BIT manuscript No.
(will be inserted by the editor)

Quadrature processes for efficient calculation of the Clausen functions

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Received: date / Accepted: date

Abstract The Clausen functions arise in numerous applications. An efficient summation/integration method for the numerical calculation of these functions of arbitrary order is proposed in this paper. The method is based on a modification of an earlier method and it does not require the construction of the coefficients in the three-term recurrence relation for the corresponding orthogonal polynomials, but only a transformation of orthogonal polynomials from the real line to the positive semiaxis. Numerical experiments are also included.

Keywords Clausen functions · Numerical computation · Gaussian quadrature · Error estimate

Mathematics Subject Classification (2010) 65D30 · 41A55 · 33C45

1 Introduction

In many problems in quantum field theory, especially in quantum electrodynamics on vacuum polarization, scattering of light by light, etc. the function

$$\phi(x) = \int_1^x \frac{\log|1+t|}{t}\mathrm{d}t$$

appears very often, and it can be expressed for x on the unit circle (cf. [1]) as

$$\phi\big(\mathrm{e}^{\mathrm{i}\theta}\big) = -\frac{\theta^2}{4} + \mathrm{i}\psi(\theta),$$

Dedicated to the Memory of Professor Germund Dahlquist (1925 – 2005).

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where

$$\psi(\theta)=\sum_{k=1}^{+\infty}\frac{\sin(k\theta)}{k^2}. \tag{1}$$

In 1832 this function (1), known as the Spence function, was tabulated by Clausen [3] for $\theta = j\pi/180$, $j = 1, 2, \ldots, 180$, to sixteen decimal places. The series (1) is slowly convergent, and a better way for evaluating its values is a numerical integration of its integral representation (cf. [1])

$$\psi(\theta)=-\int_0^\theta \log\left|2\sin\frac{t}{2}\right|\mathrm{d}t. \tag{2}$$

In [18] Linciano considered the problem of the numerical evaluation of the integral (2) and showed that Gaussian quadrature rules, for functions with a singularity of the type $\sqrt{x}\log(1/x)$ in $(0,1)$, can be applied to an auxiliary function in order to solve the problem. In the literature many papers were devoted to this integral, which is just an element of the sequence known as Clausen's functions,

$$\mathrm{Cl}_n(\theta)=\begin{cases}\displaystyle\sum_{k=1}^{+\infty}\frac{\sin(k\theta)}{k^n}, & n \text{ even},\\ \displaystyle\sum_{k=1}^{+\infty}\frac{\cos(k\theta)}{k^n}, & n \text{ odd}.\end{cases} \tag{3}$$

Evidently, $\mathrm{Cl}_2(\theta)$ is the integral (2). Otherwise, this function can be expressed in terms of the dilogarithm function $\mathrm{Li}_2(z)$ as

$$\mathrm{Cl}_2(\theta)=\mathrm{Im}\left\{\mathrm{Li}_2(\mathrm{e}^{\mathrm{i}\theta})\right\}=-\mathrm{Im}\left\{\int_0^{\mathrm{e}^{\mathrm{i}\theta}}\frac{\log(1-t)}{t}\,\mathrm{d}t\right\}=\mathrm{Im}\left\{\sum_{k=1}^{+\infty}\frac{\mathrm{e}^{\mathrm{i}k\theta}}{k^2}\right\}\quad(\theta\in\mathbb{R})$$

(cf. [16]). In a similar way, other Clausen's functions of higher order $n \geq 3$ can be expressed also in terms of the functions $\mathrm{Li}_n(z)$ for $z = \mathrm{e}^{\mathrm{i}\theta}$ (for details see the book by Lewin [17]). In the mentioned paper [16], Kölbig derived twenty-digit Chebyshev coefficients for the Clausen function $\mathrm{Cl}_2(\theta)$, allowing a fast computation of this function for real values of the argument θ, when $\theta \in [-\pi/2, \pi/2]$ and $\theta \in [\pi/2, 3\pi/2]$. Earlier in 1968, Wood [28] used the Chebyshev expansion for $t \mapsto \theta\cot(\pi\theta/2)$ and integration by parts in the integral (2) to obtain a Chebyshev series expansion, with coefficients given as numerical series, involving Bernoulli numbers. For some other approaches from 1984 see [9] and [14]. As a remarkable paper on this subject for $n \geq 2$, we mention the work by Wu, Zhang, and Liu [29], which introduces a certain sequence of approximants $\mathrm{Cl}_n^N(\theta)$ for $\mathrm{Cl}_n(\theta)$, proving an error estimate, and give some numerical comparision with with Wood's method [28] for $\mathrm{Cl}_2(\theta)$, when $\theta = \pi/3$ and $\pi/2$, including a comparision of CPU time. Also, they compare their two algorithms with 10 digits accuracy, when $\theta = \pi/3$, and $n = 3$ and $n = 4$, with regard to CPU.

For $n = 1$ the summation in the sequence of Clausen's functions (3) can be expressed in the explicit form

$$\mathrm{Cl}_1(\theta)=-\log\left|2\sin\frac{\theta}{2}\right|. \tag{4}$$

We note that for even n, $\mathrm{Cl}_n(0) = \mathrm{Cl}_n(\pi) = 0$, while for odd $n \geq 3$, $\mathrm{Cl}_n(0) = \zeta(n)$ and $\mathrm{Cl}_n(\pi) = -(1-2^{1-n})\zeta(n)$, where $z \mapsto \zeta(z)$ is the Riemann zeta function (cf. Ivić [15]).

Let $n \in \mathbb{N}$. Because of the *periodicity* $\mathrm{Cl}_n(\theta) = \mathrm{Cl}_n(\theta + 2m\pi)$ $(m \in \mathbb{Z})$ and the *parity* $\mathrm{Cl}_n(-\theta) = (-1)^{n+1}\mathrm{Cl}_n(\theta)$, the computation of the Clausen functions can be limited to the interval $[0, \pi]$, so that, in the sequel we consider only the cases when $\theta \in [0, \pi]$ (cf. [29, Remark 1.1]).

In this paper we give an efficient numerical summation/integration method [25] for calculating Clausen's functions (3) for each $n \geq 2$. This simple method is based on a modification of the method given in [22] (see also [23]). Our modification does not require the construction of the coefficients in the three-term recurrence relation for the corresponding orthogonal polynomials, but only a transformation of orthogonal polynomials from $\mathbb{R}$ to $\mathbb{R}_+$ (see [21, pp. 102–103]). Such an approach was presented at the 2nd Meeting of the Serbian Academy of Sciences and Arts, Belgrade, held on March 26, 2021, by the second author of this paper. At the same time, an application to the calculation of the values of the Riemann zeta function was given (see [26]). Our interest in this kind of summation/quadrature methods on $\mathbb{R}$ and $\mathbb{R}_+$ was inspired by the remarkable paper of Germund Dahlquist in three parts [6–8], and deepened by a very intensive epistolary correspondence of the second author with him in that period. Therefore this paper is dedicated to the memory of this very prominent scientist. The paper is organized as follows. In Section 2 we give basic facts on summation/integration methods for slowly convergent series written in the form

$$\sum_{k=1}^{+\infty} f(k) = \sum_{k=1}^{m-1} f(k) + \sum_{k=m}^{+\infty} f(k), \tag{5}$$

where $z \mapsto f(z)$ is a given function having certain properties in the complex plane. The function f may also depend on several other parameters (cf. [24, 25]). The second sum on the right of (5) is then exactly transformed to an integral weighted by the hyperbolic function $w(t) = 1/\cosh^2 t$ on $\mathbb{R}_+$ (see [22]). Finally, this integral is approximated by a corresponding N-point Gaussian quadrature rule, i.e.,

$$\sum_{k=m}^{+\infty} f(k) = \int_{\mathbb{R}_+} g(t)w(t)\mathrm{d}t = \sum_{\nu=1}^{N} A_\nu^{(N)} g\big(\tau_\nu^{(N)}\big) + R_N(g; w), \tag{6}$$

where the function g is related to f in some way, and $R_N(g; w)$ is the remainder term of this N-point Gaussian formula, which is exact for all algebraic polynomials of degree at most $2N - 1$. Properties and representations of certain functions connected to Clausen's functions $\mathrm{Cl}_n(\theta)$ are given in Section 3, and they will be used in Section 4 for the efficient calculation of Clausen's functions $\mathrm{Cl}_n(\theta)$ by summation/quadrature formulas. Numerical examples are given in Section 5.

