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Nonstandard Gaussian quadrature formulae based on operator values

**Gradimir V. Milovanović ·
Aleksandar S. Cvetković**

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Abstract In this paper, we develop the theory of so-called nonstandard Gaussian quadrature formulae based on operator values for a general family of linear operators, acting of the space of algebraic polynomials, such that the degrees of polynomials are preserved. Also, we propose a stable numerical algorithm for constructing such quadrature formulae. In particular, for some special classes of linear operators we obtain interesting explicit results connected with theory of orthogonal polynomials.

Keywords Gaussian quadrature · Interval quadrature · Linear operator · Zeros · Weight · Measure · Degree of exactness · Orthogonal polynomial · Linear functional

Mathematics Subject Classifications (2000) 41A55 · 33C45 · 33D45 · 65D30 · 65D32

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G. V. Milovanović (✉)
Faculty of Computer Sciences, Megatrend University, Bulevar umetnosti 29,
11070 Novi Beograd, Serbia
e-mail: gvm@megatrend.edu.rs

A. S. Cvetković
Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia
e-mail: aleksandarcvetkovic@pmf.ni.ac.yu

1 Introduction and preliminaries

Let $d\mu$ be a finite positive Borel measure on the real line such that its support $\mathrm{supp}(d\mu)$ is an infinite set, and all its moments $\mu_k = \int_{\mathbb{R}} x^k \, d\mu(x)$, $k = 0, 1, \ldots$, exist and are finite. With $\mathcal{P}$ we denote the set of all algebraic polynomials and with $\mathcal{P}_n$ its subset formed by all polynomials of degree at most n $(\in \mathbb{N}_0)$.

The n-point quadrature formula

$$\int_{\mathbb{R}} f(x) d\mu(x) = \sum_{k=1}^{n} w_k f(x_k) + R_n(f), \tag{1.1}$$

which is exact on the set $\mathcal{P}_{2n-1}$, i.e., $R_n(\mathcal{P}_{2n-1}) = 0$, is known as the Gauss quadrature formula (cf. Gautschi [19, p. 29]). His famous method of approximate integration, Gauss [17] discovered for the Legendre measure $d\mu(t) = dt$ on $[-1, 1]$ in 1814, and he obtained numerical values of quadrature parameters, the nodes x_k and the weights w_k, $k = 1, \ldots, n$, by solving nonlinear systems of equations for $n \le 7$. Computationally, today there are very stable methods for generating Gaussian rules. The most popular of them is one due to Golub and Welsch [22]. Their method is based on determining the eigenvalues and the first components of the eigenvectors of a symmetric tridiagonal Jacobi matrix $J_n(d\mu)$, with elements formed from the coefficients in the three-term recurrence relation for the monic polynomials $\{\pi_n(d\mu;\,\cdot)\}_{n=0}^{+\infty}$ orthogonal with respect to the inner product

$$(f, g) = (f, g)_{d\mu} = \int_{\mathbb{R}} f(x) g(x) \, d\mu(x) \quad (f, g \in L^2(d\mu)). \tag{1.2}$$

Namely, the nodes x_k in (1.1) are the eigenvalues of the Jacobi matrix $J_n(d\mu)$ (i.e., zeros of $\pi_n(d\mu;\,\cdot)$) and the weights w_k are given by $w_k = \mu_0 v_{k,1}^2$, where $v_{k,1}$ is the first component in the corresponding (normalized) eigenvector $\mathbf{v}_k \; (= [v_{k,1}\; v_{k,2}\; \ldots\; v_{k,n}]^T)$, $\mathbf{v}_k^T \mathbf{v}_k = 1$.

The Gaussian quadrature formulae were generalized in several ways. The first idea of numerical integration involving multiple nodes appeared in the middle of the last century (Chakalov [10–12], Turán [49], Popoviciu [43], Ghizzetti and Ossicini [20, 21], etc.). A survey on quadratures with multiple nodes of the form

$$\int_{\mathbb{R}} f(x) d\mu(x) \approx \sum_{k=1}^{n} \sum_{i=0}^{2s_k} w_{k,i} f^{(i)}(x_k)$$

was recently published by Milovanović [28]. Further extensions dealing with quadratures with multiple nodes for ET (Extended Tschebycheff) systems are given by Karlin and Pinkus [23], Barrow [2], Bojanov, Braess, and Dyn [6], Bojanov [5], etc. Recently, a method for the construction of the generalized Gaussian quadrature rules for Müntz polynomials on $(0, 1)$ is given in [32].

The mentioned quadrature rules use the information on the integrand only at some selected points x_k, $k = 1, \ldots, n$ (the values of the function f and its derivatives in the cases of rules with multiple nodes). Such quadratures will be called the *standard quadrature formulae*.

However, in many cases in physics and technics it is not possible to measure the exact value of the function f at points x_k, so that a standard quadrature cannot be applied. On the other side, some other information on f can be available, as

1° the averages

$$\frac{1}{2h_k}\int_{I_k} f(x)\,dx$$

of this function over some non-overlapping subintervals I_k, with length of I_k equals $2h_k$, and their union which is a proper subset of supp$(d\mu)$;

2° a fixed linear combination of the function values, e.g.

$$af(x-h)+bf(x)+cf(x+h)$$

at some points x_k, where a, b, c are constants and h is sufficiently small positive number, etc.

Thus, if the information data $\{f(x_k)\}_{k=1}^n$ in the standard quadrature (1.1) is replaced by $\{(\mathscr{A}^{h_k}f)(x_k)\}_{k=1}^n$, where $\mathscr{A}^h$ is an extension of some linear operator $\mathscr{A}^h:\mathcal{P}\to\mathcal{P}$, $h\geq 0$, we get a *non-standard quadrature formula*

$$\int_{\mathbb{R}} f(x)d\mu(x)=\sum_{k=1}^{n} w_k(\mathscr{A}^{h_k}f)(x_k)+R_n(f). \tag{1.3}$$

Notice that we use the same notation for the linear operator defined on the space of all algebraic polynomials and for its extension to the certain class of integrable functions X ($f\in X$). As a typical example for such operators is the *average* (*Steklov*) *operator* mentioned before in 1°, i.e.,

$$(\mathscr{A}^h p)(x)=\frac{1}{2h}\int_{x-h}^{x+h} p(x)\,dx,\qquad h>0,\ p\in\mathcal{P}. \tag{1.4}$$

The first idea on so-called *interval quadratures*, which are an example of non-standard quadrature rules, appeared a few decades ago. In 1976 Omladič et al. [40] considered quadratures with the average operator (1.4) (see also Pitnauer and Reimer [41]). Some further investigations were given by Kuz'mina [25], Sharipov [45], Babenko [1], and Motornyi [38].

Let $h_1,\ldots,h_n$ be nonnegative numbers such that

$$a<x_1-h_1\leq x_1+h_1<x_2-h_2\leq x_2+h_2<\cdots<x_n-h_n\leq x_n+h_n<b, \tag{1.5}$$

and let $w(x)$ be a given weight function on $[a,b]$. Using the previous inequalities it is obvious that we have $2(h_1+\cdots+h_n)<b-a$.

Recently, Bojanov and Petrov [7] proved that the Gaussian interval quadrature rule of the maximal algebraic degree of exactness $2n-1$ exists, i.e.,

$$\int_a^b f(x)w(x)\,dx = \sum_{k=1}^{n} \frac{w_k}{2h_k} \int_{x_k-h_k}^{x_k+h_k} f(x)w(x)\,dx + R_n(f), \tag{1.6}$$

where $R_n(f)=0$ for each $f \in \mathcal{P}_{2n-1}$. Under conditions $h_k = h$, $1 \le k \le n$, they also proved the uniqueness of (1.6). Moreover, in [8] Bojanov and Petrov proved the uniqueness of (1.6) for the Legendre weight ($w(x)=1$) for any set of lengths $h_k \ge 0$, $k=1,\ldots,n$, satisfying the condition (1.5). The question of the existence for bounded a, b is proved in [7] in much broader context for a given Chebyshev system of functions.

Recently in [29], using properties of the topological degree of non-linear mappings (see [42, 44]), it was proved that Gaussian interval quadrature formula is unique for the Jacobi weight function $w(x)=(1-x)^\alpha(1+x)^\beta$, α, $\beta > -1$, on $[-1,1]$ and an algorithm for numerical construction was proposed. For the special case of the Chebyshev weight of the first kind and the special set of lengths an analytic solution can be given [29]. Interval quadrature rules of Gauss–Radau and Gauss–Lobatto type with respect to the Jacobi weight functions are considered in [33].

