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ON A CLASS OF COMPLEX POLYNOMIALS HAVING ALL ZEROS IN A HALF DISC

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ABSTRACT:

We study the location of the zeros of the polynomial $p_n(z) = \pi_n(z) - i\,\theta_{n-1}\pi_{n-1}(z)$, *where* $\{\pi_k\}$ *is a system of monic polynomials orthogonal with respect to an even weight function on* $(-a,a)$, $0<a<\infty$, *and* θ_{n-1} *is a real constant. We show that all zeros of* p_n *lie in the upper half disc* $|z|<a \wedge \operatorname{Im} z>0$, *if* $0<\theta_{n-1}<\pi_n(a)/\pi_{n-1}(a)$, *and in the lower half disc* $|z|<a \wedge \operatorname{Im} z<0$, *if* $-\pi_n(a)/\pi_{n-1}(a)<\theta_{n-1}<0$. *The ultraspherical weight function is considered as an example.*

O KLASI KOMPLEKSNIH POLINOMA KOJI IMAJU SVE NULE U POLUKRUGU. *U radu se razmatra problem lokalizacije nula polinoma* $p_n(z) = \pi_n(z) - i\theta_{n-1}\pi_{n-1}(z)$, *gde je* $\{\pi_k\}$ *sistem moničnih polinoma ortogonalnih u odnosu na parnu težinsku funkciju na* $(-a,a)$, $0<a<\infty$, *a* θ_{n-1} *realna konstanta. Dokazujemo da sve nule polinoma* p_n *leže u gornjem polukrugu* $|z|<a \wedge \operatorname{Im} z>0$, *ako je* $0<\theta_{n-1}<\pi_n(a)/\pi_{n-1}(a)$, *a u donjem polukrugu* $|z|<a \wedge \operatorname{Im} z<0$, *ako je* $-\pi_n(a)/\pi_{n-1}(a)<\theta_{n-1}<0$. *Kao primer razmatrana je ultrasferna težinska funkcija.*

1. INTRODUCTION

In a series of papers, Specht [2] studied the location of the zeros of polynomials expressed as linear combinations of orthogonal polynomials. He obtained various bounds for the modulus of the imaginary part of an arbitrary zero in terms of the expansion coefficients and certain quantities depending only on the respective orthogonal polynomials. Giroux [1] sharpened some of these results by providing bounds for the sum of the moduli of the imaginary parts of all zeros. In the process of doing so, he also stated as a corollary the following result.

Theorem A. *Let*

$$f(x) = (x-x_1)(x-x_2)\ldots(x-x_n),$$

$$g(x) = (x-y_1)(x-y_2)\ldots(x-y_n),$$

with $x_1 < y_1 < x_2 < \ldots < y_{n-1} < x_n$. *Then, for any real number* c, *the zeros of the polynomial* $h(x) = f(x) + icg(x)$ *are all in the half strip* $\operatorname{Im} z \geq 0$, $x_1 \leq \operatorname{Re} z \leq x_n$, *or all are in the conjugate half strip.*

Here we consider special linear combinations of the form

$$p_n(z) = \pi_n(z) - i\theta_{n-1}\pi_{n-1}(z), \tag{1.1}$$

where $\{\pi_k\}$ is a system of monic polynomials orthogonal with respect to an even weight function on $(-a,a)$, $0 < a < \infty$, and θ_{n-1} is a real constant. We combine Theorem A with Rouché's theorem to show, in this case, that all zeros of p_n, under appropriate restrictions on θ_{n-1}, are contained in a half disc of radius a. The result is illustrated in the case of Gegenbauer polynomials.

2. LOCATION OF THE ZEROS OF $p_n(z)$

Let $\omega(x)$ be an even weight function on $(-a,a)$, $0<a<\infty$. Then the monic polynomials orthogonal with respect to $\omega(x)$ satisfy a three-term recurrence relation of the form

$$\begin{cases} \pi_{k+1}(z) = z\pi_k(z) - \beta_k\pi_{k-1}(z), & k=0,1,\ldots, \\ \pi_{-1}(z) = 0, \quad \pi_0(z) = 1, \end{cases} \tag{2.1}$$

where $\beta_k > 0$. Since $\pi_k(-z) = (-1)^k\pi_k(z)$, $k=0,1,\ldots$, the polynomial (1.1) can be expanded in the form

$$p_n(z) = z^n - i\theta_{n-1}z^{n-1} + \ldots,$$

so that

$$\sum_{k=1}^{n} \zeta_k = i\theta_{n-1},$$

hence

$$\sum_{k=1}^{n} \operatorname{Im}\zeta_k = \theta_{n-1},$$

here $\zeta_1, \zeta_2, \ldots, \zeta_n$ are the zeros of the polynomial (1.1).

By Theorem A and (2.2) all zeros of the polynomial (1.1) lie in the half strip

(2.3) $\quad \operatorname{Im} z > 0, \quad -a < \operatorname{Re} z < a \quad \text{if } \theta_{n-1} > 0,$

or

(2.3′) $\quad \operatorname{Im} z < 0, \quad -a < \operatorname{Re} z < a \quad \text{if } \theta_{n-1} < 0,$

strict inequality holding in the imaginary part, since $p_n(z)$ for $\theta_{n-1} \neq 0$ cannot have real zeros. Of course, if $\theta_{n-1} = 0$, all zeros lie in $(-a,a)$.

Let D_a be the disc $D_a = \{z : |z| < a\}$ and ∂D_a its boundary. We first prove the following auxiliary result.