2 Basic facts on summation/quadrature methods

After the Laplace transform method presented by Gautschi and Milovanović [12], where the general term of the series is expressible in terms of the Laplace transform, or its derivative, of a known function, Milovanović [25] developed a method for summing slowly convergent series $\sum_{k=m}^{+\infty}(\pm 1)^k f(k)$, based on a contour integration over a rectangle $\mathit{\Gamma}$ in the complex plane in which the weight function w is one of the hyperbolic functions $w_1(t) = 1/\cosh^2 t$ (as in (6)) or $w_2(t) = \sinh t/\cosh^2 t$ (for alternating series). The function g can be expressed in terms of the indefinite integral F of f chosen so as to satisfy the following decay properties in the complex region (see [19], [22]):

(C1) *F is a holomorphic function in the region*

$$D = \left\{ z \in \mathbb{C} \;\middle|\; \operatorname{Re} z \geq \alpha,\ m-1 < \alpha < m \right\}; \tag{7}$$

(C2) $\lim\limits_{|t|\to+\infty} \mathrm{e}^{-c|t|} F\left(x + \mathrm{i}t/\pi\right) = 0$, *uniformly for* $x \geq \alpha$;

(C3) $\lim\limits_{x\to+\infty} \displaystyle\int_{\mathbb{R}} \mathrm{e}^{-c|t|} \left| F\left(x + \mathrm{i}t/\pi\right) \right| \mathrm{d}t = 0$,

where $c = 2$ (*or* $c = 1$ *for "alternating" series*).

In this paper we consider only the case for the series $\sum_{k=m}^{+\infty} f(k)$, which appears in (6).

Taking the rectangular contour $\mathit{\Gamma} = \mathit{\Gamma}_{\alpha,\beta,\delta} = \partial G$, with $m-1 < \alpha < m$, $n < \beta < n+1$ (n is an integer greater than m), $\delta > 0$ and

$$G = \left\{ z \in \mathbb{C} \;\middle|\; \alpha \leq \operatorname{Re} z \leq \beta,\ |\operatorname{Im} z| \leq \frac{\delta}{\pi} \right\} \subset D,$$

then an integration of the function $z \mapsto f(z)(\pi/\tan \pi z)$ over $\mathit{\Gamma}$, after integration by parts, leads to

$$T_{m,n} = \sum_{k=m}^{n} f(k) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathit{\Gamma}} \left(\frac{\pi}{\sin \pi z} \right)^2 F(z) \mathrm{d}z, \tag{8}$$

where F is an integral of f, such that $F(\infty) = 0$. Setting $\alpha = m - 1/2$, $\beta = n + 1/2$, and letting $\delta \to +\infty$ and $n \to +\infty$, under conditions (C1), (C2), and (C3), Milovanović [22] transformed the integral (8) to

$$T_m = T_{m,\infty} = \sum_{k=m}^{+\infty} f(k) = -\frac{1}{2} \int_{-\infty}^{+\infty} F\left(m - \frac{1}{2} + \mathrm{i}\frac{t}{\pi} \right) \frac{\mathrm{d}t}{\cosh^2 t}, \tag{9}$$

i.e.,

$$T_m = \int_0^{+\infty} \mathit{\Phi}\left(m - \frac{1}{2}, \frac{t}{\pi} \right) \frac{\mathrm{d}t}{\cosh^2 t}, \tag{10}$$

where

$$\mathit{\Phi}(x, y) = -\frac{1}{2} \left[F(x + \mathrm{i}y) + F(x - \mathrm{i}y) \right]. \tag{11}$$

The integral (10) can be efficiently calculated by using Gaussian quadrature rules with respect to the hyperbolic weight function $w(t) = 1/\cosh^2 t$, but their construction by the Golub-Welsch algorithm [13] requires knowing the corresponding symmetric tridiagonal Jacobi matrix

$$J_N(w) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathbf{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{N-1}} \\ \mathbf{O} & & & \sqrt{\beta_{N-1}} & \alpha_{N-1} \end{bmatrix}, \tag{12}$$

i.e., the coefficients α_k and β_k in the three-term recurrence relation for the corresponding monic orthogonal polynomials $\pi_k(t) \equiv \pi_k(w;t)$,

$$\begin{aligned} &\pi_{k+1}(t) = (t-\alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t), \quad k = 0,1,2,\ldots, \\ &\pi_{-1}(t) = 0, \ \ \pi_0(t) = 1. \end{aligned} \tag{13}$$

The nodes $\tau_\nu^{(N)}$ of this N-point Gaussian quadrature rule given by (6) are the eigenvalues of the Jacobi matrix $J_N(w)$ (or zeros of $\pi_N(t)$), and the weight coefficients (Christoffel numbers) can be calculated from the the first components of the corresponding normalized eigenvectors $\mathbf{v}_\nu = [v_{\nu,1} \ \ldots \ v_{\nu,N}]^T$ ($\mathbf{v}_\nu^T\mathbf{v}_\nu = 1$) of this Jacobi matrix $J_N(w)$ (cf. Mastroianni and Milovanović [21, p. 326]) in the form $A_\nu^{(N)} = \beta_0 v_{\nu,1}^2$, $\nu = 1,\ldots,N$.

Now, when we have the quadrature formula (6), using (10) we get the following approximation of the sum (5),

$$\sum_{k=1}^{+\infty} f(k) \approx \sum_{k=1}^{m-1} f(k) + \sum_{\nu=1}^{N} A_\nu^{(N)} \Phi\left(m - \frac{1}{2}, \frac{\tau_\nu^{(N)}}{\pi}\right). \tag{14}$$

For generating the recursion coefficients α_k and β_k in (13), Milovanović [22, §3] used the discretized Stieltjes-Gautschi procedure [10], with the discretization based on the Gauss-Laguerre quadrature rule.

Remark 1 Recent progress in variable-precision arithmetic and symbolic computation now makes it possible to generate these coefficients α_k and β_k directly by using the original Chebyshev method in sufficiently high precision. Respective symbolic/variable-precision software for orthogonal polynomials and quadrature formulas of Gaussian type is available: Gautschi's package `SOPQ` in Matlab (see [11]) and the Mathematica package `OrthogonalPolynomials` (see [5], [27]).