Recently, Bojanov and Petrov [9] proved the existence and uniqueness of the weighted Gaussian interval quadrature formula for a given system of continuously differentiable functions, which constitute an ET system of order two on $[a,b]$.

The cases with interval quadratures on unbounded intervals with the classical generalized Laguerre and Hermite weights have been recently investigated by Milovanović and Cvetković in [30] and [34].

Another approach in quadrature formulae of Gaussian type for intervals of the same length with the average operator (1.4) appeared in 1992 in Omladič's paper [39]. The middle points of the intervals are zeros of some kind of orthogonal polynomials. More precisely, Omladič proved that the nodes x_k, $k=1,\ldots,n$, of his quadratures are zeros of the *average Legendre polynomials* $p_n^h(x) \equiv p_n(x)$, which satisfy the three-term recurrence relation

$$p_{n+1}(x) = x\,p_n(x) - \frac{n^2\left(1-n^2h^2\right)}{4n^2-1} p_{n-1}(x), \quad n \ge 1.$$

In this paper we follow this idea and develop the theory and numerical construction of nonstandard quadratures of Gaussian type for a general family of linear operators, acting of the space of algebraic polynomials, such that the degrees of polynomials are preserved. In particular, we consider some special linear operators, for which we can get some interesting explicit results connected with theory of orthogonal polynomials.

The paper is organized as follows. In the next section the formulation and the proof of the main result are given. Section 3 contain further refinement of the theory, developed for some special classes of operators. Finally, using the results presented in Section 3, Section 4 resolves the problem of construction of this kind of quadrature rules.

2 Nonstandard Gaussian quadrature formulae

Let $H = H_\delta$ be any right δ-neighborhood of the number zero, i.e., $H_\delta = [0, \delta)$, $\delta > 0$. We consider families of linear operators $\mathscr{A}^h$, $h \in H$, acting on the space of all algebraic polynomials $\mathcal{P}$, such that the degrees of polynomials are preserved, i.e.,

$$\deg(\mathscr{A}^h p) = \deg(p), \tag{2.1}$$

and

$$\lim_{h\to 0^+} (\mathscr{A}^h p)(x) = p(x), \quad x \in \mathbb{C}, \tag{2.2}$$

for any $p \in \mathcal{P}$ and each $h \in H$. Concerning degree preserving property for the convenience we define $\deg(0) = -1$, so that degree preserving property also means that the zero polynomial is the image only of the zero polynomial.

For a given family of linear operators $\mathscr{A}^h$, $h \in H$, we consider the nonstandard interpolatory quadrature of *Gaussian type*

$$\int_{\mathbb{R}} f(x)d\mu(x) = \sum_{k=1}^{n} w_k(\mathscr{A}^h f)(x_k) + R_n(f), \tag{2.3}$$

which is exact for each polynomial of degree at most $2n-1$, i.e., $R_n(\mathcal{P}_{2n-1}) = 0$.

Our main result can be stated in the following form:

Theorem 2.1 *Let $\mathscr{A}^h$, $h \in H$, be a family of linear operators satisfying the conditions (2.1) and (2.2) and $d\mu$ be a finite positive Borel measure on the real line with its support* $\operatorname{supp}(d\mu) \subset \mathbb{R}$. *For any $n \in \mathbb{N}$ there exists $\varepsilon > 0$, such that for every $h \in H_\varepsilon = [0, \varepsilon)$ there exists the unique interpolatory quadrature formula (2.3) of Gaussian type, with nodes $x_k \in$* $\operatorname{Co}(\operatorname{supp}(\mu))$ *and positive weights $w_k > 0$, $k = 1, \ldots, n$.*

We are going to prove existence and uniqueness property of the nonstandard Gaussian quadrature formula (2.3) for a general family $\mathscr{A}^h$, $h \in H$, satisfying the previous conditions. For some special classes of operators, such as

$$(\mathscr{A}^h p)(x) = \frac{1}{2h}\int_{x-h}^{x+h} p(t)\,dt, \tag{2.4}$$

$$(\mathscr{A}^h p)(x) = \sum_{k=-m}^{m} a_k p(x+kh) \quad \text{or} \quad (\mathscr{A}^h p)(x) = \sum_{k=-m}^{m-1} a_k p\left(x + \left(k+\frac{1}{2}\right)h\right), \tag{2.5}$$

and

$$(\mathscr{A}^h p)(x) = \sum_{k=0}^{m} \frac{b_k h^k}{k!} \mathcal{D}^k p(x), \tag{2.6}$$

we give more properties of the Gaussian quadrature formula. In the previous formulae we assume that m is a fixed natural number and $\mathcal{D}^k = d^k/dx^k$, $k \in \mathbb{N}_0$.

2.1 Some auxiliary results

According to the fact that we are working with families of linear operators, we can represent the action of $\mathscr{A}^h$ applied to any given $p \in \mathcal{P}$ if we know the values of this operator $\mathscr{A}^h$ to monomials x^k, $k \in \mathbb{N}_0$. Suppose that we have

$$\left(\mathscr{A}^h x^n\right)(t) = \sum_{k=0}^{n} \alpha_k^n(h) t^k, \quad n \in \mathbb{N}_0.$$

Then it follows from (2.2) that

$$\lim_{h \to 0^+} \alpha_k^n(h) = \delta_{n,k}, \quad 0 \le k \le n,$$

where $\delta_{n,k}$ is the Kronecker's delta. To simplify the notation, we introduce the convention $\alpha_k^n(h) = 0$ for $k > n$. In this way, we can state this continuity property as follows:

Lemma 2.1 *The family of linear operators $\mathscr{A}^h$, $h \in H$, is continuous, i.e.,*

$$\lim_{h \to 0^+} (\mathscr{A}^h p)(t) = p(t), \quad p \in \mathcal{P},\ t \in \mathbb{C},$$

if and only if the functions $\alpha_k^n(h)$, $k, n \in \mathbb{N}$, defined by

$$(\mathscr{A}^h x^n)(t) = \sum_{k \in \mathbb{N}_0} \alpha_k^n(h) t^k, \quad n \in \mathbb{N}_0,$$

are continuous at zero, with the property

$$\lim_{h \to 0^+} \alpha_k^n(h) = \delta_{n,k}, \quad n, k \in \mathbb{N}_0.$$

Note that this continuity property in a certain sense can be formally written in the form $\lim_{h \to 0^+} \mathscr{A}^h = \mathscr{I}$, where, as usual, $\mathscr{I}$ is the identity operator. We will use this simple notation to denote the continuity property.

Lemma 2.2 *A degree preserving linear operator $\mathscr{A}^h : \mathcal{P} \mapsto \mathcal{P}$ is bijective.*

Proof Suppose we have two polynomials p_1 and p_2 such that $\mathscr{A}^h p_1 = \mathscr{A}^h p_2$. According to the linearity $\mathscr{A}^h(p_1 - p_2) = 0$ and the degree preserving property $\deg(p_1 - p_2) = -1$, i.e., $p_1 - p_2 = 0$, we conclude that $\mathscr{A}^h$ is injective.

If we suppose that $\mathscr{A}^h$ is not surjective, then there exists some polynomial $q \in \mathcal{P}$ $\left(\text{for example, } q(t) := \sum q_k t^k\right)$, such that it is not an image by $\mathscr{A}^h$ of any $p \in \mathcal{P}$. But degree preservation means that in formulae

$$(\mathscr{A}^h x^n)(t) = \sum_{k\in\mathbb{N}_0} \alpha_k^n t^k, \quad n \in \mathbb{N}_0,$$

we have $\alpha_n^n \neq 0, n \in \mathbb{N}_0$. Suppose $\deg(q) = N$. Then we can solve the triangular system of equations

$$(\mathscr{A}^h x^n)(t) = \sum_{k\in\mathbb{N}_0} \alpha_k^n t^k, \quad n = 0, 1, \ldots, N,$$

for the values t^k, $k = 0, 1, \ldots, N$, so that we have

$$t^k = \sum_{n=0}^{k} \beta_n^k (\mathscr{A}^h x^n)(t), \quad k = 0, 1, \ldots, N.$$

Then, the polynomial

$$\sum_{k=0}^{N} q_k \sum_{n=0}^{k} \beta_n^k x^n$$

is mapped by $\mathscr{A}^h$ into q, which is a contradiction. □

This lemma shows that we can treat our family of linear operators as a family of isomorphisms. Since every operator $\mathscr{A}^h$ in the family is bijective, it has the inverse operator $(\mathscr{A}^h)^{-1}$, which is also linear (cf. [26, p. 9]).

