_{n=1}^{\infty} \frac{\sin((2n+1)x)}{(2n+1)^s} \quad \text{and} \quad \mathsf{C}(s,x) = \sum_{n=1}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^s}.$$

8 Reciprocity Law

As we mentioned in Section 3, for positive h and k and $(h,k) = 1$ the reciprocity law

$$s(h,k) + s(k,h) = \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} + \frac{1}{hk}\right) - \frac{1}{4}$$

holds. For the sums $s_p(a,b)$, defined by (13) in Corollary 2, the reciprocity law is given by

$$hk^n s_n(h,k) + kh^n s_n(k,h)$$

$$= \frac{1}{n+1}\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j B_j h^j B_{n+1-j}k^{n+1-j} + \frac{n}{n+1}B_{n+1},$$

where $(h,k)=1$ and B_n is the nth Bernoulli number (cf. [2], [4], [5]).

In the sequel we mention some reciprocity theorems for Hardy sums:

Theorem 12. *Let h and k be coprime positive integers. Then if $h+k$ is odd,*

$$S(h,k)+S(k,h)=1. \tag{37}$$

Theorem 13. *Let h and k be coprime integers. Then if h and k are odd,*

$$s_5(h,k)+s_5(k,h)=\frac{1}{2}-\frac{1}{2hk}. \tag{38}$$

Theorem 14. *Let h and k be coprime integers. If h is even, then*

$$s_1(h,k)-2s_2(k,h)=\frac{1}{2}-\frac{1}{2}\left(\frac{1}{hk}+\frac{h}{k}\right). \tag{39}$$

Theorem 15. *Let h and k be coprime integers. If k is odd, then*

$$2s_3(h,k)-s_4(k,h)=1-\frac{h}{k}. \tag{40}$$

These reciprocity theorems appear in Hardy's list [48], as Eqs. (viii), (vii), (vi), (vi') and (ix) on pages 122–123. Berndt [13] deduced (37), (39), and (40), and Goldberg [40] deduced (38) from Berndt's transformation formulae. For other proofs which do not depend on transformation theory, we refer to Sitaramachandrarao [91]. Otherwise, all reciprocity theorems can be proved by using contour integration and Cauchy Residue Theorem.

The reciprocity law of the sums $T_p(h,k)$, defined by (28) in Definition 1, is proved in [52]:

Theorem 16. *Let $(h,k)=1$ and $h,k\in\mathbb{N}$ with $h\equiv 1\,(\mathrm{mod}\ 2)$ and $k\equiv 1\,(\mathrm{mod}\ 2)$. Then we have*

$$\begin{aligned}&k^pT_p(h,k)+h^pT_p(k,h)\\ &=2\sum_{\substack{j=0\\ j-[\frac{hj}{k}]\equiv 1 \bmod 2}}^{k-1}\left(kh\left(E+\frac{j}{k}\right)+k\left(E+h-\left[\frac{hj}{k}\right]\right)\right)^p+(hE+kE)^p+(p+2)E_p,\end{aligned}$$

where

$$(hE+kE)^{n+1}=\sum_{j=1}^{n+1}\binom{n+1}{j}h^jE_jk^{n+1-j}E_{n+1-j}.$$

The first proof of reciprocity law of the Dedekind sums does not contain the theory of the Dedekind eta function related to Rademacher [73]. The other proofs of the reciprocity law of the Dedekind sums were given by Rademacher and Grosswald [76]. Berndt [11] and Berndt and Goldberg [14]) gave various types of Dedekind sums and their reciprocity laws. Berndt's methods are of three types. The first method uses contour integration which was first given by Rademacher [73]. This method has been used by many authors for example Grosswald [43], Hardy [48], his method is a different technique in contour integration. The second method is the Riemann-Stieltjes integral, which was invented by Rademacher [74]. The third method of Berndt is (periodic) Poisson summation formula. For the method and technique see also the references cited in each of these earlier works.

The famous property of the all arithmetic sums is the reciprocity law. In the sequel we prove reciprocity law of (36), by using contour integration.

Theorem 17. *Let h, k, $m \in \mathbb{N}$ with $h \equiv 1 \pmod 2$ and $k \equiv 1 \pmod 2$ and $(h,k) = 1$. Then we have*

$$\begin{aligned} hk^{2m+1}T_{2m}(h,k) \,&+\, kh^{2m+1}T_{2m}(k,h) \\ &= 2E_{2m+1} - \sum_{j=0}^{m-1} \binom{2m}{2j+1} E_{2j+1}E_{2m-2j-1}h^{2j+2}k^{2m-2j}, \end{aligned}$$

where E_n are Euler numbers of the first kind.