Remark 2 If we keep integration over $\mathbb{R}$ (instead of reducing to the positive half-line $(0,+\infty)$) and change the variable $t := t/2$, (9) can be reduced to an integral with the *logistic weight* $w^{\log}(t) = \mathrm{e}^{-t}/\left(1+\mathrm{e}^{-t}\right)^2$, i.e.,

$$T_m = -\int_{-\infty}^{+\infty} F\left(m - \frac{1}{2} + \mathrm{i}\frac{t}{2\pi}\right) w^{\log}(t)\,\mathrm{d}t,$$

for which the coefficients in the three-term recurrence relation for the corresponding orthogonal polynomials are known in the explicit form (see [22])

$$\alpha_k = 0, \quad \beta_0 = 1, \quad \beta_k = \frac{\pi^2 k^4}{4k^2-1}, \quad k \geq 1,$$

so we do not need a procedure to generate these coefficients. However, as was shown in [22], such Gaussian formulas over $\mathbb{R}$ converge considerably slower than the ones for one-sided integration.

Milovanović [26] has recently studied the slightly more complicated hyperbolic weight function

$$x \mapsto w(x) = \frac{1}{\sqrt{x}\cosh^2 \dfrac{\pi\sqrt{x}}{2}} \tag{15}$$

on $(0, +\infty)$ by the one in (10) and determined explicit expressions for the coefficients α_k and β_k in the corresponding three-term recurrence relation (13). In this way, he avoided the most difficult part of the construction, which is otherwise accompanied by numerical instability. These coefficients are given by

$$\alpha_0 = \frac{1}{3}, \quad \alpha_k = \frac{32k^4 + 32k^3 + 8k^2 - 1}{(4k-1)(4k+3)}, \quad \beta_k = \frac{16(2k-1)^4 k^4}{(4k-3)(4k-1)^2(4k+1)}, \tag{16}$$

for $k = 1, 2, \ldots$, and $\beta_0 = 4/\pi$.

Remark 3 Depending on the properties of the function f it is often appropriate to extract and calculate a finite number of initial terms in the series (5) and then apply the procedure only to the series starting with the index $k = m$. A rapidly increasing speed of convergence of a summation process is achieved when m is increased (see [22, 25]).

3 Properties of functions connected to Clausen's functions $\mathrm{Cl}_n(\theta)$

Let $z \mapsto \mathrm{Ci}(z)$ be a complex function defined as a *cosine integral* by

$$\mathrm{Ci}(z) = -\int_z^\infty \frac{\cos t}{t}\,\mathrm{d}t,$$

with a branch cut discontinuity in the complex z plane running from $-\infty$ to 0. In Wolfram's MATHEMATICA this function, implemented as `CosIntegral[z]`, is suitable for both symbolic and numerical manipulation, and it can be evaluated to arbitrary numerical precision.

In our computation of the Clausen functions (3) we define

$$f_{2\nu+1}(z, \theta) = \frac{\cos(\theta z)}{z^{2\nu+1}} \quad \text{and} \quad f_{2\nu+2}(z, \theta) = \frac{\sin(\theta z)}{z^{2\nu+2}}, \quad \nu = 0, 1, \ldots . \tag{17}$$

The summation/quadrature method needs their corresponding primitive functions (integrals) $F_{2\nu+1}(z, \theta)$ and $F_{2\nu+2}(z, \theta)$, $\nu = 0, 1, \ldots$, which vanish at infinity.

It is easy to see that

$$F_1(z,\theta) = \mathrm{Ci}(z\theta), \quad F_2(z,\theta) = \frac{1}{z}\left[\theta z\,\mathrm{Ci}(z\theta) - \sin(\theta z)\right], \tag{18}$$

$$F_3(z,\theta) = -\frac{1}{2z^2}\left[\theta^2 z^2\,\mathrm{Ci}(z\theta) + \cos(\theta z) - \theta z\,\sin(\theta z)\right],$$

$$F_4(z,\theta) = -\frac{1}{6z^3}\left[\theta^3 z^3\,\mathrm{Ci}(z\theta) + \left(2-\theta^2 z^2\right)\sin(\theta z) + \theta z\cos(\theta z)\right],$$

$$F_5(z,\theta) = \frac{1}{24z^4}\left[\theta^4 z^4\,\mathrm{Ci}(z\theta) + \theta z\left(2-\theta^2 z^2\right)\sin(\theta z) + \left(\theta^2 z^2 - 6\right)\cos(\theta z)\right],$$

$$F_6(z,\theta) = \frac{1}{120z^5}\Big[\theta^5 z^5\,\mathrm{Ci}(z\theta) + \theta z\left(\theta^2 z^2 - 6\right)\cos(\theta z) - \left(\theta^4 z^4 - 2\theta^2 z^2 + 24\right)\sin(\theta z)\Big].$$

In order to find the general formulas for the primitive functions $F_{2\nu+1}(z,\theta)$ and $F_{2\nu+2}(z,\theta)$, we first use integration by parts $\big(u=\cos(\theta z),\ \mathrm{d}v = z^{-(2\nu+1)}\mathrm{d}z\big)$, so that

$$F_{2\nu+1}(z,\theta) = \int \frac{\cos(\theta z)}{z^{2\nu+1}}\mathrm{d}z = -\frac{1}{2\nu}\cdot\frac{\cos(\theta z)}{z^{2\nu}} - \frac{\theta}{2\nu}\int\frac{\sin(\theta z)}{z^{2\nu}}\mathrm{d}z,$$

hence,

$$2\nu F_{2\nu+1}(z,\theta) = -\frac{\cos(\theta z)}{z^{2\nu}} - \theta F_{2\nu}(z,\theta). \tag{19}$$

Similarly, we have

$$F_{2\nu}(z,\theta) = \int\frac{\sin(\theta z)}{z^{2\nu}}\mathrm{d}z = -\frac{1}{2\nu-1}\cdot\frac{\sin(\theta z)}{z^{2\nu-1}} + \frac{\theta}{2\nu-1}\int\frac{\cos(\theta z)}{z^{2\nu-1}}\mathrm{d}z,$$

from which we get

$$(2\nu-1)F_{2\nu}(z,\theta) = -\frac{\sin\theta z}{z^{2\nu-1}} + \theta F_{2\nu-1}(z,\theta). \tag{20}$$

Thus, from (19) and (20), we conclude that the following recurrence relations hold,

$$F_{2\nu+1}(z,\theta) = \frac{1}{2\nu(2\nu-1)z^{2\nu}}\left\{-(2\nu-1)\cos(\theta z) + \theta z\sin(\theta z) - \theta^2 z^{2\nu}F_{2\nu-1}(z,\theta)\right\} \tag{21}$$

and

$$F_{2\nu+2}(z,\theta) = -\frac{1}{(2\nu+1)2\nu z^{2\nu+1}}\left\{2\nu\sin(\theta z) + \theta z\cos(\theta z) + \theta^2 z^{2\nu+1}F_{2\nu}(z,\theta)\right\}, \tag{22}$$

and we use them for proving the following general results:

Proposition 1 *Let $\xi = z\theta$, $0 < \theta < \pi$. The sequence of functions $\{F_n(z,\theta)\}_{n=1}^{+\infty}$ can be expressed as*

$$F_{2\nu+1}(z,\theta) = \frac{(-1)^\nu}{(2\nu)!z^{2\nu}}\Big\{\xi^{2\nu}\,\mathrm{Ci}(\xi) + a_{\nu-1}\big(\xi^2\big)\cos\xi - \xi b_{\nu-1}\big(\xi^2\big)\sin\xi\Big\} \tag{23}$$

and

$$F_{2\nu+2}(z,\theta) = \frac{(-1)^\nu}{(2\nu+1)!z^{2\nu+1}}\Big\{\xi^{2\nu+1}\,\mathrm{Ci}(\xi) + \xi a_{\nu-1}\big(\xi^2\big)\cos\xi - b_{\nu}\big(\xi^2\big)\sin\xi\Big\}, \tag{24}$$

with $a_{-1}(\xi) = b_{-1}(\xi) = 0$, where a_ν and b_ν are monic polynomials in ξ of degree ν (≥ 0), given by

$$a_\nu(\xi) = \sum_{j=0}^{\nu}(-1)^j(2j+1)!\xi^{\nu-j} \quad \text{and} \quad b_\nu(\xi) = \sum_{j=0}^{\nu}(-1)^j(2j)!\xi^{\nu-j}, \tag{25}$$

respectively.