The following result is related to the inverse family of operators $(\mathscr{A}^h)^{-1}$, $h \in H$.

Lemma 2.3 *Let $\mathscr{A}^h$, $h \in H$, be a given family of isomorphisms acting on the space of all algebraic polynomials, such that any operator $\mathscr{A}^h$ preserves the degree of a polynomial and that $\lim_{h\to 0^+} \mathscr{A}^h = \mathscr{I}$. Then, the family of inverse operators $(\mathscr{A}^h)^{-1}$, $h \in H$, satisfies the same properties.*

Proof Suppose that for some $h \in H$, the operator $(\mathscr{A}^h)^{-1}$ does not preserve the degree of polynomials, i.e., there exists some $p \in \mathcal{P}$, with $\deg(p) = n$, such that $\deg((\mathscr{A}^h)^{-1} p) \neq n$. Then, $\deg((\mathscr{A}^h)^{-1} p) \neq n$ implies $\deg(\mathscr{A}^h((\mathscr{A}^h)^{-1} p)) = n$, which means that $\mathscr{A}^h$ does not preserve degree of the polynomial which is a contradiction.

As in the proof of Lemma 2.2 we can solve the triangular system of equations

$$(\mathscr{A}^h x^n)(t) = \sum_{k\in\mathbb{N}_0} \alpha_k^n(h) t^k, \quad n = 0, 1, \ldots, N,$$

so that we have

$$t^k = \sum_{n=0}^{k} \beta_n^k(h)\big(\mathscr{A}^h x^n\big)(t), \quad k = 0, 1, \ldots, N.$$

Applying the inverse operator $(\mathscr{A}^h)^{-1}$, we get

$$\big((\mathscr{A}^h)^{-1} t^k\big)(x) = \sum_{n=0}^{k} \beta_n^k(h) x^n, \quad k = 0, 1, \ldots, N.$$

Using Cramer's formulae, the functions $\beta_n^k(h)$, $0 \le n \le k$, can be expressed as rational functions of $\alpha_k^n(h)$, which denominator is given by $\prod_{\nu=0}^{N} \alpha_\nu^\nu(h) \neq 0$, $h \in H$, according to the degree preserving property. Since $\alpha_k^n(h)$, $k, n \in \mathbb{N}_0$, are continuous at $h = 0$, the functions $\beta_n^k(h)$, $0 \le k \le n$, are also continuous at $h = 0$. It means, according to Lemma 2.1, that the family $(\mathscr{A}^h)^{-1}$, $h \in H$, is continuous. □

We also adopt the definition $\beta_n^k(h) = 0$, $n > k$. As a direct consequence, we have the following result:

Lemma 2.4 *The functions $\alpha_k^n(h)$ and $\beta_n^k(h)$ satisfy the following simple property*

$$\sum_{\nu \in \mathbb{N}_0} \alpha_\nu^n(h) \beta_k^\nu(h) = \delta_{n,k}, \quad k, n \in \mathbb{N}_0.$$

Proof Applying the inverse operator to

$$\big(\mathscr{A}^h x^n\big)(t) = \sum_{k \in \mathbb{N}_0} \alpha_k^n(h) t^k,$$

we get

$$x^n = \sum_{k=0}^{n} x^k \sum_{\nu=k}^{n} \alpha_\nu^n(h) \beta_k^\nu(h).$$

To complete the proof we need only to read the terms with given powers of x. □

Now, for a given family $\mathscr{A}^h$, $h \in H$, and a positive measure $d\mu$, with $\mathrm{supp}(\mu) \subset \mathbb{R}$, we define the following family of the linear functionals

$$\mathcal{L}^h(p) = \int ((\mathscr{A}^h)^{-1} p)(x)\, d\mu(x), \quad p \in \mathcal{P}. \tag{2.7}$$

It is obvious that the functional $\mathcal{L}^h$, for a given $h \in H$, is linear. We are now able to define the following bilinear functional

$$\langle p, q \rangle^h = \mathcal{L}^h(pq), \quad p, q \in \mathcal{P}, \tag{2.8}$$

with properties which are summarized in the following lemma.

Lemma 2.5 *The bilinear functionals* (2.8) *satisfy the following properties*

$$\langle p, q\rangle^h = \langle q, p\rangle^h, \quad \langle \alpha p + \beta r, q\rangle^h = \alpha\langle p, q\rangle^h + \beta\langle r, q\rangle^h, \quad p, q, r \in \mathcal{P}.$$

Proof Direct calculation. □

According to these properties, the bilinear functional $\langle\cdot,\cdot\rangle^h$ can be understood, in a general case, as a formal or non-Hermitian inner product on $\mathcal{P}$ (cf. [3, 46]). In a general case, the linear functional $\mathcal{L}^h$ is not regular, i.e., the sequence of (formal) orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle^h$ does not exist. But, under certain assumptions the functional $\mathcal{L}^h$ could be regular, and we have the following lemma.

Lemma 2.6 *Let the family $\mathscr{A}^h$, $h \in H$, satisfy the property $(\mathscr{A}^h)^{-1} : (\mathcal{P}, \mathbb{R}) \mapsto (\mathcal{P}, \mathbb{R})$. Then, for every $n \in \mathbb{N}$ there exists an $\varepsilon > 0$, such that every linear functional $\mathcal{L}^h$, $h \in [0, \varepsilon)$, is positive definite on $(\mathcal{P}_{2n}, \mathbb{R})$ and the sequence of orthogonal polynomials π_k^h, $k = 0, 1, \ldots, n$, exists with respect to $\mathcal{L}^h$, $h \in [0, \varepsilon)$.*

Proof As in the proofs of the previous lemmas, we adopt the notation

$$\big((\mathscr{A}^h)^{-1}t^k\big)(x) = \sum_{\nu\in\mathbb{N}_0} \beta_\nu^k(h)x^\nu, \quad k = 0, 1, \ldots, n,$$

where the functions $\beta_\nu^k(h)$, according to Lemma 2.1, have the property

$$\lim_{h\to 0^+} \beta_\nu^k(h) = \delta_{k,\nu}.$$

For the moments of the linear functional $\mathcal{L}^h$, we have

$$m_\nu(h) = \mathcal{L}^h(x^\nu) = \int \big((\mathscr{A}^h)^{-1}t^\nu\big)(x)\, d\mu(x) = \sum_{k=0}^{\nu} \beta_\nu^k(h)m_k, \quad \nu = 0, 1, \ldots, 2n,$$

where m_k, $k = 0, 1, \ldots, 2n$, are the moments of the measure μ, i.e., practically the moments of the linear functional $\mathcal{L}^0$. According to the property $(\mathscr{A}^h)^{-1} : (\mathcal{P}, \mathbb{R}) \mapsto (\mathcal{P}, \mathbb{R})$, we know that all moments are real. The moments $m_\nu(h)$, $\nu = 0, 1, \ldots, 2n$, are continuous functions of h at the point $h = 0$, and therefore we have

$$\lim_{h\to 0^+} m_\nu(h) = m_\nu, \quad \nu = 0, 1, \ldots, 2n.$$

According to the Theorem about positive definiteness of the linear functionals (see [13, p. 15]), we can conclude that a linear functional is positive definite on $(\mathcal{P}_{2n}, \mathbb{R})$, provided all moments are real and the corresponding Hankel determinants

$$\Delta_\nu(h) = \begin{vmatrix} m_0(h) & m_1(h) & \cdots & m_{\nu-1}(h) \\ m_1(h) & m_2(h) & & m_\nu(h) \\ \vdots & & & \\ m_{\nu-1}(h) & m_\nu(h) & & m_{2\nu-2}(h) \end{vmatrix}, \quad \nu = 1, \ldots, n+1, \qquad (2.9)$$

are positive ($\Delta_0(h) := 1$). For $h = 0$ all determinants $\Delta_\nu(0)$, $\nu = 1, \ldots, n+1$, are positive, since the measure μ is positive and the corresponding linear functional $\mathcal{L}^0$ is positive definite. The determinants $\Delta_\nu(h)$, $\nu = 1, \ldots, n+1$, are continuous functions of h at the point $h = 0$, and therefore there exist ε_ν, $\nu = 1, \ldots, n+1$, such that $\Delta_\nu(h) > 0$ for $h \in [0, \varepsilon_\nu)$, $\nu = 1, \ldots, n+1$.