Proof. Following [11, Theorem 4.2], [14, Theorem 3], [43], [44], [76], we give the proof of this theorem. We use the contour integration method with the function

$$F_m(z) = \frac{\tan \pi hz \tan \pi kz}{z^{2m+1}}$$

over the contour C_N,

$$I_N = \frac{1}{2\pi \mathrm{i}} \int_{C_N} F_m(z)\,\mathrm{d}z,$$

where C_N is a positively oriented circle of radius R_N, with $1 \leq N < \infty$, centered at the origin. The sequence of radii R_N is increasing to ∞ and is chosen so that the poles of $F_m(z)$ are at a distance from C_N greater than some fixed positive number for each N.

From the above, we have

$$I_N = \frac{1}{2\pi R_N^{2m}} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}2m\theta} \tan\left(\pi h R_N \mathrm{e}^{\mathrm{i}\theta}\right) \tan\left(\pi k R_N \mathrm{e}^{\mathrm{i}\theta}\right) \mathrm{d}\theta.$$

By the condition on C_N, if $R_N \to \infty$, then $\tan\left(R_N \mathrm{e}^{\mathrm{i}\theta}\right)$ is bounded, and threfore $\lim_{N\to\infty} I_N = 0$ as $R_N \to \infty$ for each $m \in \mathbb{N}$.

The function $F_m(z)$ has a pole of order $2m-1$ at the origin, whose the residue can be determined from the corresponding Laurent series at $z = 0$. Using the expansion

$$\tan z = \sum_{\nu=0}^{\infty} \tau_\nu z^{2\nu-1}, \quad |z| < \frac{\pi}{2},$$

where

$$\tau_\nu = (-1)^{\nu-1}\frac{4^\nu(4^\nu-1)B_{2\nu}}{(2\nu)!} = (-1)^\nu \frac{4^\nu E_{2\nu-1}}{2(2\nu-1)!} \quad \text{(cf. Eq. (27))}$$

we can get the Laurent expansion of $F_m(z)$ at $z=0$ in the following form

$$F_m(z) = \frac{hk\pi^2}{z^{2m-1}}\sum_{n=0}^{\infty}(-1)^n\pi^{2n}4^{n+1}f_n z^{2n}, \tag{41}$$

where

$$f_n = \sum_{j=0}^{n}\frac{E_{2j+1}E_{2n-2j+1}h^{2j}k^{2n-2j}}{(2j+1)!(2n-2j+1)!}.$$

Then, from (41) for $n=m-1$, we get the residue of the function $F_m(z)$ at the pole $z=0$ as

$$\begin{aligned}\operatorname*{Res}_{z=0} F_m(z) &= hk\pi^2(-1)^{m-1}\pi^{2m-2}4^m f_{m-1}\\ &= -\frac{(-1)^m(2\pi)^{2m}}{(2m)!\,hk}\sum_{j=0}^{m-1}\binom{2m}{2j+1}E_{2j+1}E_{2m-2j-1}h^{2j+2}k^{2m-2j}.\end{aligned} \tag{42}$$

The other singularities of the function $F_m(z)$ in the interior of the contour C_N are points of the sets

$$X_N = \left\{\xi_j = \frac{2j-1}{2h} \,:\, |\xi_j| < R_N,\ j\in\mathbb{Z}\right\}$$

and

$$Y_N = \left\{\eta_\ell = \frac{2\ell-1}{2k} \,:\, |\eta_\ell| < R_N,\ \ell\in\mathbb{Z}\right\}.$$

Since h and k are odd positive integers, then the sets X_N and Y_N has an intersection

$$Z_N = X_N\cap Y_N = \left\{\zeta_j = \frac{2j-1}{2} \,:\, |\zeta_j| < R_N,\ j\in\mathbb{Z}\right\},$$

with double poles of the function $F_m(z)$ in the interior of C_N. Their residues are

$$\operatorname*{Res}_{z=\zeta_j} F_m(z) = \lim_{z\to\zeta_j}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-\zeta_j)^2F_m(z)\right] = -\frac{(2m+1)4^{m+1}}{(2j-1)^{2m+2}hk\pi^2}. \tag{43}$$

Te residues of the simple poles $z=\xi_j\in X_N\setminus Z_N$ and $\eta_\ell\in Y_N\setminus Z_N$ are easily found to be

$$\operatorname*{Res}_{z=\xi_j} F_m(z) = \lim_{z\to\xi_j}(z-\xi_j)F_m(z) = -\frac{2^{2m+1}h^{2m}}{\pi(2j-1)^{2m+1}}\tan\left(\frac{(2j-1)\pi k}{2h}\right) \tag{44}$$

and

$$\operatorname*{Res}_{z=\eta_\ell} F_m(z) = \lim_{z\to\eta_\ell}(z-\eta_\ell)F_m(z) = -\frac{2^{2m+1}k^{2m}}{\pi(2\ell-1)^{2m+1}}\tan\Big(\frac{(2\ell-1)\pi h}{2k}\Big), \quad (45)$$

respectively. Now, by the Cauchy residue theorem, we have

$$I_N = \sum_{\xi_j\in X_N\setminus Z_N}\operatorname*{Res}_{z=\xi_j}F_m(z) + \sum_{\eta_\ell\in Y_N\setminus Z_N}\operatorname*{Res}_{z=\eta_\ell}F_m(z) + \operatorname*{Res}_{z=0}F_m(z) + \sum_{\zeta_j\in Z_N}\operatorname*{Res}_{z=\zeta_j}F_m(z). \quad (46)$$