Proof We put $\xi = z\theta$, where $0 < \theta < \pi$. For $\nu = 0$ the formulas (23) and (24) are true, because they reduce to (18).

Now, we consider the recurrence relation (21) when $\nu := \nu + 1$, i.e.,

$$F_{2\nu+3}(z,\theta) = \frac{1}{(2\nu+2)(2\nu+1)z^{2\nu+2}}\Big[-(2\nu+1)\cos\xi + \xi\sin\xi - \xi^2 z^{2\nu}F_{2\nu+1}(z,\theta)\Big],$$

and suppose that (23) holds for some ν (≥ 0). Then, we have

$$\begin{aligned} F_{2\nu+3}(z,\theta) &= \frac{1}{(2\nu+2)(2\nu+1)z^{2\nu+2}}\Big\{-(2\nu+1)\cos\xi + \xi\sin\xi \\ &\qquad - \frac{(-1)^\nu\xi^2}{(2\nu)!}\Big[\xi^{2\nu}\mathrm{Ci}(\xi) + a_{\nu-1}(\xi^2)\cos\xi - \xi b_{\nu-1}(\xi^2)\sin\xi\Big]\Big\} \\ &= \frac{(-1)^{\nu+1}}{(2\nu+2)!z^{2\nu+2}}\Big\{\xi^{2\nu+2}\mathrm{Ci}(\xi) + \big[(-1)^\nu(2\nu+1)! + \xi^2 a_{\nu-1}(\xi^2)\big]\cos\xi \\ &\qquad - \xi\big[(-1)^\nu(2\nu)! + \xi^2 b_{\nu-1}(\xi^2)\big]\sin\xi\Big\}. \end{aligned}$$

Since, according to (25),

$$(-1)^\nu(2\nu+1)! + \xi^2 a_{\nu-1}(\xi^2) = \sum_{j=0}^{\nu}(-1)^j(2j+1)!\xi^{2\nu-2j} = a_\nu\big(\xi^2\big)$$

and

$$(-1)^\nu(2\nu)! + \xi^2 b_{\nu-1}(\xi^2) = \sum_{j=0}^{\nu}(-1)^j(2j)!\xi^{2\nu-2j} = b_\nu\big(\xi^2\big),$$

we obtain

$$F_{2\nu+3}(z,\theta) = \frac{(-1)^{\nu+1}}{(2\nu+2)!z^{2\nu+2}}\Big\{\xi^{2\nu+2}\,\mathrm{Ci}(\xi) + a_\nu\big(\xi^2\big)\cos\xi - \xi b_\nu\big(\xi^2\big)\sin\xi\Big\},$$

thus completing the inductive proof of (23) for each ν (≥ 0).

In a similar way, using (22) for $\nu := \nu + 1$, we can prove the formula (24) for each ν (≥ 0). □

Remark 4 Note that $F_n(\infty,\theta)=0$, as well as that all terms in $F_n(z,\theta)$ tend to zero as $z\to\infty$. For example, for the terms in (23) $(n=2\nu+1)$ we have

$$\lim_{z\to\infty}\frac{\xi^{2\nu}\,\mathrm{Ci}(\xi)}{z^{2\nu}}=0,\quad \lim_{z\to\infty}\frac{a_{\nu-1}\big(\xi^2\big)\cos\xi}{z^{2\nu}}=0,\quad \lim_{z\to\infty}\frac{\xi b_{\nu-1}\big(\xi^2\big)\sin\xi}{z^{2\nu}}=0,$$

for each $0<\theta<\pi$, where $\xi=z\theta$, as well as for the terms in (24) where $n=2\nu+2$. Also, we can check that these functions $F_n(z,\theta)$, $n\ge 2$, satisfy the conditions (C1)–(C3), given at the beginning of Section 2.

Remark 5 The explicit form of the monic polynomials $\xi\mapsto a_\nu(\xi)$ and $\xi\mapsto b_\nu(\xi)$, $\nu=0,1,\ldots,8$, are

$$\begin{aligned}
&a_0(\xi)=1,\quad a_1(\xi)=\xi-6,\quad a_2(\xi)=\xi^2-6\xi+120,\\
&a_3(\xi)=\xi^3-6\xi^2+120\xi-5040,\quad a_4(\xi)=\xi^4-6\xi^3+120\xi^2-5040\xi+362880,\\
&a_5(\xi)=\xi^5-6\xi^4+120\xi^3-5040\xi^2+362880\xi-39916800,\\
&a_6(\xi)=\xi^6-6\xi^5+120\xi^4-5040\xi^3+362880\xi^2-39916800\xi+6227020800,\\
&a_7(\xi)=\xi^7-6\xi^6+120\xi^5-5040\xi^4+362880\xi^3-39916800\xi^2+6227020800\xi\\
&\qquad\quad-1307674368000,\\
&\alpha_8(\xi)=\xi^8-6\xi^7+120\xi^6-5040\xi^5+362880\xi^4-39916800\xi^3+6227020800\xi^2\\
&\qquad\quad-1307674368000\xi+355687428096000
\end{aligned}$$

and

$$\begin{aligned}
&b_0(\xi)=1,\quad b_1(\xi)=\xi-2,\quad b_2(\xi)=\xi^2-2\xi+24,\quad b_3(\xi)=\xi^3-2\xi^2+24\xi-720,\\
&b_4(\xi)=\xi^4-2\xi^3+24\xi^2-720\xi+40320,\\
&b_5(\xi)=\xi^5-2\xi^4+24\xi^3-720\xi^2+40320\xi-3628800,\\
&b_6(\xi)=\xi^6-2\xi^5+24\xi^4-720\xi^3+40320\xi^2-3628800\xi+479001600,\\
&b_7(\xi)=\xi^7-2\xi^6+24\xi^5-720\xi^4+40320\xi^3-3628800\xi^2+479001600\xi\\
&\qquad\quad-87178291200,\\
&b_8(\xi)=\xi^8-2\xi^7+24\xi^6-720\xi^5+40320\xi^4-3628800\xi^3+479001600\xi^2\\
&\qquad\quad-87178291200\xi+20922789888000,
\end{aligned}$$

respectively.

4 Calculation of Clausen's functions $\mathrm{Cl}_n(\theta)$ using quadrature formulas

According to (17) and (5), Clausen's functions $\mathrm{Cl}_n(\theta)$, defined in (3), can be written as

$$\mathrm{Cl}_n(\theta)=\sum_{k=1}^{+\infty}f_n(k,\theta)=\sum_{k=1}^{m-1}f_n(k,\theta)+\sum_{k=m}^{+\infty}f_n(k,\theta),\tag{26}$$

where a finite sum of the first $m-1$ terms is extracted. Using (10) and (11), the infinite series on the right hand side in (26) can be transformed to

$$\sum_{k=m}^{+\infty}f_n(k,\theta)=\int_0^{+\infty}\Phi_n\left(m-\frac{1}{2},\frac{t}{\pi},\theta\right)\frac{\mathrm{d}t}{\cosh^2 t},\tag{27}$$

where

$$\Phi_n(x,y,\theta) = -\frac{1}{2}\left[F_n(x+\mathrm{i}y,\theta)+F_n(x-\mathrm{i}y,\theta)\right], \tag{28}$$

and the functions $z \mapsto F_n(z,\theta)$ are given in Proposition 1.