We can identify the set $[0, \varepsilon)$ in the following form

$$[0, \varepsilon) = \bigcap_{k=1}^{n+1} [0, \varepsilon_k), \qquad \varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_{n+1}\}.$$

Therefore, the family of the linear functionals $\mathcal{L}^h$, $h \in [0, \varepsilon)$, is positive definite on $(\mathcal{P}_{2n}, \mathbb{R})$; hence, there exists the sequence of orthogonal polynomials π_k^h, $k = 0, 1, \ldots, n$, with respect to each $\mathcal{L}^h$, $h \in [0, \varepsilon)$. □

According to the proof of this lemma it is obvious that ε depends of n. Namely, the following implication

$$n_1 > n_2 \;\Rightarrow\; [0, \varepsilon(n_1)) \subset [0, \varepsilon(n_2)),$$

is an immediate consequence. In another words, we can expect that ε is a non-increasing function of n. It is interesting to pose a question whether there exists the case in which $\varepsilon(n) = +\infty$, $n \in \mathbb{N}$. Later, we prove that such families of operators exist, for example, one family of operators fulfilling this property is

$$(\mathscr{A}^h p)(x) = 2p(x + h/2) - p(x - h/2), \qquad p \in \mathcal{P}.$$

For the families of linear functionals for which $(\mathscr{A}^h)^{-1} : (\mathcal{P}, \mathbb{R}) \mapsto (\mathcal{P}, \mathbb{C})$, in the general case we cannot claim that the linear functional is positive definite, but we can prove that some linear functionals $\mathcal{L}^h$ are regular.

Lemma 2.7 *For every given $n \in \mathbb{N}_0$ there exists an $\varepsilon > 0$ such that the linear functional $\mathcal{L}^h$, $h \in [0, \varepsilon)$, is regular on the space $(\mathcal{P}_{2n}, \mathbb{C})$, i.e., there exists a sequence of polynomials π_ν^h, $\nu = 0, 1, \ldots, n$, orthogonal with respect to $\mathcal{L}^h$, $h \in [0, \varepsilon)$.*

Proof Like in the proof of the previous lemma, we start with the moments of the linear functional $\mathcal{L}^h$,

$$m_\nu(h) = \mathcal{L}^h(t^\nu) = \int \big((\mathscr{A}^h)^{-1} t^\nu\big)(x)\, d\mu(x), \qquad \nu = 0, 1, \ldots, 2n,$$

and then we form the Hankel determinants $\Delta_\nu(h)$, $\nu = 1, 2, \ldots, n+1$, as in (2.9). Since the measure μ is positive, the functional $\mathcal{L}^0$ is regular. Using a continuity argument as in the previous lemma, we prove that there exists an $\varepsilon > 0$, such that $\Delta_\nu(h) \neq 0$, $\nu = 1, 2 \ldots, n+1$, for $h \in [0, \varepsilon)$.

Since the linear functionals $\mathcal{L}^h$ are regular on $(\mathcal{P}_{2n}, \mathbb{C})$, we conclude that the corresponding sequence of orthogonal polynomials exists. □

Theorem 2.2 *Suppose two families of linear operators $\mathscr{A}_1^h$ and $\mathscr{A}_2^h$ are given such that*

$$\left(\mathscr{A}_1^h - \mathscr{A}_2^h\right)p = 0, \qquad p \in \mathcal{P}_{2n-1}.$$

Then two different linear functionals $\mathcal{L}_1^h$ and $\mathcal{L}_2^h$, defined by (2.7), which are induced by these operators (in the case they are regular) have the same first n members of the orthogonal polynomial sequence. In another words if we denote sequence of orthogonal polynomials with respect to inner product (2.7) for $\mathscr{A}^h = \mathscr{A}_1^h$ with p_k^1, $k \in \mathbb{N}_0$ and for $\mathscr{A}^h = \mathscr{A}_2^h$ with p_k^2, $k \in \mathbb{N}_0$, then

$$p_k^1 = p_k^2, \quad k = 0, 1, \ldots, n.$$

Proof It is enough to prove that first $2n-1$ moments are the same for two inner products. Hence, it is enough to prove that the values of the operators $(\mathscr{A}_1^h)^{-1}$ and $(\mathscr{A}_2^h)^{-1}$ on $1, x, \ldots, x^{2n-1}$ are the same. As in the proofs of previous lemmas, we have

$$\left(\mathscr{A}_1^h t^k\right)(x) = \left(\mathscr{A}_2^h t^k\right)(x) = \sum_{\nu=0}^{k} \alpha_\nu^k x^\nu, \quad k = 0, 1, \ldots, 2n-1.$$

Using this system of equations we can solve for x^ν, $\nu = 0, 1, \ldots, n$, since it is triangular system of equations with $\alpha_k^k \neq 0$. Using the fact that $\mathscr{A}_1^h$ and $\mathscr{A}_2^h$ are linear, we get

$$x^\nu = \mathscr{A}_1^h \left(\sum_{k=0}^{\nu} \beta_k^\nu t^k\right) = \mathscr{A}_2^h \left(\sum_{k=0}^{\nu} \beta_k^\nu t^k\right), \quad \nu = 0, 1, \ldots, 2n-1.$$

It is obvious that $((\mathscr{A}_1^h)^{-1} - (\mathscr{A}_2^h)^{-1})x^\nu = 0$, $\nu = 0, 1, \ldots, 2n-1$. Hence, first $2n-1$ moments of the linear functionals $\mathcal{L}_1^h$ and $\mathcal{L}_2^h$ are the same. Using a representation of orthogonal polynomials via moments (see [13, p. 17])

$$p_k(x) = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_k \\ \mu_1 & \mu_2 & & \mu_{k+1} \\ \vdots & & & \\ \mu_{k-1} & \mu_k & & \mu_{2k-1} \\ 1 & x & & x^k \end{vmatrix},$$

we conclude that $p_k^1 = p_k^2$, $k = 0, 1, \ldots, n$. □

This result enables us to consider the family (2.4) as a special case of the family (2.5), since we can always construct a quadrature rule to such that

$$\frac{1}{2h}\int_{x-h}^{x+h} p(x)dx = \sum_{k=-n+1}^{n-1} w_k p(x+kh),$$

for each $p \in \mathcal{P}_{2n-1}$.

It also gives us a machinery to treat operator families of the following form

$$(\mathscr{A}^h p)(x) = \frac{1}{\mu([-h,h])}\int_{-h}^{h} p(x+t)d\mu(t).$$

Again, this family can be reduced to the family (2.5) on the space $\mathcal{P}_{2n-1}$, if we apply a similar interpolation quadrature rule.

Actually, it can be proved that the families (2.5) and (2.6) are equivalent, in the sense given in the following lemma.

Lemma 2.8 *Let $\mathscr{A}_1^h$ be the family of operators given by* (2.5)*. Given $n \in \mathbb{N}$ there always exists family of operators $\mathscr{A}_2^h$, given by* (2.6)*, such that*

$$\big(\mathscr{A}_1^h - \mathscr{A}_2^h\big)p = 0, \quad p \in \mathcal{P}_{2n}.$$

Let $\mathscr{A}_1^h$ be the family of operators given by (2.6)*. Given $n \in \mathbb{N}$ there always exists family of operators $\mathscr{A}_2^h$, given by* (2.5)*, such that*

$$\big(\mathscr{A}_1^h - \mathscr{A}_2^h\big)p = 0, \quad p \in \mathcal{P}_{2n}.$$

Proof It is enough to construct an operator $\mathscr{A}_2^h$, given by (2.6), which satisfies

$$\big(\mathscr{A}_1^h - \mathscr{A}_2^h\big)x^k = 0, \quad k = 0, 1, \ldots, 2n.$$

The previous reduces to the linear system of equations

$$\begin{aligned}\sum_{\nu=0}^{k}\binom{k}{\nu} b_\nu h^\nu t^{k-\nu} &= \sum_{\ell=-m_1}^{m_1} a_\ell (t+\ell h)^k \\ &= \sum_{\ell=-m_1}^{m_1} a_\ell \sum_{\nu=0}^{k}\binom{k}{\nu} t^{k-\nu}(\ell h)^\nu, \quad k = 0, 1, \ldots, 2n,\end{aligned}$$

i.e.,

$$b_\nu = \sum_{\ell=-m_1}^{m_1} a_\ell \ell^\nu, \quad \nu = 0, 1, \ldots, 2n. \tag{2.10}$$

For the proof of the second part of the statement it is enough to note that equation (2.10) has the unique solution for a_ℓ, $\ell = -m_1, \ldots, m_1$, for the case $m_1 = n$, since the matrix of the system is a regular Vandermonde matrix. □

It is well-known that the construction of a Gaussian quadrature formula is connected with an interpolation problem. In our case, for any set of real distinct numbers x_k, $k = 1, \ldots, n$, and any operator $\mathscr{A}^h$ from our family, we are concerned with the solution of the following interpolation problem: *Find* *the polynomial P of degree less than $2n$ which solves the following system of equations*

$$(\mathscr{A}^h P)(x_k) = f_{0,k}, \quad [(\mathscr{A}^h P)(x)]'_{x=x_k} = f_{1,k}, \quad k = 1, \ldots, n, \tag{2.11}$$

where $f_{m,k}$ $(m = 0, 1;\ k = 1, \ldots, n)$ are any two given sequences of numbers.