Using the residues (45), (44), (42), (43), and letting $N\to\infty$ in (46), we have that $I_N \to 0$ and

$$\begin{aligned}
&\frac{4^{m+1}k^{2m}}{\pi}\sum_{\substack{\ell=1\\ 2\ell-1\not\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{\tan\Big(\frac{(2\ell-1)\pi h}{2k}\Big)}{(2\ell-1)^{2m+1}} + \frac{4^{m+1}h^{2m}}{\pi}\sum_{\substack{j=1\\ 2j-1\not\equiv 0\,(\mathrm{mod}\,h)}}^{\infty}\frac{\tan\Big(\frac{(2j-1)\pi k}{2h}\Big)}{(2j-1)^{2m+1}}\\
&\quad = -\frac{(-1)^m(2\pi)^{2m}}{(2m)!\,hk}\sum_{j=0}^{m-1}\binom{2m}{2j+1}E_{2j+1}E_{2m-2j-1}h^{2j+2}k^{2m-2j}\\
&\quad\quad -\frac{(2m+1)4^{m+2}}{2hk\pi^2}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^{2m+2}},
\end{aligned}$$

where the last sum is given by (25).

Multiplying this equality by the factor $hk(-1)^m(2m)!/(2\pi)^{2m}$, we arrive at the desired result.

Corollary 4. *Let For each $m\in\mathbb{N}$ and each positive odd integer $k\geq 3$, we have*

$$k^{2m+1}T_{2m}(1,k) = 2E_{2m+1} - \sum_{j=0}^{m-1}\binom{2m}{2j+1}E_{2j+1}E_{2m-2j-1}h^{2j+2}k^{2m-2j}.$$

This result can be proved in a similar way as Theorem 17. For the sets of singularities here $Z_N = X_N$, so that the first term on the right hand side in (46) vanishes.

We now give a relation between Hurwitz zeta function, $\tan z$ and the sum $T_{2m}(h,k)$.

Hence, substituting $n = rk+j$, $0\leq r\leq\infty$, $1\leq j\leq k$ $(j\neq(k+1)/2)$ into (36), and recalling that $\tan(\pi+\alpha) = \tan\alpha$, then we have

$$\begin{aligned}
T_{2m}(h,k) &= \frac{4(-1)^m(2m)!}{\pi^{2m+1}}\sum_{\substack{j=1\\ j\neq(k+1)/2}}^{k}\sum_{r=0}^{\infty}\frac{\tan\Big(\frac{\pi h2(rk+j)+1}{2k}\Big)}{(2(rk+j)+1)^{2m+1}}\\
&= \frac{4(-1)^m(2m)!}{\pi^{2m+1}(2k)^{2m+1}}\sum_{\substack{j=1\\ j\neq(k+1)/2}}^{k}\tan\Big(\frac{(2j-1)\pi h}{2k}\Big)\sum_{r=0}^{\infty}\frac{1}{\Big(r+\frac{2j-1}{2k}\Big)^{2m+1}},
\end{aligned}$$

where the last sum can be identified as the Hurwitz zeta function $\zeta(s,x)$ at $s = 2m+1$ and $x = (2j-1)/2$. Note that $j \neq (k+1)/2$ provides the condition $2n-1 \not\equiv 0 \,(\mathrm{mod}\, k)$ in the summation process in (**??**).

In this way, we now arrive at the following result:

Theorem 18. *Let h and k be coprime positive integers and $m \in \mathbb{N}$. Then*

$$T_{2m}(h,k) = \frac{4(-1)^m(2m)!}{(2k\pi)^{2m+1}} \sum_{\substack{j=1 \\ j \neq (k+1)/2}}^{k} \tan\Big(\frac{(2j-1)\pi h}{2k}\Big)\zeta\Big(2m+1, \frac{2j-1}{2k}\Big).$$

Remark 3. Finally, we mention the sums $Y(h,k)$ defined by Simsek [88] (see also [86] and [65, p. 211]) as $Y(h,k) = 4ks_5(h,k)$, where h and k are odd with $(h,k) = 1$. If we integrate the function $F(z) = \cot(\pi z)\tan(\pi hz)\tan(\pi kz)$ over a contour, obtained from the rectangle with vertices at $\pm \mathrm{i}B$, $\frac{1}{2} \pm \mathrm{i}B$ $(B > 0)$, we see that $F(z)$ has the poles $z = 0$ and $z = 1/2$ on this contour; therefore, reciprocity of the sums $Y(h,k)$ is given by

$$hY(h,k) + kY(k,h) = 2hk - 2.$$

Remark 4. Using the definitions of the q-analogues of some classical arithmetic functions (Riemann zeta functions, Dirichlet L-functions, Hurwitz zeta functions, Dedekind sums), Simsek [87] defined q-analogues of these functions and gave some relations among them.