Theorem 1 *Let $z \mapsto f_n(z,\theta)$ and $z \mapsto F_n(z,\theta)$ be functions of the complex variable, defined in* (17) *and in Proposition 1 by* (23)–(25)*, respectively. If $\left(\xi_k^{(N)}, A_k^{(N)}\right)$, $k = 1,\ldots,N$, are the parameters* (*nodes and weights*) *of the N-point Gaussian quadrature rule with respect to the weight function* (15) *on* $(0,+\infty)$*, then*

$$\mathrm{Cl}_n(\theta) = Q_n^{N,m}(\theta) + E_n^{N,m}(\theta), \tag{29}$$

where

$$Q_n^{N,m}(\theta) = \sum_{k=1}^{m-1} f_n(k,\theta) - \frac{\pi}{4}\sum_{\nu=1}^{N} A_\nu^{(N)} \mathrm{Re}\left\{F_n\left(m-\frac{1}{2}+\frac{\mathrm{i}}{2}\sqrt{\xi_\nu^{(N)}},\theta\right)\right\} \tag{30}$$

and $E_n^{N,m}(\theta)$ is the error in the Gaussian quadrature formula depending on number of points N, the number of extracted terms, and θ.

Proof We start with (27) and change the variable $t = \pi\sqrt{x}/2$. Then (27) becomes

$$\sum_{k=m}^{+\infty} f_n(k,\theta) = \frac{\pi}{4}\int_0^{+\infty} \Phi_n\left(m-\frac{1}{2},\frac{\sqrt{x}}{2},\theta\right)\frac{\mathrm{d}x}{\sqrt{x}\cosh^2\dfrac{\pi\sqrt{x}}{2}}.$$

Now, because of (26) and (28), we get

$$\mathrm{Cl}_n(\theta) = \sum_{k=1}^{m-1} f_n(k,\theta) - \frac{\pi}{4}\int_0^{+\infty} \mathrm{Re}\left\{F_n\left(m-\frac{1}{2}+\mathrm{i}\frac{\sqrt{x}}{2},\theta\right)\right\} w(x)\mathrm{d}x,$$

where $w(x)$ is the weight function given by (15). Using the N-point Gaussian quadrature formula with respect to this weight function (with the nodes $\xi_k^{(N)}$ and the weights $A_k^{(N)}$, $k = 1,\ldots,N$), we obtain the desired result. □

Since the recursion coefficients for the respective orthogonal polynomials with the weight function (15) on $(0,+\infty)$ are known in explicit form, an application of the Golub-Welsch algorithm [13] to the symmetric tridiagonal Jacobi matrix (12), with α_k and β_k given in (16), provide us with the quadrature parameters $\left(\xi_k^{(N)}, A_k^{(N)}\right)$, $k = 1,\ldots,N$, of the corresponding N-point Gaussian quadrature rule.

Remark 6 The Golub-Welsch algorithm is included in the MATHEMATICA package `OrthogonalPolynomials` (see [5], [27]), so that we need only one command `"aGaussianNodesWeights"`, with the corresponding parameters. In Tables 1 and 2 we give the parameters of the Gaussian quadratures $\left(\xi_k^{(N)}, A_k^{(N)}\right)$, $k = 1,\ldots,N$, for $N = 10$ and $N = 20$, rounded to 18 and 35 decimal digits, respectively. Numbers in parentheses indicate decimal exponents.

ν	$\xi_\nu^{(N)}$	$A_\nu^{(N)}$
1	1.02058363572825669(−1)	1.03011687962788351
2	1.10087335344093285	2.25802208525057270(−1)
3	3.87538426676163313	1.67040618192678355(−2)
4	9.49089900532915982	6.04607037721632010(−4)
5	1.92369217680950509(+1)	1.16718411051188568(−5)
6	3.48489807870071239(+1)	1.15360113152844190(−7)
7	5.88422971202513756(+1)	5.23125205372251291(−10)
8	9.52057687864990326(+1)	8.88585842372452499(−13)
9	1.51245498588511132(+2)	3.78823474931504689(−16)
10	2.44769266678480451(+2)	1.45749170427449731(−20)

Table 1 Gaussian parameters: nodes $\xi_\nu^{(N)}$ and weights $A_\nu^{(N)}$, $\nu = 1, 2, \ldots, N$, for $N = 10$

ν	$\xi_\nu^{(N)}$	$A_\nu^{(N)}$
1	7.833274118021773985942959203421512 1(−2)	9.452184477021381352273949875108444 6(−1)
2	7.994805575802808688850430596576445 5(−1)	2.867373091000518027256673420868165 8(−1)
3	2.633535984370767605306302460068185 1	3.799071588613489203535639981405482 2(−2)
4	6.067125354800522713778706639834993 9	3.105368885793693681274890110374275 6(−3)
5	1.157801430801497286725204216097385 3(+1)	1.797952398563473526796541359017282 8(−4)
6	1.967574219682007399786022352565215 5(+1)	7.659906125418737564460511363627378 5(−6)
7	3.092984029327314426292033841364265 1(+1)	2.422439802111304250254419452081149 9(−7)
8	4.598890514567295807853142947036874 6(+1)	5.672464161661686603385118458719734 7(−9)
9	6.560285479680404034132525111840303 9(+1)	9.740052786086906643974014704473124 2(−11)
10	9.065215290372941319400720493956387 2(+1)	1.206882452454118503417979373450382 6(−12)
11	1.221877334669234815296730432128338 5(+2)	1.054992916074715193006563744491290 4(−14)
12	1.614876365903447488423575894129443 6(+2)	6.310786164952170882446597654284721 7(−17)
13	2.101406248787482514044326311807549 3(+2)	2.480674096194818608279088970715087 9(−19)
14	2.701753845342719824863204315494047 8(+2)	6.068944075643647849080969319008971 0(−22)
15	3.442714685319780377460114515566925 4(+2)	8.577099728671171031948567203976207 8(−25)
16	4.361286415924854688584081300800657 2(+2)	6.296750414836763110694582719147875 0(−28)
17	5.511767063988322601219426308520696 7(+2)	2.045570607190395882317996000414779 5(−31)
18	6.981320809615157311200688193700647 6(+2)	2.258006419692752335532584347360038 0(−35)
19	8.931889848952827736674475594859238 0(+2)	5.121471290722713488311623944189831 4(−40)
20	1.176573108297750619389054779893729 8(+3)	6.655405593345552125656015293110494 0(−46)

Table 2 Gaussian parameters: nodes $\xi_\nu^{(N)}$ and weights $A_\nu^{(N)}$, $\nu = 1, 2, \ldots, N$, for $N = 20$

Remark 7 According to [21, Eq. (2.2.5)] and (16), we have

$$\|\pi_N\|^2 = \beta_0\beta_1\cdots\beta_N = \frac{4}{\pi}\prod_{k=1}^{N}\frac{16(2k-1)^4k^4}{(4k-3)(4k-1)^2(4k+1)},$$

that is

$$\|\pi_N\|^2 = \frac{2^{1-4N}\Gamma(2N+1)^4}{\Gamma\left(2N+\frac{1}{2}\right)\Gamma\left(2N+\frac{3}{2}\right)} \le \frac{(2N)!^2}{2^{4N-1}},$$

due to the inequality $\Gamma(2N+1)^2 \le \Gamma(2N+1/2)\,\Gamma(2N+3/2)$ (the logarithmic convexity of Euler's gamma function).