Lemma 2.9 *The interpolation problem* (2.11) *has the unique solution $P \in \mathcal{P}_{2n-1}$.*

Proof At first, for real distinct numbers x_k, $k = 1, \ldots, n$, we construct polynomials

$$H(x) = \prod_{k=1}^{n}(x - x_k), \qquad M_k(x) = \frac{H(x)}{(x - x_k)H'(x_k)},$$

$$S_k(x) = (1 - 2M_k'(x_k)(x - x_k))(M_k(x))^2, \quad T_k(x) = (x - x_k)(M_k(x))^2,$$

$$U_k(x) = ((\mathscr{A}^h)^{-1} S_k)(x), \qquad V_k(x) = ((\mathscr{A}^h)^{-1} T_k)(x),$$

for which the following properties can be verified by direct calculations

$$\big(\mathscr{A}^h U_k\big)(x_\nu) = S_k(x_\nu) = \delta_{k,\nu}, \quad \big[\big(\mathscr{A}^h U_k\big)(x)\big]'_{x=x_\nu} = S_k'(x_\nu) = 0,$$

$$\big(\mathscr{A}^h V_k\big)(x_\nu) = T_k(x_\nu) = 0, \quad \big[\big(\mathscr{A}^h V_k\big)(x)\big]'_{x=x_\nu} = T_k'(x_\nu) = \delta_{k,\nu}.$$

Hence, we can identify the polynomial P in the following form

$$P(x) = \sum_{k=1}^{n} [f_{0,k} U_k(x) + f_{1,k} V_k(x)], \quad P \in \mathcal{P}_{2n-1}.$$

It remains to prove that our interpolation problem has the unique solution. Here it is enough to verify that the corresponding homogenous problem has only the trivial solution in $\mathcal{P}_{2n-1}$. Suppose that the homogenous problem has a solution P which is not trivial. Then, according to the system of equations

$$(\mathscr{A}^h P)(x_k) = 0, \quad [(\mathscr{A}^h P)(x)]'_{x=x_k} = 0, \quad k = 1, \ldots, n,$$

we conclude easily that the polynomial $\mathscr{A}^h P$ has n distinct double zeros at the points x_k, $k = 1, \ldots, n$, which means that the polynomial $\mathscr{A}^h P$ is of degree at least $2n$. However, the operator $\mathscr{A}^h$ preserves degree of polynomials, hence, the polynomial P has degree at least $2n$, which is a contradiction. Thus, our interpolation problem has the unique solution. □

2.2 Proof of the main result

Now, we are ready to prove our main result in the following reformulated form:

Theorem 2.3 *For every positive measure μ, with* $\operatorname{supp}(\mu) \subset \mathbb{R}$, *and any given family of isomorphisms $\mathscr{A}^h$, $h \in H$, which preserves degree of the polynomial, is continuous and, has the property $(\mathscr{A}^h)^{-1} : (\mathcal{P}_{2n}, \mathbb{R}) \mapsto (\mathcal{P}_{2n}, \mathbb{R})$, for every*

$n \in \mathbb{N}$ there exists an $\varepsilon > 0$, such that the quadrature formula (2.3) exists uniquely for $h \in [0, \varepsilon)$, i.e.,

$$\int p\, d\mu = \sum_{k=1}^{n} w_k(\mathscr{A}^h p)(x_k), \quad p \in \mathcal{P}_{2n-1},$$

with nodes $x_k \in \mathrm{Co}(\mathrm{supp}(\mu))$ and positive weights w_k, $k = 1, \ldots, n$.

Proof As we proved in Lemma 2.6, the linear functional $\mathcal{L}^h$, defined by (2.7), is positive definite on $(\mathcal{P}_{2n}, \mathbb{R})$ for $h \in [0, \varepsilon_1)$, i.e., there exists a sequence of polynomials orthogonal with respect to $\mathcal{L}^h$. We can express the monic orthogonal polynomials π_n^h in the following form (see [13, p. 17], [48, p. 97])

$$\pi_0^h(x) = 1, \quad \pi_n^h(x) = \frac{1}{\Delta_n(h)} \begin{vmatrix} m_0(h) & m_1(h) & \cdots & m_n(h) \\ m_1(h) & m_2(h) & & m_{n+1}(h) \\ \vdots & & & \\ m_{n-1}(h) & m_n(h) & & m_{2n-1}(h) \\ 1 & x & & x^n \end{vmatrix}, \quad n \geq 1,$$

where $\Delta_n(h)$ is defined by (2.9). From this formula we conclude that the coefficients of polynomials $\pi_n^h(x)$ are continuous functions of h at the point $h = 0$. Since the zeros of polynomials are continuous functions of their coefficients (see [35, p. 177]), we conclude that the zeros of the polynomial π_n^h are continuous functions of h at $h = 0$. Since all zeros of the polynomial π_n^0 are contained in the set $\mathrm{Co}(\mathrm{supp}(\mu)) \subset \mathbb{R}$ (see [47, p. 4], [27]), then according to the mentioned continuity property there exists an $\varepsilon_2 > 0$ such that for $h \in [0, \varepsilon_2)$ the zeros of π_n^h are contained in $\mathrm{Co}(\mathrm{supp}(\mu))$. Thus, for any $h \in [0, \varepsilon)$, where $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we have that all zeros of π_n^h are contained in $\mathrm{Co}(\mathrm{supp}(\mu))$.

Now, take $n \in \mathbb{N}$ and a corresponding ε such that the linear functional $\mathcal{L}^h$ is positive definite on $(\mathcal{P}_{2n}, \mathbb{R})$ and that all zeros of π_n^h are contained in $\mathrm{Co}(\mathrm{supp}(\mu))$. Choose some polynomial $P \in \mathcal{P}_{2n-1}$. According to Lemma 2.9, we have uniquely

$$P(x) = \sum_{k=1}^{n} [f_{0,k} U_k(x) + f_{1,k} V_k(x)], \tag{2.12}$$

with $f_{0,k} = (\mathscr{A}^h P)(x_k)$, $f_{1,k} = [(\mathscr{A}^h P)(x)]'_{x=x_k}$, $k = 1, \ldots, n$, where the polynomials U_k and V_k are constructed for the set of points x_k, $k = 1, \ldots, n$, which are zeros of π_n^h. Using the definition of V_k we have

$$\int V_k\, d\mu = \int \big((\mathscr{A}^h)^{-1} T_k\big)(x)\, d\mu(x) = \mathcal{L}^h(T_k) = \frac{1}{(\pi_n^h)'(x_k)} \mathcal{L}^h\big(\pi_n^h M_k\big) = 0,$$

according to the orthogonality property, since M_k is of degree $n - 1$. The previous equality is true for every $k = 1, \ldots, n$.

If we integrate (2.12), we get

$$\int P\,d\mu = \sum_{k=1}^{n}\left[f_{0,k}\int U_k\,d\mu + f_{1,k}\int V_k\,d\mu\right] = \sum_{k=1}^{n} w_k(\mathscr{A}^h P)(x_k),$$

with an identification

$$w_k = \int U_k\,d\mu, \quad k = 1,\ldots,n.$$

In other words, we have just constructed a quadrature rule which is exact for each $P \in \mathcal{P}_{2n-1}$. Its nodes are zeros of the polynomial π_n^h orthogonal with respect to $\mathcal{L}^h$ and, as we know, these zeros belong to $\mathrm{Co}(\mathrm{supp}(\sigma))$. For the weight coefficients we have

$$w_k = \int U_k\,d\mu = \mathcal{L}^h(S_k) = \mathcal{L}^h\big((M_k)^2\big) - \frac{2M_k'(x_k)}{(\pi_n^h)'(x_k)}\mathcal{L}^h\big(\pi_n^h M_k\big) = \mathcal{L}^h\big((M_k)^2\big) > 0,$$

where we have used the orthogonality property and positive definiteness of the linear functional $\mathcal{L}^h$.