Remark 5. The higher multiple elliptic Dedekind sums and the reciprocity law have been introduced and considered by Bayad and Simsek [7] (see also [6], [8], [29], [83], [46]).

9 Sums Obtained From Gauss-Chebyshev Quadratures

It is well-known that Gaussian quadrature formulas with respect to the Chebyshev weight functions of the first, second, third, and fourth kind,

$$w_1(t) = \frac{1}{\sqrt{1-t^2}}, \quad w_2(t) = \sqrt{1-t^2}, \quad w_3(t) = \sqrt{\frac{1+t}{1-t}}, \quad w_4(t) = \sqrt{\frac{1-t}{1+t}},$$

respectively (cf. [18], [35], [58, p. 122]) have nodes expressible by trigonometric functions. In a short note in 1884 Stieltjes [95] gave the explicit expressions for these quadrature formulas for the weights w_1, w_2, and w_4,

$$\int_{-1}^{1} w_1(t)f(t)\,\mathrm{d}t = \frac{\pi}{n}\sum_{k=1}^{n} f\left(\cos\frac{(2k-1)\pi}{2n}\right) + R_{n,1}[f], \tag{47}$$

$$\int_{-1}^{1} w_2(t)f(t)\,\mathrm{d}t = \frac{\pi}{n+1}\sum_{k=1}^{n} \sin^2\frac{k\pi}{n+1}\, f\left(\cos\frac{k\pi}{n+1}\right) + R_{n,2}[f], \tag{48}$$

$$\int_{-1}^{1} w_4(t)f(t)\,\mathrm{d}t = \frac{4\pi}{2n+1}\sum_{k=1}^{n}\sin^2\frac{k\pi}{2n+1}\,f\left(\cos\frac{2k\pi}{2n+1}\right)+R_{n,4}[f], \quad (49)$$

where $R_{n,\nu}(f)=0$, $\nu=1,2,4$, for all algebraic polynomials of degree at most $2n-1$. In the class of functions $C^{2n}[-1,1]$, the remainder term of these Gaussian formulas can be done in the form (cf. [58, p. 333])

$$R_{n,\nu}[f] = \frac{\|\pi_{n,\nu}\|^2}{(2n)!}f^{(2n)}(\xi), \quad -1<\xi<1,$$

where the norms of the corresponding orthogonal polynomials $\pi_{n,\nu}$ can be expressed by the coefficients $\beta_{k,\nu}$ in their three-term recurrence relations

$$\pi_{k+1,\nu}(t) = (t-\alpha_{k,\nu})\pi_{k,\nu}(t)-\beta_{k,\nu}\pi_{k-1,\nu}(t), \quad k=1,2,\dots,$$
$$\pi_{0,\nu}(t)=1, \ \ \pi_{-1,\nu}(t)=0,$$

as $\|\pi_{n,\nu}\|^2 = \beta_{0,\nu}\beta_{1,\nu}\cdots\beta_{n,\nu}$, with

$$\beta_{0,\nu}=\mu_{0,\nu}=\int_{-1}^{1} w_\nu(t)\,\mathrm{d}t.$$

The recurrence coefficients for these Chebyshev weights, as well as the corresponding values of $\|\pi_{n,\nu}\|^2$ are presented in Table 1 (cf. [35, p. 29]). For completeness, we also give parameters for the Chebyshev weight of the third kind w_3. The

Table 1 Recurrence coefficients for different kind of Chebyshev polynomials

Weight function	Recurrence coefficients	$\|\pi_{n,\nu}\|^2$
Chebyshev I $(\nu=1)$	$\alpha_{k,1}=0\ \ (k\ge 0)$ $\beta_{0,1}=\pi,\ \ \beta_{1,1}=\frac{1}{2},\ \ \beta_{k,1}=\frac{1}{4}\ \ (k\ge 1)$	$\frac{2\pi}{4^n}$
Chebyshev II $(\nu=2)$	$\alpha_{k,2}=0\ \ (k\ge 0)$ $\beta_{0,2}=\frac{\pi}{2},\ \ \beta_{k,2}=\frac{1}{4}\ \ (k\ge 1)$	$\frac{\pi}{2^{2n+1}}$
Chebyshev III $(\nu=3)$	$\alpha_{0,3}=\frac{1}{2},\ \ \alpha_{k,3}=0\ \ (k\ge 1)$ $\beta_{0,3}=\pi,\ \ \beta_{k,3}=\frac{1}{4}\ \ (k\ge 1)$	$\frac{\pi}{4^n}$
Chebyshev IV $(\nu=4)$	$\alpha_{0,4}=-\frac{1}{2},\ \ \alpha_{k,4}=0\ \ (k\ge 1)$ $\beta_{0,4}=\pi,\ \ \beta_{k,4}=\frac{1}{4}\ \ (k\ge 1)$	$\frac{\pi}{4^n}$