Since

$$\begin{aligned}\frac{\partial}{\partial x}\Phi_n\Big(m-\frac{1}{2},\frac{\sqrt{x}}{2},\theta\Big) &= \frac{\partial}{\partial x}\left\{-\frac{1}{2}\left[F_n\Big(m-\frac{1}{2}+\mathrm{i}\frac{\sqrt{x}}{2},\theta\Big)+F_n\Big(m-\frac{1}{2}-\mathrm{i}\frac{\sqrt{x}}{2},\theta\Big)\right]\right\}\\ &= \frac{1}{8\mathrm{i}\sqrt{x}}\left[f_n\Big(m-\frac{1}{2}+\mathrm{i}\frac{\sqrt{x}}{2},\theta\Big)-f_n\Big(m-\frac{1}{2}-\mathrm{i}\frac{\sqrt{x}}{2},\theta\Big)\right],\end{aligned}$$

where

$$f_n(z,\theta)=\frac{\partial}{\partial z}F_n(z,\theta)=\frac{1}{z^n}\begin{cases}\cos\theta z, & n \text{ is odd},\\ \sin\theta z, & n \text{ is even},\end{cases}$$

we obtain the error estimate in (29) in the form

$$\left|E_n^{N,m}(\theta)\right| \le \frac{(2N)!\pi}{2^{4N+3}}\max_{x>0}\left|\frac{\partial^{2N-1}}{\partial x^{2N-1}}\left\{\frac{1}{\sqrt{x}}f_n\Big(\frac{1}{2}(2m-1+\mathrm{i}\sqrt{x}),\theta\Big)\right\}\right|,$$

but such an estimate has no practical value. A direct calculation of the error term will be given in the next section.

5 Numerical experiments

In this section we show the application of the formula (30) and present the error term $\theta \mapsto E_n^{N,m}(\theta)$ as a function of $\theta \in [0,\pi]$. For calculating the quadrature approximations $Q_n^{N,m}(\theta)$ of $\mathrm{Cl}_n(\theta)$ for $n = 2,3,4,5,6$ we take the number of quadrature nodes $N = 10,20,30$ and $m = 1,2,3,6,11,16$ (the number of extracted terms is $m-1$). These graphics for

$$\mathrm{Err}_n(\theta) \equiv |E_n^{N,m}(\theta)| = \left|Q_n^{N,m}(\theta)-\mathrm{Cl}_n(\theta)\right|, \quad n = 2,3,\dots,6,$$

are given in Figures 1,2, …, 5, respectively. As the exact values $\mathrm{Cl}_n(\theta)$ we take values obtained by using the Mathematica package, with `WorkingPrecision->90`. Alternatively, we can use the quadrature approximations for sufficiently large N and m (here $N = 100$ and $m = 21$). All computations were performed in Mathematica, Ver. 13.2 on MacOS Ventura 13.1.

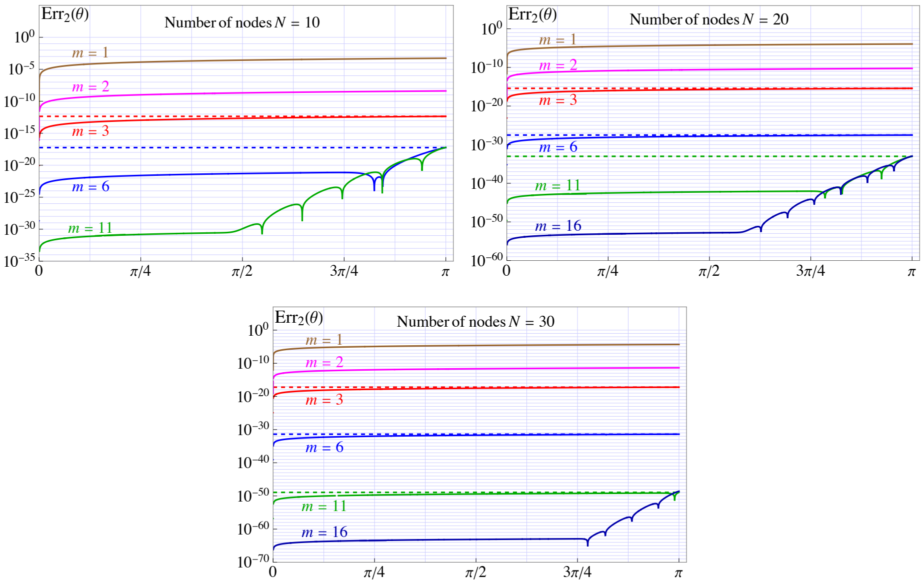

Fig. 1 Absolute errors $\mathrm{Err}_2(\theta)$ in quadrature approximations $Q_2^{N,m}(\theta)$ of $\mathrm{Cl}_2(\theta)$, $0 \le \theta \le \pi$, for $N = 10, 20, 30$ quadrature nodes and some selected m

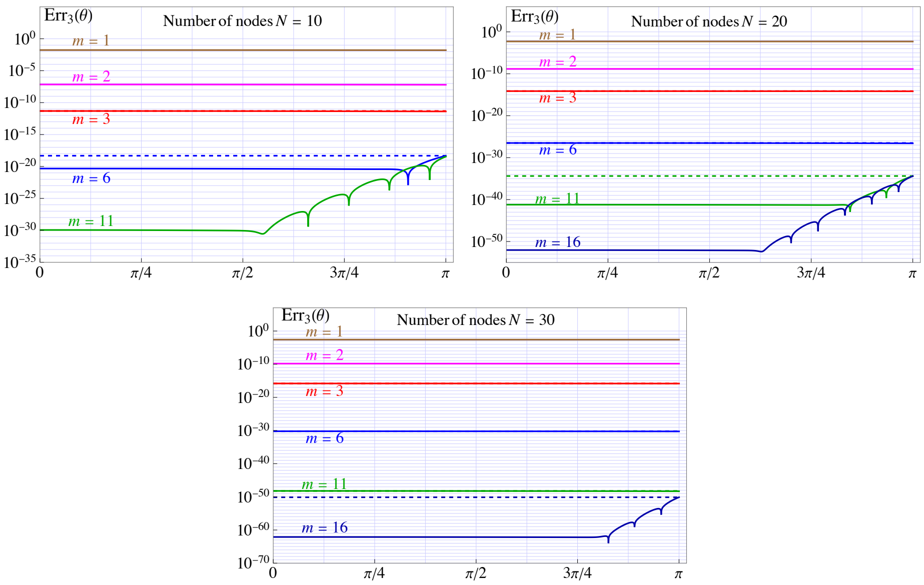

Fig. 2 Absolute errors $\mathrm{Err}_3(\theta)$ in quadrature approximations $Q_3^{N,m}(\theta)$ of $\mathrm{Cl}_3(\theta)$, $0 \le \theta \le \pi$, for $N = 10, 20, 30$ quadrature nodes and some selected m