The uniqueness property of our quadrature formula is identified easily, since the monic orthogonal polynomial π_n^h is determined uniquely for the positive definite linear functional $\mathcal{L}^h$, hence, its zeros are too. □

This completes the existence and the uniqueness property of the quadrature formula (2.3). However, there exists still problem of the construction of such a quadrature formula. It can be very instructive if we are able to give an algorithm for such a construction or if we are able to derive a procedure using which we can find the family $(\mathscr{A}^h)^{-1}$, $h \in H$. In the next section we present the procedure which can resolve the mentioned questions for certain special families of operators.

We finish this section with an illustrative example.

Theorem 2.4 *Suppose we have the family of operators*

$$(\mathscr{A}^h p)(t) = p(t) + ht\frac{dp(t)}{dt}, \quad p \in \mathcal{P}.$$

Then this is a family of continuous, degree preserving, isomorphism of $\mathcal{P}$. The inverse family can be represented in the following form

$$\big((\mathscr{A}^h)^{-1}p\big)(x) = \begin{cases} \dfrac{x^{-1/h}}{h}\displaystyle\int_0^x t^{1/h-1}p(t)\,dt, & x > 0,\\ p(0), & x = 0,\\ \dfrac{(-x)^{-1/h}}{h}\displaystyle\int_x^0 (-t)^{1/h-1}p(t)\,dt, & x < 0.\end{cases}$$

Proof Since the family is degree preserving it is a family of isomorphisms. Trivially we check it is continuous. We find easily the values of our family on the polynomial natural basis,

$$\big(\mathscr{A}^h x^k\big)(t) = (1+kh)t^k \quad \text{i.e.,} \quad \big((\mathscr{A}^h)^{-1}t^k\big)(x) = \frac{x^k}{1+kh}, \qquad k \in \mathbb{N}_0.$$

What is left to do is just to check that given representation of $(\mathscr{A}^h)^{-1}$ matches given sequence of values on the natural basis. □

For example, this theorem combined with results already presented guarantees the existence of the following quadrature rule

$$\int_{\mathbb{R}^+} p(x)\,d\mu(x) = \sum_{k=1}^{n} w_k \int_{x_{k-1}}^{x_k} x^{1/h-1} p(x)\,dx, \qquad p \in \mathcal{P},$$

where $x_0 = 0$, for every measure μ and h small enough, depending on μ and n, where we can obtain the nodes of the quadrature formula as the zeros of the n-th polynomial orthogonal with respect to the linear functional

$$\mathcal{L}^h(p) = \int_{\mathbb{R}^+} \big(\mathscr{A}^h p\big)(x)\,d\mu(x) = \int_{\mathbb{R}^+} \left(p(x) + hx\frac{dp(x)}{dx}\right) d\mu(x), \qquad p \in \mathcal{P}.$$

It is also interesting to note that the sequence of moments can be given in the form

$$\mathcal{L}^h\big(x^k\big) = (1+kh)\mathcal{L}^0(x^k) = (1+kh)\int_{\mathbb{R}^+} x^k\,d\mu(x), \qquad k \in \mathbb{N}_0.$$

3 Special families of linear functionals

3.1 Basic consideration

In order to construct the quadrature formula (2.3) we need the zeros of the polynomial π_n^h orthogonal with respect to the functional $\mathcal{L}^h$ defined by (2.7). Hence, the first problem is how to compute the values of the functional $\mathcal{L}^h$. According to the fact that $\mathcal{L}^h$ is linear, we need only the moments of the linear functional $\mathcal{L}^h$, i.e., we should know how to compute $(\mathscr{A}^h)^{-1}x^k$, $k = 0, 1, \ldots, 2n$. If we know the action of $\mathscr{A}^h$, i.e.,

$$\big(\mathscr{A}^h t^n\big)(x) = \sum_{k\in\mathbb{N}_0} \alpha_k^n x^k, \qquad n \in \mathbb{N}_0,$$

then we can calculate

$$t^n = \sum_{k\in\mathbb{N}_0} \alpha_k^n \big((\mathscr{A}^h)^{-1}x^k\big)(t), \qquad n \in \mathbb{N}_0,$$

and we are able to calculate the moments of the inverse operator $(\mathscr{A}^h)^{-1}$, because the system of linear equations is triangular with elements on the main diagonal which are not zeros. Since $(\mathscr{A}^h)^{-1}$ is a linear operator, we know its action on the whole $\mathcal{P}$.

For a special families of linear operators $\mathscr{A}^h$, we are able to give more precise results on the interpretation of $(\mathscr{A}^h)^{-1}$.

In the sequel, we use the moments of the operators $\mathscr{A}^h$ and $(\mathscr{A}^h)^{-1}$, denoted by

$$\eta_{n,h}(x) = \big(\mathscr{A}^h t^n\big)(x) \quad \text{and} \quad \mu_{k,h}(t) = \big((\mathscr{A}^h)^{-1} x^k\big)(t) \quad (n, k \in \mathbb{N}_0),$$

respectively. Note that the first moment of $\mathscr{A}^h$ is a constant different from zero.

Theorem 3.1 *Assume that the moments of the operator $\mathscr{A}^h$ can be expressed in the form*

$$\frac{\eta_{n,h}(x)}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!} \frac{C^h_{n-k}}{(n-k)!}, \quad n \in \mathbb{N}_0, \tag{3.1}$$

where C^h_k, $k \in \mathbb{N}_0$, are constants. Then

$$\sum_{k=0}^{n} \frac{\eta_{k,h}(x)}{k!} \frac{\mu_{n-k,h}(t)}{(n-k)!} = \frac{(x+t)^n}{n!}, \quad n \in \mathbb{N}_0, \tag{3.2}$$

and the moments of the inverse operators $(\mathscr{A}^h)^{-1}$ satisfy the property (3.1).

The moments of the operator $(\mathscr{A}^h)^{-1}$ are determined uniquely by the moments of $\mathscr{A}^h$ and vice-versa.

Proof Applying the operator $(\mathscr{A}^h)^{-1}$ to (3.1), we get

$$\frac{t^n}{n!} = \sum_{k=0}^{n} \frac{C^h_{n-k}}{(n-k)!} \frac{\mu_{k,h}(t)}{k!}, \quad n \in \mathbb{N}_0. \tag{3.3}$$

Next we have

$$\begin{aligned}
\sum_{k=0}^{n} \frac{\eta_{k,h}(x)}{k!} \frac{\mu_{n-k,h}(t)}{(n-k)!} &= \sum_{k=0}^{n} \frac{\mu_{n-k,h}(t)}{(n-k)!} \sum_{\nu=0}^{k} \frac{x^\nu}{\nu!} \frac{C^h_{k-\nu}}{(k-\nu)!} \\
&= \sum_{\nu=0}^{n} \frac{x^\nu}{\nu!} \sum_{k=\nu}^{n} \frac{\mu_{n-k,h}(t)}{(n-k)!} \frac{C^h_{k-\nu}}{(k-\nu)!} \\
&= \sum_{\nu=0}^{n} \frac{x^\nu}{\nu!} \frac{t^{n-\nu}}{(n-\nu)!} = \frac{(x+t)^n}{n!}.
\end{aligned}$$

Starting with the equality (3.3), we note that the matrix of the system of equations is lower-triangular and has constant elements on diagonals, where by diagonal we mean elements which have a constant difference of indices. We prove that the inverse matrix of such a matrix has the same properties. Trivially it is a lower-triangular matrix. Denote elements of this inverse matrix by $a_{i,j}$, $i, j \in \mathbb{N}_0$. We know that $a_{i,j} = 0$ provided $i < j$. The elements on the main diagonal of this inverse matrix are constant, since $a_{i,i} = 1/C_0^h$, $i \in \mathbb{N}_0$. Suppose that elements on the $i - j = 0, 1, \ldots, k$. Using the identity

$$\sum_{\nu=j}^{i} \frac{C_{i-\nu}^h}{(i-\nu)!} a_{\nu,j} = \delta_{i,j}, \quad i, j \in \mathbb{N}_0,$$

we have that

$$\begin{aligned} a_{j+k+1+\nu,j+\nu} &= \frac{\delta_{j+k+1+\nu,j+\nu}}{C_0^h} - \frac{1}{C_0^h} \sum_{\ell=j+\nu}^{j+k+\nu} \frac{C_{j+k+1+\nu-\ell}^h}{(j+k+1+\nu-\ell)!} a_{\ell,j+\nu} \\ &= \frac{\delta_{j+k+1,j}}{C_0^h} - \frac{1}{C_0^h} \sum_{\ell=0}^{k} \frac{C_{k+1-\ell}^h}{(k+1-\ell)!} b_\ell, \qquad j, \nu \in \mathbb{N}_0, \end{aligned}$$

which means that $a_{j+k+1+\nu,j+\nu}$, $\nu \in \mathbb{N}_0$, does not depend on ν, i.e., the elements of the inverse matrix on the diagonal $i - j = k + 1$ are constant.