corresponding quadrature formula for w_3 (cf. [61]),

$$\int_{-1}^{1} w_3(t)f(t)\,\mathrm{d}t = \frac{4\pi}{2n+1}\sum_{k=1}^{n}\cos^2\frac{(2k-1)\pi}{2(2n+1)} f\left(\cos\frac{(2k-1)\pi}{2n+1}\right) + R_{n,3}[f], \quad (50)$$

can be obtained by changing $t := -t$ and using (49), as well as

$$\int_{-1}^{1} w_3(t)f(t)\,\mathrm{d}t = \int_{-1}^{1} w_4(t)f(-t)\,\mathrm{d}t = Q_{n,4}(f(-\,\cdot\,)) + R_{n,3}[f].$$

Taking certain algebraic polynomials of degree at most $2n$ and integrating them by using some of quadrature formulas (47)–(49) and (50), we can obtain some trigonometric sums. For example, by a polynomial $p_m(t)$ of degree $m < 2n$, using the quadrature formula (47) we get

$$\sum_{k=1}^{n} p_m\left(\cos\frac{(2k-1)\pi}{2n}\right) = \frac{n}{\pi}\int_{-1}^{1}\frac{p_m(t)}{\sqrt{1-t^2}}\,\mathrm{d}t,$$

but if $p_{2n}(t) = At^{2n} + $ terms of lower degree, then

$$\sum_{k=1}^{n} p_{2n}\left(\cos\frac{(2k-1)\pi}{2n}\right) = \frac{n}{\pi}\int_{-1}^{1}\frac{p_{2n}(t)}{\sqrt{1-t^2}}\,\mathrm{d}t - \frac{2nA}{4^n}.$$

Thus, we need to compute only integrals of the forms $\int_{-1}^{1} p_m(t)w_\nu(t)\,\mathrm{d}t$ for $\nu = 1,2,3,4$. As usual $(s)_n$ is the well known Pochhammer symbol defined by

$$(s)_n = s(s+1)\cdots(s+n-1) = \frac{\Gamma(s+n)}{\Gamma(s)} \quad (\Gamma \text{ is the gamma function}).$$

If we take

$$p_m(t) = U_m(ct) = 2^m c^m t^m + \text{ terms of lower degree}, \quad c \in \mathbb{R},$$

where U_m is the Chebyshev polynomial of the second kind and degree m and use the integral (cf. [71, p. 456])

$$\int_{-1}^{1}\frac{U_{2m}(cx)}{\sqrt{1-x^2}}\,\mathrm{d}x = \pi P_m(2c^2-1),$$

where P_m is the Legendre polynomial of degree m, we get the following statement:

Theorem 19. *Let U_m be Chebyshev polynomial of the second kind and degree m and $c \in \mathbb{R}$. We have*

$$\sum_{\nu=1}^{n} U_m\left(c\cos\frac{(2\nu-1)\pi}{2n}\right) = \begin{cases} 0, & m \text{ is odd } (m \geq 1), \\ nP_{m/2}(2c^2-1), & m \text{ is even } (0 \leq m < 2n), \\ n\left(P_n(2c^2-1) - 2c^{2n}\right), & m = 2n, \end{cases}$$

where P_n is the Legendre polynomial of degree n.

Similarly, using the Legendre polynomials

$$P_m(t) = \frac{(n+1)_n}{2^n n!} t^n + \text{ terms of lower degree},$$

defined by the generating function (cf. [58, p. 129]

$$\frac{1}{\sqrt{1-2xt+x^2}} = \sum_{m=0}^{\infty} P_m(t) x^m,$$

for which the following integrals are true (see [71, p. 423])

$$\int_{-1}^{1} \frac{P_{2m}(x)}{\sqrt{1-x^2}} \,\mathrm{d}x = \pi \left(\frac{(1/2)_m}{m!} \right)^2$$

and

$$\int_{-1}^{1} \frac{x P_{2m+1}(x)}{\sqrt{1-x^2}} \,\mathrm{d}x = \pi \left(\frac{(1/2)_m}{m!} \right)^2 \frac{2m+1}{2m+2},$$

we can prove the following statement:

Theorem 20. *Let P_m be the Legendre polynomial of degree n. Then*

$$\sum_{\nu=1}^{n} P_m\left(\cos\frac{(2\nu-1)\pi}{2n}\right) = \begin{cases} 0, & m \text{ is odd } (m \ge 1), \\ \dfrac{n}{4^m}\dbinom{m}{m/2}^2, & m \text{ is even } (0 \le m < 2n), \\ \dfrac{n}{4^{2n}}\left(\dbinom{2n}{n}^2 - 2\dbinom{4n}{2n}\right), & m = 2n, \end{cases}$$

and

$$\sum_{\nu=1}^{n} \cos\frac{(2\nu-1)\pi}{2n} P_{m-1}\left(\cos\frac{(2\nu-1)\pi}{2n}\right)$$
$$= \begin{cases} 0, & m \text{ is odd } (m \ge 1), \\ \dfrac{n}{4^{m-2}}\dfrac{m-1}{m}\dbinom{m-2}{(m-1)/2}^2, & m \text{ is even } (0 < m < 2n), \\ \dfrac{1}{4^{2n-1}}\left(2(2n-1)\dbinom{2n-2}{n-1}^2 - n\dbinom{4n-2}{2n-1}\right), & m = 2n. \end{cases}$$