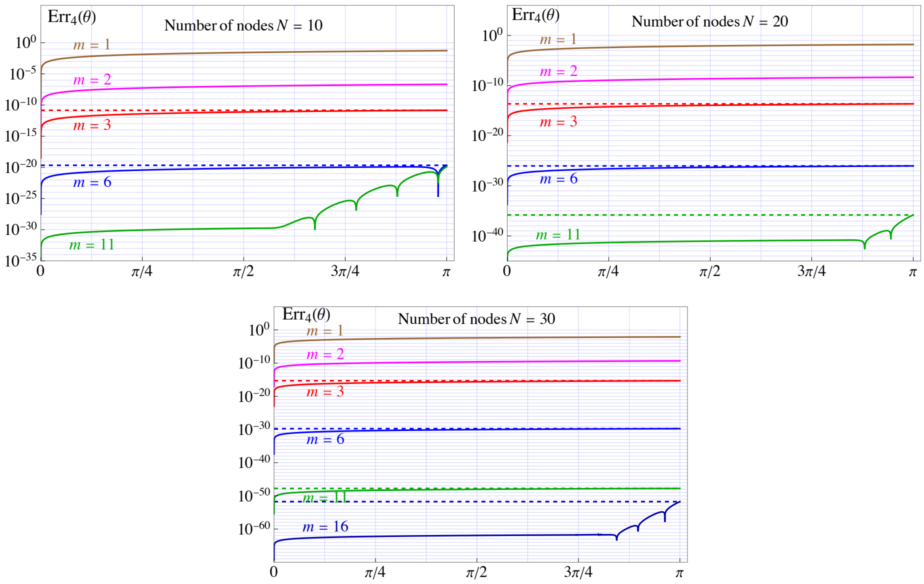

Fig. 3 Absolute errors $\mathrm{Err}_4(\theta)$ in quadrature approximations $Q_4^{N,m}(\theta)$ of $\mathrm{Cl}_4(\theta)$, $0 \le \theta \le \pi$, for $N = 10, 20, 30$ quadrature nodes and some selected m

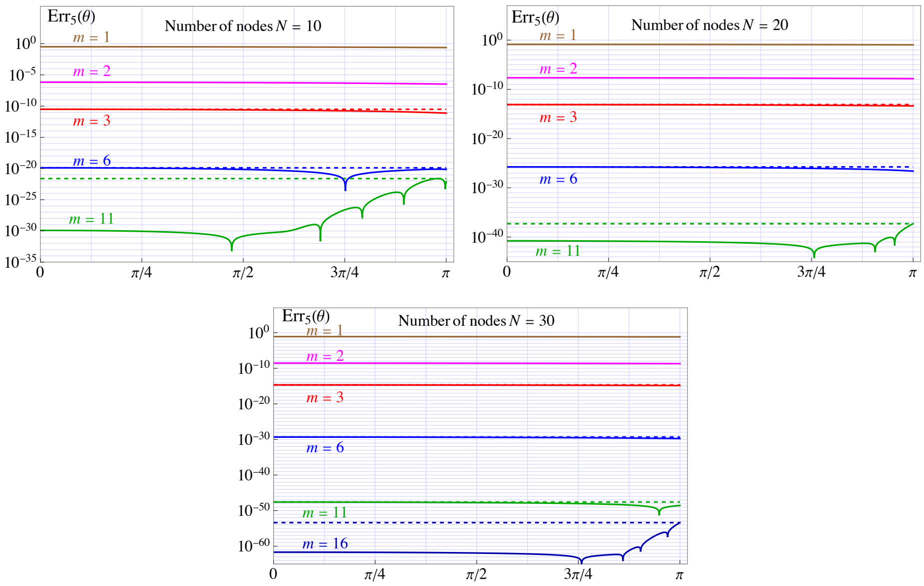

Fig. 4 Absolute errors $\mathrm{Err}_5(\theta)$ in quadrature approximations $Q_5^{N,m}(\theta)$ of $\mathrm{Cl}_5(\theta)$, $0 \le \theta \le \pi$, for $N = 10, 20, 30$ quadrature nodes and some selected m

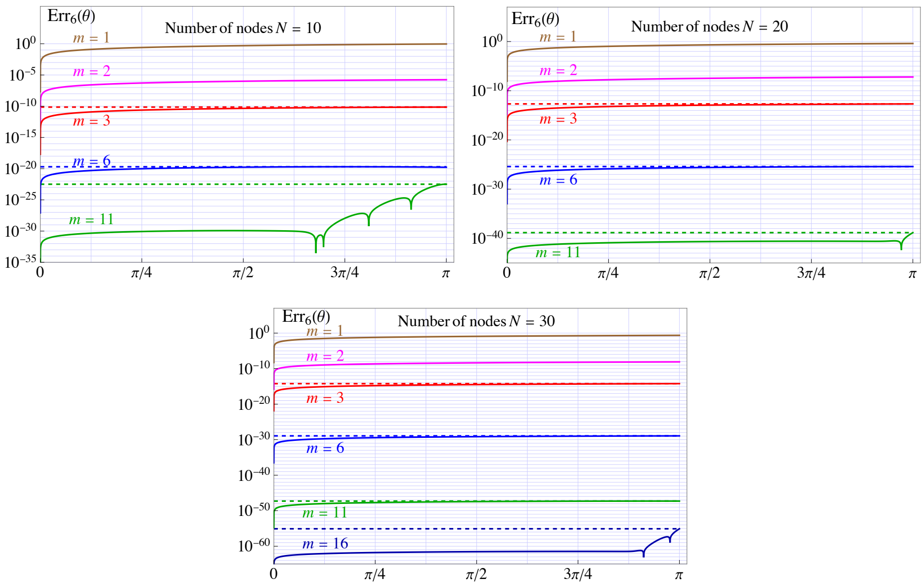

Fig. 5 Absolute errors $\mathrm{Err}_6(\theta)$ in quadrature approximations $Q_6^{N,m}(\theta)$ of $\mathrm{Cl}_6(\theta)$, $0 \le \theta \le \pi$, for $N = 10, 20, 30$ quadrature nodes and some selected m

From the obtained graphics in Figures 1–5 we can determine the error bounds, i.e., the maximal values of the errors, $\max_{0\leq\theta\leq\pi} \mathrm{Err}_n(\theta)$, for some typical vales of the number of nodes N in the quadrature formula (30), as well as of the number of the extracted terms $(m-1)$ in the series. These error bounds are presented in Table 3.

n	N	$m=3$	$m=6$	$m=11$	$m=16$
2	10	4.39×10^{-13}	5.66×10^{-18}		
	20	3.80×10^{-16}	3.08×10^{-28}	9.52×10^{-34}	
	30	6.03×10^{-18}	4.06×10^{-32}	1.18×10^{-49}	
3	10	4.88×10^{-12}	4.82×10^{-19}		
	20	7.39×10^{-15}	3.15×10^{-27}	4.19×10^{-35}	
	30	1.44×10^{-16}	5.30×10^{-31}	4.87×10^{-49}	7.42×10^{-51}
4	10	1.46×10^{-11}	2.16×10^{-20}		
	20	2.26×10^{-14}	9.39×10^{-27}	1.55×10^{-36}	
	30	4.74×10^{-16}	1.80×10^{-30}	1.76×10^{-48}	1.85×10^{-52}
5	10	3.09×10^{-11}	1.29×10^{-20}	2.42×10^{-22}	
	20	7.92×10^{-14}	1.76×10^{-26}	5.08×10^{-38}	
	30	2.14×10^{-15}	4.68×10^{-30}	2.58×10^{-48}	4.08×10^{-54}
6	10	7.24×10^{-11}	2.00×10^{-20}	3.22×10^{-23}	
	20	2.11×10^{-13}	3.92×10^{-26}	1.45×10^{-39}	
	30	5.93×10^{-15}	1.17×10^{-29}	5.08×10^{-48}	7.98×10^{-56}

Table 3 Error bounds $\max_{0\leq\theta\leq\pi} \mathrm{Err}_n(\theta)$ for quadrature approximations $Q_n^{N,m}(\theta)$ of $\mathrm{Cl}_n(\theta)$, $0\leq\theta\leq\pi$, for $N=10,20,30$ quadrature nodes and some selected values of m

It can be seen that we can get all values of the Clausen functions $\mathrm{Cl}_n(\theta)$ with at least 17 correct decimal digits (D-arithmetic) if we take $N=10$ nodes in the quadrature formula (30) (see Table 1 for the corresponding parameters, nodes and weight coefficients) and $m=6$, i.e., five extracted terms. The number of correct decimal digits increases when the order n of Clausen's function $\mathrm{Cl}_n(\theta)$ becomes larger. For getting exact results in the so-called Q-arithmetic (with about 33 decimal digits) we need a quadrature formula with $N=20$ nodes (see Table 2 for nodes and weight coefficients) and $m=11$ (ten extracted terms).