To prove the rest of this theorem, we note that the previous system of equations is triangular. If the moments of the operator $\mathscr{A}^h$ are given, we have

$$\frac{\mu_{n,h}(t)}{n!} = \frac{1}{\eta_{0,h}(x)} \left(\frac{(x+t)^n}{n!} - \sum_{k=1}^{n} \frac{\eta_{k,h}(x)}{k!} \frac{\mu_{n-k,h}(t)}{(n-k)!} \right), \quad n \in \mathbb{N}_0,$$

where we use the fact that $\eta_{0,h}(x) \neq 0$, since $\mathscr{A}^h$ is a degree preserving operator. According to the fact that $((\mathscr{A}^h)^{-1})^{-1} = \mathscr{A}^h$, the moments of $\mathscr{A}^h$ are given uniquely by the moments of $(\mathscr{A}^h)^{-1}$. □

Now, we introduce two generating functions for the moments of the operators $\mathscr{A}^h$ and $(\mathscr{A}^h)^{-1}$ by

$$f_{\mathscr{A}^h}(u, x) = \sum_{k \in \mathbb{N}_0} \frac{\eta_{k,h}(x)}{k!} u^k, \quad f_{(\mathscr{A}^h)^{-1}}(u, t) = \sum_{k \in \mathbb{N}_0} \frac{\mu_{k,h}(t)}{k!} u^k, \tag{3.4}$$

respectively. These functions are defined at least at the point $u = 0$. Also formally, we can form the Cauchy product of these two series which represent the generating functions

$$\begin{aligned} f_{\mathscr{A}^h}(u, x) f_{(\mathscr{A}^h)^{-1}}(u, t) &= \sum_{k \in \mathbb{N}_0} u^k \sum_{\nu=0}^{k} \frac{\eta_{\nu,h}(x)}{\nu!} \frac{\mu_{k-\nu,h}(t)}{(k-\nu)!} \\ &= \sum_{k \in \mathbb{N}_0} \frac{(u(x+t))^k}{k!} = \exp(u(x+t)). \end{aligned}$$

Regarding this we have the following result:

Theorem 3.2 *Let $\mathscr{D}_{\mathscr{A}^h} \neq \{0\}$ be the domain of the absolute convergence of the series representing the generating function $f_{\mathscr{A}^h}(u, x)$ from (3.4). Then, the series which represents the generating function $f_{(\mathscr{A}^h)^{-1}}(u, t)$ has a domain of the* *absolute convergence* $\mathscr{D}_{(\mathscr{A}^h)^{-1}} \neq \{0\}$ *and the both generating functions satisfy the following equality*

$$f_{\mathscr{A}^h}(u, x)\, f_{(\mathscr{A}^h)^{-1}}(u, t) = \exp(u(x + t)), \tag{3.5}$$

where $u \in \mathscr{D}_{\mathscr{A}^h} \cap \mathscr{D}_{(\mathscr{A}^h)^{-1}}$*, and this intersection does not equal to* $\{0\}$.

Proof Define the function

$$g_h(u, t) := \frac{\exp(u(x + t))}{f_{\mathscr{A}^h}(u, x)},$$

which is quotient of two analytic functions on $\mathscr{D}_{\mathscr{A}^h}$. Therefore, g_h is a meromorphic function of u on $\mathscr{D}_{\mathscr{A}^h}$. Because of $f_{\mathscr{A}^h}(0, x) = \eta_0(x) \neq 0$, there exists some neighborhood $\mathscr{D}_{(\mathscr{A}^h)^{-1}}$ of the point $u = 0$, such that the function $f_{\mathscr{A}^h}(u, x) \neq 0$ for $u \in \mathscr{D}_{(\mathscr{A}^h)^{-1}}$. If the function $f_{\mathscr{A}^h}(\,\cdot\,, x)$ does not take zero on the whole $\mathscr{D}_{\mathscr{A}^h}$, then we can take $\mathscr{D}_{(\mathscr{A}^h)^{-1}} = \mathscr{D}_{\mathscr{A}^h}$. The previous means that the function g_h is analytic on $\mathscr{D}_{(\mathscr{A}^h)^{-1}}$, i.e., it has the series representing it in the neighborhood of $u = 0$.

Hence, it must be

$$f_{\mathscr{A}^h}(u, x) g_h(u, t) = \exp(u(x + t)), \quad u \in \mathscr{D}_{(\mathscr{A}^h)^{-1}},$$

where all functions in the formula are analytic. If we expand these functions in potential series, we get

$$\sum_{k \in \mathbb{N}_0} \frac{\eta_{k,h}(x)}{k!} u^k \sum_{\nu \in \mathbb{N}_0} \frac{g_{\nu,h}(t)}{\nu!} u^\nu = \sum_{k \in \mathbb{N}_0} \frac{(x + t)^k}{k!} u^k.$$

For every $u \in \mathscr{D}_{(\mathscr{A}^h)^{-1}}$, the product of series can be calculated using the Cauchy product for $f_{\mathscr{A}^h}$ and g_h. According to this fact, for $u \in \mathscr{D}_{(\mathscr{A}^h)^{-1}}$ we have

$$\sum_{k \in \mathbb{N}_0} u^k \sum_{\nu \in \mathbb{N}_0} \frac{\eta_{\nu,h}(x)}{\nu!} \frac{g_{k-\nu,h}(t)}{(k - \nu)!} = \exp(u(x + t)).$$

Finally, if we multiply it by u^{-j-1}, $j \in \mathbb{N}_0$, and then integrate it over the circle $\{u \mid |u| = r\} \subset \mathscr{D}_{(\mathscr{A}^h)^{-1}}$, using Cauchy's theorem, we obtain

$$\sum_{\nu=0}^{j} \frac{\eta_{\nu,h}(x)}{\nu!} \frac{g_{j-\nu,h}(t)}{(j - \nu)!} = \frac{(x + t)^j}{j!}, \quad j \in \mathbb{N}_0.$$

According to Theorem 3.1, the previous system of equations uniquely determines the moments of $(\mathscr{A}^h)^{-1}$, so that $f_{(\mathscr{A}^h)^{-1}} = g$ and the function $f_{(\mathscr{A}^h)^{-1}}$ is analytic on $\mathscr{D}_{(\mathscr{A}^h)^{-1}}$. □

3.2 Special cases of families (2.4), (2.5), and (2.6)

Knowing the generating function of moments of the operator $\mathscr{A}^h$, we can determine the corresponding generating function of moments of the operator $(\mathscr{A}^h)^{-1}$. For the families given by (2.4), (2.5), and (2.6), we can prove that they satisfy the previous property (3.1).

Theorem 3.3 *The families of operators given by* (2.4), (2.5), *and* (2.6) *satisfy the property* (3.1).

Proof Using a direct calculation, for the family given by (2.4) we have

$$\frac{\eta_{k,h}(x)}{k!} = \frac{1}{2k!h}\int_{x-h}^{x+h} t^k\,dt = \frac{1}{2h}\frac{(x+h)^{k+1}-(x-h)^{k+1}}{(k+1)!} = \sum_{\nu=0}^{k}\frac{x^\nu}{\nu!}\frac{C_{k-\nu}}{(k-\nu)!},$$

where

$$C_\nu^h = \frac{1+(-1)^\nu}{2(\nu+1)}h^\nu, \qquad \nu \in \mathbb{N}_0.$$

For the first family given in (2.5), we have

$$\frac{\eta_{k,h}(x)}{k!} = \sum_{\nu=-m}^{m} a_\nu \sum_{j=0}^{k}\frac{x^j}{j!}\frac{(\nu h)^{k-j}}{(k-j)!} = \sum_{j=0}^{k}\frac{x^j}{j!}\frac{C_{k-j}^h}{(k-j)!},$$

where $C_\nu^h = \sum_{j=-m}^{m} a_j (jh)^\nu$, $\nu \in \mathbb{N}_0$. Similarly, we have for the second family given in (2.5).