Now, we use the following weighted integral for Legendre polynomials, expressed in terms of the hypergeometric function (see [71, p. 422]),

$$I_\beta(m) = \int_{-1}^{1} (1-x^2)^{\beta-1} P_m(x) \,\mathrm{d}x = (-1)^m \sqrt{\pi} \frac{\Gamma(\beta)}{\Gamma\left(\beta+\frac{1}{2}\right)} \, {}_3F_2\left[\begin{matrix} -m, \; m+1, \; \beta \\ 2\beta, \; 1 \end{matrix} ; 1 \right].$$

For odd m this integral vanishes. We need now its value for $\beta = 3/2$ and even m, i.e.,

$$
\begin{aligned}
I_{3/2}(m) &= \frac{\pi}{2}\, {}_3F_2\left[\begin{matrix} -m,\ m+1,\ 3/2 \\ 3,\ 1 \end{matrix}\ ; 1\right] \\
&= \frac{\pi}{2} \cdot \frac{\pi}{2\Gamma\left(\frac{1}{2}-\frac{m}{2}\right)\Gamma\left(\frac{3}{2}-\frac{m}{2}\right)\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{m}{2}+2\right)}.
\end{aligned}
$$

Since $\Gamma(z) = \Gamma(z+k)/(z)_k$ $(k \in \mathbb{N})$, we have

$$
\Gamma\left(\frac{1}{2}-\frac{m}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(\frac{1}{2}-\frac{m}{2}\right)_{m/2}} = (-1)^{m/2} \frac{2^m\sqrt{\pi}}{m!}\left(\frac{m}{2}\right)!,
$$

so that the previous integral becomes

$$
I_{3/2}(m) = -\frac{\pi}{4^m(m-1)(m+2)}\binom{m}{m/2}^2.
$$

Setting $p_m(t) = P_m(t)$ in (48) we get the following result:

Theorem 21. *Let P_m be the Legendre polynomial of degree n. Then*

$$
\sum_{\nu=1}^{n} \sin^2\frac{k\pi}{n+1} P_m\left(\cos\frac{k\pi}{n+1}\right)
= \begin{cases}
0, & m \text{ is odd } (m \geq 1), \\[4pt]
-\dfrac{n+1}{4^m(m+2)(m-1)}\dbinom{m}{m/2}^2, & m \text{ is even } (0 \leq m < 2n), \\[4pt]
-\dfrac{2}{4^{2n+1}}\left(\dfrac{1}{2n-1}\dbinom{2n}{n}^2 + (n+1)\dbinom{4n}{2n}\right), & m = 2n.
\end{cases}
$$

In the sequel we give trigonometric sums obtained by monomials $p_m(t) = t^m$ in the quadrature formulas (47)–(49). Because of that, we need the moments

$$
\mu_{m,\nu} = \int_{-1}^{1} w_\nu(t) t^m \, \mathrm{d}t, \quad \nu = 1,2,3,4.
$$

These moments are

$$
\mu_{m,1} = \frac{\sqrt{\pi}\left((-1)^m+1\right)\Gamma\left(\frac{m+1}{2}\right)}{m\Gamma\left(\frac{m}{2}\right)} = \begin{cases}
0, & m \text{ is odd}, \\[4pt]
\dfrac{\pi}{2^m}\dbinom{m}{m/2}, & m \text{ is even};
\end{cases}
$$

$$\mu_{m,2} = \frac{\sqrt{\pi}\,((-1)^m+1)\,\Gamma\left(\frac{m+1}{2}\right)}{4\Gamma\left(\frac{m}{2}+2\right)} = \begin{cases} 0, & m \text{ is odd},\\ \dfrac{\pi}{2^m(m+2)}\dbinom{m}{m/2}, & m \text{ is even}; \end{cases}$$

$$\mu_{m,4} = \frac{2^m\left(((-1)^m-1)\,\Gamma\left(\frac{m}{2}+1\right)^2 + ((-1)^m+1)\,\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+3}{2}\right)\right)}{\Gamma(m+2)}$$

$$= \begin{cases} -\dfrac{\pi}{2^{m+1}}\dbinom{m+1}{(m+1)/2}, & m \text{ is odd},\\ \dfrac{\pi}{2^m}\dbinom{m}{m/2}, & m \text{ is even}. \end{cases}$$

Note that $\mu_{m,3} = (-1)^m \mu_{m,4}$.

According to (47)–(49) we conclude that the following results hold.