Lemma. For each $z \in \partial D_a$ one has

(2.4) $$\left| \frac{\pi_k(z)}{\pi_{k-1}(z)} \right| \geqq \frac{\pi_k(a)}{\pi_{k-1}(a)}, \qquad k = 1,2,\ldots .$$

Proof. Let $r_k(z) = \pi_k(z)/\pi_{k-1}(z)$ and $z \in \partial D_a$. We seek lower bounds r_k (not depending on z) of $|r_k(z)|$ for $z \in \partial D_a$,

$$|r_k(z)| \geqq r_k, \quad z \in \partial D_a.$$

From the recurrence relation (2.1) there follows

(2.5) $$r_k(z) = z - \frac{\beta_{k-1}}{r_{k-1}(z)}, \qquad k = 2,3,\ldots,$$

where $r_1(z) = z$. We can take, therefore,

(2.6) $$r_1 = a, \quad r_k = a - \frac{\beta_{k-1}}{r_{k-1}}, \quad k = 2,3,\ldots .$$

Using the usual notation of continued fraction, we obtain from (2.6)

$$r_k = a - \frac{\beta_{k-1}}{a -}\;\frac{\beta_{k-2}}{a -}\;\cdots\;\frac{\beta_1}{a} .$$

It is easily seen that $r_k = \pi_k(a)/\pi_{k-1}(a)$, $k \geqq 1$. Indeed, using (2.1),

$$r_k = a - \frac{\beta_{k-1}}{r_{k-1}} = a - \beta_{k-1}\frac{\pi_{k-2}(a)}{\pi_{k-1}(a)} = \frac{\pi_k(a)}{\pi_{k-1}(a)}. \square$$

By a similar argument one could show that

$$2a - \frac{\pi_k(a)}{\pi_{k-1}(a)} \geqq \left|\frac{\pi_k(z)}{\pi_{k-1}(z)}\right|, \quad z \in \partial D_a,$$

but this will not be needed in the following.

Theorem. _If the constant_ θ_{n-1} _satisfies_ $0 < \theta_{n-1} < \pi_n(a)/\pi_{n-1}(a)$, _then all zeros of the polynomial_ (1.1) _lie in the upper half disc_

$$|z| < a \quad \wedge \quad \operatorname{Im} z > 0 .$$

_If_ $-\pi_n(a)/\pi_{n-1}(a) < \theta_{n-1} < 0$, _then all zeros of_ (1.1) _are in the lower half disc_

$$|z| < a \quad \wedge \quad \operatorname{Im} z < 0 .$$

Proof. By (2.4) we have

$$\left|\frac{\pi_n(z)}{\pi_{n-1}(z)}\right| \geqq \frac{\pi_n(a)}{\pi_{n-1}(a)}, \quad z \in \partial D_a, \tag{2.7}$$

hence, if $\pi_n(a)/\pi_{n-1}(a) > |\theta_{n-1}|$,

$$|\pi_n(z)| > |\theta_{n-1}\pi_{n-1}(z)|, \quad z \in \partial D_a .$$

Applying Rouché's theorem to (1.1), we conclude that all zeros of the polynomial p_n lie in the open disc D_a. Combining this with (2.3) or (2.3′), we obtain the assertions of the theorem. $\square$

3. EXAMPLE: GEGENBAUER POLYNOMIALS

We now consider the ultraspherical weight function $\omega(x) = (1-x^2)^{\lambda-1/2}$ $(\lambda > -1/2)$ on $(-1,1)$. In this case, $a=1$, and $\pi_k(z) = \frac{k!}{2^k(\lambda)_k} C_k^\lambda(x)$, where $C_k^\lambda(x)$ is the Gegenbauer polynomial and $(\lambda)_k$ Pochhammer's symbol, $(\lambda)_k = \lambda(\lambda+1)\ldots(\lambda+k-1)$.

Since

$$\frac{\pi_k(1)}{\pi_{k-1}(1)} = \frac{k}{2(\lambda+k-1)} \cdot \frac{C_k^\lambda(1)}{C_{k-1}^\lambda(1)} = \frac{2\lambda+k-1}{2(\lambda+k-1)}$$

and

$$D^m C_k^\lambda(x) = 2^m(\lambda)_m C_{k-m}^{\lambda+m}(x), \qquad m \leq k,$$

where D is the differentiation operator, our theorem implies the following

Corollary. *Let $\pi_k(z)$ denote the monic Gegenbauer polynomial of degree* k *with parameter* λ. *If the constant* θ_{n-1} *satisfies* $0 < \theta_{n-1} < \dfrac{2\lambda+n-1}{2(\lambda+n-1)}$, *then all zeros of the polynomial* [illegible]$_n(z) = \pi_n(z) - i\theta_{n-1}\pi_{n-1}(z)$ *and of its derivatives lie in the upper half disc* $|z|<1 \wedge \operatorname{Im} z > 0$. *If* $-\dfrac{2\lambda+n-1}{2(\lambda+n-1)} < \theta_{n-1} < 0$, *then they are all in the lower half disc* $|z|<1 \wedge \operatorname{Im} z < 0$.

The upper bound $(2\lambda+n-1)/(2(\lambda+n-1))$ for $|\theta_{n-1}|$ becomes $n/(2n-1)$ in the case of Legendre polynomials ($\lambda=1/2$), and $1/2$ in the case of Chebyshev polynomials ($\lambda=0$).

REFERENCES

1. GIROUX A.: *Estimates for the imaginary parts of the zeros of a polynomial.* Proc. Amer. Math. Soc. 44(1974), 61 - 67.
2. SPECHT W.: *Die Lage der Nullstellen eines Polynomes, I - IV.* Math. Nachr. 15(1956), 353 - 374; 16(1957), 257 - 263; 16 (1957), 369 - 389: 21(1960), 201 - 222.