The corresponding graphics of $\mathrm{Cl}_n(\theta)$, $n=2,\dots,6$, for $\theta\in[0,\pi]$ are displayed in Figure 6 (left).

Example 1 As an illustration of our method we consider the integral (see [4, Theorem 1] and Figure 6 (right)),

$$I=\int_{\pi/3}^{\pi/2}\log\left|\frac{\tan t+\sqrt{7}}{\tan t-\sqrt{7}}\right|\mathrm{d}t=\mathrm{Cl}_2(\theta_+)+\frac{1}{2}\left[\mathrm{Cl}_2(2\omega_+)-\mathrm{Cl}_2(2\omega_++2\theta_+)\right],\quad(31)$$

where $\theta_+=\arctan(\sqrt{7}/3)$ and $\omega_+=\arctan(\sqrt{7})-2\pi/3$. This and other related integrals originate from hyperbolic geometry and quantum field theory (cf. [2], [20]).

For calculating the values of $\mathrm{Cl}_2(\theta)$ we use the quadrature approximation $Q_2^{N,m}(\theta)$ in D-arithmetic (with $N=10$ and $m=6$) and Q-arithmetic (with $N=20$ and $m=11$).

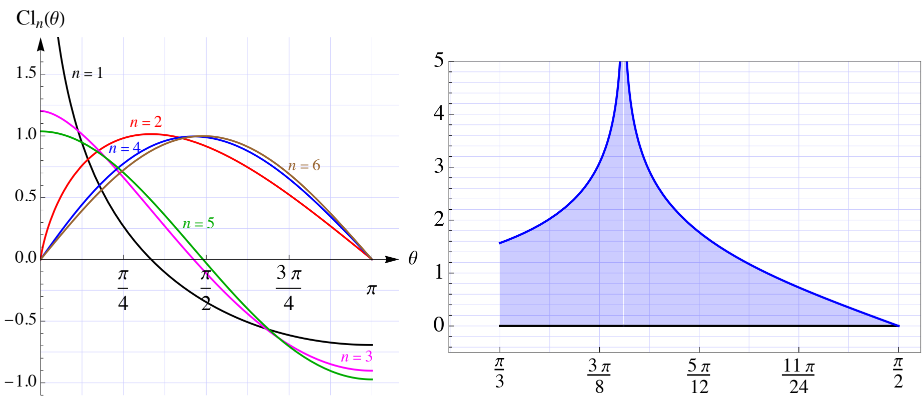

Fig. 6 (left) The Clausen functions $\theta \mapsto \mathrm{Cl}_n(\theta)$, $n = 1(1)6$, for $0 \le \theta \le \pi$; (right) The integrand of I

Since $\tau_1 = \theta_+ \in (0,\pi)$, $\tau_2 = 2\omega_+ \in (-\pi,0)$, $\tau_3 = \tau_2 + 2\tau_1 \in (-\pi,0)$, and $\mathrm{Cl}_2(-\theta) = -\mathrm{Cl}_2(\theta)$, the integral I can be expressed in the form

$$I = \mathrm{Cl}_2(\tau_1) - \frac{1}{2}\left[\mathrm{Cl}_2(-\tau_2) - \mathrm{Cl}_2(-\tau_3)\right],$$

In this way, we get the approximation $\tilde{I}$ of the previous integral I in D- and Q-arithmetic (see Table 4) and, as we can see, the obtained results are accurate to the level of machine precision!

θ	$Q_2^{10,6}(\theta)$ (D-arithmetic)	$Q_2^{20,11}(\theta)$ (Q-arithmetic)
τ_1	0.962673014616618041	0.96267301461661804142143261997207522
$-\tau_2$	0.837664473558190622	0.83766447355819062193124505652118547
$-\tau_3$	0.690148299957661066	0.69014829995766106628618812498413506
$\tilde{I}$	0.88891492781635326	0.8889149278163532635989041542035500

Table 4 Approximation of the integral I in D- and Q-arithmetic

Example 2 In a paper on massive 3-loop Feynman diagrams reducible to SC* primitives of algebras of the sixth root of unity, Broadhurst [2] mentioned the so-called *two-loop constant* defined by

$$S_2 = \sum_{n=0}^{\infty} \frac{2n+1}{(3n+1)^2(3n+2)^2} = \frac{2}{3^{3/2}}\mathrm{Cl}_2(2\pi/3) = \frac{4}{3^{5/2}}\mathrm{Cl}_2(\pi/3).$$

The relative errors of the partial sums $S_2(N)$ (from $n = 0$ to $n = N = 10^k$) of this series are 3.91×10^{-4}, 4.65×10^{-6}, 4.73×10^{-8}, 4.74×10^{-10}, 4.74×10^{-12}, when $k = 1, 2, 3, 4, 5$, respectively.

Using the Laplace transform method ([12], [25]), we can reduce this series to

$$S_2 = \frac{1}{4} + \int_0^{\infty} \frac{t}{\mathrm{e}^t - 1} f(t)\,\mathrm{d}t,$$

where $t \mapsto f(t)$ is the inverse Laplace transform of the function $p \mapsto F(p)$, and $-F'(p) = (2p+1)/((3p+1)^2(3p+2)^2)$. In our case, $f(t) = \frac{2}{27}\mathrm{e}^{-t/2}\sinh(t/6)$.

Now, applying Gaussian quadrature with respect to the Bose-Einstein weight function $t \mapsto t/(\mathrm{e}^t - 1)$ on $(0, \infty)$ to the last integral with N nodes, e.g., when $N = 5, 10, 15, \ldots$ (see Table 5 of the Appendix in [12]), we get a sequence of the quadrature approximations with a fast convergence. For example, for $N = 10$ and $N = 20$ nodes, the relative errors in the the quadrature approximations are 4.57×10^{-12} and 1.03×10^{-23}, respectively.

On the other hand our method is much more efficient. As before (Example 2), the quadrature approximations $4\times3^{-5/2}Q_2^{10,6}(\pi/3)$ and $4\times3^{-5/2}Q_2^{20,11}(\pi/3)$ of the sum S_2 are accurate at the level of machine precision in D- and Q-arithmetic, respectively. However, if we take sufficiently high arithmetic, e.g., `WorkingPrecision -> 24` for the first quadrature approximation the relative error is 9.89×10^{-22}, while with `WorkingPrecision -> 43` for the second approximation it is 1.20×10^{-42}.

Acknowledgements

The authors are deeply grateful to the referees for their valuable comments and constructive suggestions for improvements of this paper and its better presentation.

Declarations

Ethical Approval Not applicable.

Conflicts of interest The authors declared that they have no conflicts of interest to this work.

Authors' contributions The authors contributed equally to the writing of this paper. They read and approved the final manuscript.

Funding The first author was partially supported by MPNTR grant No. 174017, Serbia. The work of the second author was supported in part by the Serbian Academy of Sciences and Arts, Belgrade (Project Φ-96).

Availability of data and materials Not applicable.

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