Theorem 22. *We have*

$$\sum_{\nu=1}^{n} \cos^m \frac{(2\nu-1)\pi}{2n} = \begin{cases} 0, & m \text{ is odd } (m \ge 1),\\ \dfrac{n}{2^m}\dbinom{m}{m/2}, & m \text{ is even } (0 \le m < 2n),\\ \dfrac{n}{4^n}\left(\dbinom{2n}{n} - 2\right), & m = 2n; \end{cases}$$

$$\sum_{\nu=1}^{n} \sin^2 \frac{\nu\pi}{n+1} \cos^m \frac{\nu\pi}{n+1} = \begin{cases} 0, & m \text{ is odd } (m \ge 1),\\ \dfrac{n+1}{(m+2)2^m}\dbinom{m}{m/2}, & m \text{ is even } (0 \le m < 2n),\\ \dfrac{1}{2^{2n+1}}\left(\dbinom{2n}{n} - n - 1\right), & m = 2n; \end{cases}$$

$$\sum_{\nu=1}^{n} \sin^2 \frac{\nu\pi}{2n+1} \cos^m \frac{2\nu\pi}{2n+1} = \begin{cases} \dfrac{2n+1}{2^{m+2}}\dbinom{m}{m/2}, & m \text{ is even } (0 \le m < 2n),\\ -\dfrac{2n+1}{2^{m+3}}\dbinom{m+1}{(m+1)/2}, & m \text{ is odd } (0 \le m < 2n),\\ \dfrac{2n+1}{4^{n+1}}\left(\dbinom{2n}{n} - 1\right), & m = 2n. \end{cases}$$

Trigonometric sums can be also obtained in a similar way using quadrature formulas of Radau and Lobatto type with respect to Chebyshev weights. We mention that shortly after Stieltjes' results [95], Markov [57] (see also [61]) obtained the explicit expressions for Gauss-Radau and Gauss-Lobatto formulas, with respect to the Chebyshev weight of the first kind $w_1(t)$ (for both endpoints),

$$\int_{-1}^{1} w_1(t) f(t)\,\mathrm{d}t = \frac{2\pi}{2n+1}\left[\frac{1}{2} f(-1) + \sum_{k=1}^{n} f\left(\cos\frac{(2k-1)\pi}{2n+1}\right)\right] + R_{n+1,1}^{(-1)}[f],$$

$$\int_{-1}^{1} w_1(t) f(t)\,\mathrm{d}t = \frac{2\pi}{2n+1}\left[\frac{1}{2}f(1) + \sum_{k=1}^{n} f\left(\cos\frac{2k\pi}{2n+1}\right)\right] + R_{n+1,1}^{(+1)}[f],$$

and

$$\int_{-1}^{1} w_1(t) f(t)\,\mathrm{d}t = \frac{\pi}{n+1}\left[\frac{1}{2}f(-1) + \sum_{k=1}^{n} f\left(\cos\frac{k\pi}{n+1}\right) + \frac{1}{2}f(1)\right] + R_{n+2,1}^{L}[f],$$

respectively, where

$$R_{n+1,1}^{(\mp 1)}[f] = \pm\frac{\pi f^{(2n+1)}(\xi)}{(2n+1)!2^{2n}} \qquad (f \in C^{2n+1}[-1,1])$$

and

$$R_{n+2,1}^{L}[f] = -\frac{\pi f^{(2n+2)}(\xi)}{(2n+2)!2^{2n+1}}, \qquad (f \in C^{2n+2}[-1,1])$$

and $\xi \in (-1,1)$. These Gauss-Radau formulas are exact for all algebraic polynomials of degree at most $2n$, while the Gauss-Lobatto is exact for polynomials up to degree $2n+1$, so that we can obtain trigonometric sums taking monomials x^m in the previous formulas for all $m \le 2n+1$ and $m \le 2n+2$, respectively.

There are also similar quadrature formulas for other Chebyshev weights $w_\nu(t)$, $\nu = 2,3,4$. For details see [36], [37], [34], [78], [67], [69]. A general approach in construction of Gauss-Radau and Gauss-Lobatto formulas can be found in [58, pp. 328–332]. In some of these cases the nodes of quadratures can be expressed in terms of trigonometric functions, and such quadratures can be used for getting trigonometric sums.

Some more complex trigonometric sums can be obtained using quadrature formulas of Turán type (quadrature with multiple nodes) with respect to the Chebyshev weight functions. For some details on such quadrature formulas see [97], [39], [59], [38], [60], [64], [79], [80], [81].

10 Sums Obtained From Trigonometric Quadrature Rules

Another way for getting trigonometric sums is based on quadrature rules with maximal trigonometric degree of exactness. With $\mathscr{T}_d$ we denote the linear space of all trigonometric polynomials of degree less than or equal to d,

$$t_d(x) = \frac{a_0}{2} + \sum_{k=1}^{d}\big(a_k \cos kx + \sin kx\big) \qquad (a_k, b_k \in \mathbb{R}).$$

If $|a_d| + |b_d| > 0$ the degree of t_d is strictly d.

We say that a quadrature formula of the form

$$\int_0^{2\pi} f(x)w(x)\,\mathrm{d}x = \sum_{\nu=0}^{n} w_\nu f(x_\nu) + R_n(f), \quad 0 \le x_0 < x_1 < \cdots < x_n < 2\pi,$$

has trigonometric degree of exactness equal to d if the remainder term $R_n(f) = 0$ for all $f \in \mathscr{T}_d$ and there exists some g in $\mathscr{T}_{d+1}$ such that $R_n(g) \neq 0$.

Quadrature rules with a maximal trigonometric degree of exactness are known in the literature as *trigonometric quadrature rules of Gaussian type*. Maximal trigonometric degree of exactness for quadrature rule with $n+1$ nodes is n.

A brief historical survey of available approaches for the construction of quadrature rules with maximal trigonometric degree of exactness has been given in [61] (see also [62], [68]). Here, we consider only a simple case of the $(2n+1)$-point trigonometric quadrature formula

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = \frac{2\pi}{2n+1}\sum_{k=0}^{2n} f(x_k) + R_{2n+1}[f], \tag{51}$$

with the nodes

$$x_k = \theta + \frac{2k\pi}{2n+1}, \qquad k = 0,1,\ldots,2n,$$

where $0 \le \theta < 2\pi/(2n+1)$. Formula (51) is exact for every trigonometric polynomial of degree at most $2n$ (cf. [98]). Such kind of quadratures have applications in numerical integration of 2π-periodic functions. Two special cases of the quadrature formula (51) for which $\theta = 0$ and $\theta = \pi/(2n+1)$ are very interesting in applications. Their quadrature sums are

$$Q_{2n+1}^{T}(f) = \frac{2\pi}{2n+1}\sum_{k=0}^{2n} f\left(\frac{2k\pi}{2n+1}\right) \tag{52}$$

and

$$Q_{2n+1}^{M}(f) = \frac{2\pi}{2n+1}\sum_{k=0}^{2n} f\left(\frac{(2k+1)\pi}{2n+1}\right), \tag{53}$$

respectively. Some details on $Q_{2n+1}^{T}(f)$ and its applications in the trigonometric approximation can be found in [58, Chap. 3]. The second formula $Q_{2n+1}^{M}(f)$ has been analyzed in [62].

Remark 6. Putting $h = 2\pi/(2n+1)$ and $f_\alpha \equiv f(\alpha h)$, we can write the formulas (52) and (53) in the forms

$$Q_{2n+1}^{T}(f) = h\left\{\frac{1}{2}f_0 + f_1 + \cdots + f_{2n} + \frac{1}{2}f_{2n+1}\right\}$$

and

$$Q_{2n+1}^{M}(f) = h\left\{f_{1/2} + f_{3/2} + \cdots + f_{2n} + f_{2n+1/2}\right\},$$

where, because of periodicity, we introduced $f_{2n+1} = f(2\pi) = f(0) = f_0$. These quadratures (52) and (53) are symmetric with respect to the point $x = \pi$, and they are, in fact, the composite *trapezoidal* and *midpoint* rules, respectively. Also, they are equivalent to the trigonometric version of the Gauss-Radau formulas with respect to the Chebyshev weight of the first kind on $(-1,1)$ (cf. [61]).

Taking $f(x) = \cos^{2m-2\nu} x \sin^{2\nu} x$ in (51), where $0 \le m \le n$ and $0 \le \nu \le m$, we get the following result:

Theorem 23. *If* $n, m, \nu \in \mathbb{N}$ *and* $0 \le \nu \le m \le n$, *we have*

$$\sum_{k=0}^{2n} \cos^{2m-2\nu}\left(\theta + \frac{2k\pi}{2n+1}\right) \sin^{2\nu}\left(\theta + \frac{2k\pi}{2n+1}\right) = \frac{2n+1}{4^m} \frac{\binom{2m}{m}\binom{m}{\nu}}{\binom{2m}{2\ni}}$$

for each real θ.

Here, we see that $f \in \mathscr{T}_{2m}$, as well as that

$$\begin{aligned} I_{m,\nu} = \int_0^{2\pi} \cos^{2m-2\nu} x \sin^{2\nu} x \,\mathrm{d}x &= 4\int_0^{\pi/2} \cos^{2m-2\nu} x \sin^{2\nu} x \,\mathrm{d}x \\ &= 2\int_0^1 t^{\nu-1/2}(1-t)^{m-\nu-1/2}\,\mathrm{d}t \\ &= \frac{2}{m!}\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(m-\nu+\frac{1}{2}\right), \end{aligned}$$

after the change of variables $t = \sin^2 x$. Since $\Gamma(\nu + 1/2) = (2\nu - 1)!!\sqrt{\pi}/2^\nu$, by using (51), we obtain the desired result.

Selecting other functions, such that $f \in \mathscr{T}_{2n}$, we can get similar results as in Section 9.

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