Finally, for the family given by (2.6), we have

$$\frac{\eta_{k,h}(x)}{k!} = \sum_{\nu=0}^{k}\frac{b_\nu h^\nu}{\nu!}\frac{x^{k-\nu}}{(k-\nu)!} = \sum_{\nu=0}^{m}\frac{x^\nu}{\nu!}\frac{C_{k-\nu}^h}{(k-\nu)!},$$

where $C_\nu^h = b_\nu h^\nu$, $\nu \in \mathbb{N}_0$. □

In order to simplify the notation we introduce the following definition:

Definition 3.1 For the families of operators defined by

$$\mathscr{A}^h p = \sum_{k=-m}^{m} a_k p(\cdot + kh) \quad \text{and} \quad \mathscr{A}^h p = \sum_{k=-m}^{m-1} a_k p\left(\cdot + \left(k+\frac{1}{2}\right)h\right),$$

we introduce the characteristic polynomials as

$$Q(z) = \sum_{k=-m}^{m} a_k z^{k+m} \quad \text{and} \quad Q(z) = \sum_{k=-m}^{m-1} a_k z^{k+m}, \tag{3.6}$$

respectively. For the family

$$\mathscr{A}^h p = \sum_{k=0}^{m}\frac{b_k h^k}{k!}\mathscr{D}^k p,$$

we define the characteristic polynomial as

$$Q(z) = \sum_{k=0}^{m} \frac{b_k}{k!} z^k. \tag{3.7}$$

For the family given in (2.4) it is proven in [39] that it is a bijective family acting on $\mathcal{P}$. The degree preserving and continuity properties of this family are trivial. For families of the form (2.5) and (2.6), we have the following statement:

Theorem 3.4

1° *The family of operators $\mathscr{A}^h$, defined by (2.5), is bijective family of continuous and degree preserving operators if and only if for the characteristic polynomial (3.6) we have $Q(1) = 1$.*

2° *The family of operators $\mathscr{A}^h$, defined by (2.6), is bijective family of continuous and degree preserving operators if and only if for the characteristic polynomial (3.7) we have $Q(0) = 1$.*

Proof

1° It is enough to consider only the first family of operators in (2.5).
If the family (2.5) is continuous then $\mathscr{A}^0 = \mathscr{I}$, but

$$p = \mathscr{A}^0 p = \sum_{k=-m}^{m} a_k p(\cdot + k0) = Q(1)p, \quad p \in \mathcal{P},$$

hence, $Q(1) = 1$.
Now, let $Q(1) = 1$. The family (2.5) is evidently linear. It is a degree preserving as well, since the leading coefficients in the polynomials p and $\mathscr{A}^h p$ are the same. The family is bijective according to Lemma 2.2.
Finally, for the continuity we have first $\mathscr{A}^0 = \mathscr{I}$ and, because of linearity, we know that it is enough to prove the continuity only for one basis of $\mathcal{P}$, so that

$$\left(\mathscr{A}^h t^j\right)(x) = \sum_{k=-m}^{m} a_k (x + kh)^j = \sum_{k=-m}^{m} a_k \sum_{\nu=0}^{j} \binom{j}{\nu} x^\nu (kh)^{j-\nu}$$

$$= \sum_{\nu=0}^{j} x^j \binom{j}{\nu} \sum_{k=-m}^{m} a_k (kh)^{j-\nu}.$$

Since

$$\lim_{h \to 0^+} \binom{j}{\nu} \sum_{k=-m}^{m} a_k (kh)^{j-\nu} = \delta_{j,\nu},$$

we conclude that the family is continuous.

2° A similar proof can be given for the family of operators (2.6). □

Theorem 3.5

1° *The generating functions* $f_{\mathscr{A}^h}$ *and* $f_{(\mathscr{A}^h)^{-1}}$ *of the operator* (2.5) *are given by*

$$f_{\mathscr{A}^h}(u,x) = Q\big(e^{hu}\big)e^{ux}\exp(-\deg(Q)hu/2)$$

and

$$f_{(\mathscr{A}^h)^{-1}}(u,t) = \frac{\exp(u(t+\deg(Q)h/2))}{Q(e^{hu})},$$

respectively, where Q *is defined in* (3.6).

2° *The generating functions for the family* (2.6) *are given by*

$$f_{\mathscr{A}^h}(u,x) = Q(hu)e^{ux} \quad \text{and} \quad f_{(\mathscr{A}^h)^{-1}}(u,t) = \frac{e^{ut}}{Q(hu)},$$

where Q *is defined by* (3.7).

3° *The generating functions for the family* (2.4) *are given by*

$$f_{\mathscr{A}^h}(u,x) = e^{ux}\frac{\sinh(hu)}{hu} \quad \text{and} \quad f_{(\mathscr{A}^h)^{-1}}(u,t) = e^{ut}\frac{hu}{\sinh(hu)}.$$

Proof We will determine only the expressions for $f_{\mathscr{A}^h}(u,x)$. The expressions for $f_{(\mathscr{A}^h)^{-1}}(u,t)$ are obtained directly from (3.5).

1° For the first family of (2.5) we get

$$\begin{aligned} f_{\mathscr{A}^h}(u,x) &= \sum_{k=0}^{+\infty}\frac{u^k}{k!}\sum_{j=-m}^{m} a_j(x+jh)^k = \sum_{j=-m}^{m} a_j \sum_{k=0}^{+\infty}\sum_{m=0}^{k}\frac{(ux)^m}{m!}\frac{(jhu)^{k-m}}{(k-m)!} \\ &= \sum_{j=-m}^{m} a_j \sum_{k=0}^{+\infty}\frac{(ux)^k}{k!}\sum_{m=0}^{+\infty}\frac{(jhu)^m}{m!} = e^{ux}\sum_{j=-m}^{m} a_j e^{jhu} \\ &= Q\big(e^{hu}\big)e^{ux}\exp(-\deg(Q)hu/2), \end{aligned}$$

where we have used intensively Cauchy series product theorem and Fubini theorem.

2° For the family given by (2.6), we have

$$\begin{aligned} f_{\mathscr{A}^h}(u,x) &= \sum_{k=0}^{+\infty} u^k \sum_{j=0}^{\min(m,k)} \frac{b_j h^j}{j!}\frac{x^{k-j}}{(k-j)!} \\ &= \sum_{j=0}^{m}\frac{b_j(hu)^j}{j!}\sum_{k=0}^{+\infty}\frac{(ux)^k}{k!} = Q(hu)e^{ux}. \end{aligned}$$

3° Finally, for the family (2.4) we obtain

$$f_{\mathscr{A}}(u,x) = \frac{1}{2hu}\sum_{k=0}^{+\infty}\frac{u^{k+1}}{k!}\frac{(x+h)^{k+1}-(x-h)^{k+1}}{k+1} = e^{ux}\frac{\sinh(hu)}{hu}.$$

$\square$

Remark 3.1 It may be interesting to give precisely the domain on which equality (3.5) is valid. Since the functions $f_{\mathscr{A}^h}$ and exp are entire functions, the expression is valid on the neighborhood of the number 0 on which the function $f_{(\mathscr{A}^h)^{-1}}$ is analytic. For the family (2.5), the equality is valid on the following set

$$\left\{u \;\middle|\; h|u| < \min_{k\in\mathbb{N}_0,\nu=1,\dots,\deg(Q)} \{|\log|\lambda_\nu| + \mathrm{i}(\arg(\lambda_\nu) + 2k\pi)|\} = hu_{\min}\right\},$$

where λ_ν, $\nu = 1, \dots, \deg(Q)$, are zeros of the characteristic polynomial Q, counting multiplicities.

Similarly, for the family (2.6), the equality is valid on the set

$$\left\{u \;\middle|\; h|u| < \min_{\nu=1,\dots,\deg(Q)} \{|\lambda_\nu|\} = hu_{\min}\right\},$$

where, again λ_ν, $\nu = 1, \dots, \deg(Q)$, are zeros of the characteristic polynomial Q, counting multiplicities.

From these expressions we can give the following estimate

$$|\mu_{k,h}(t)|^{1/k} \approx \frac{k}{eu_{\min}},$$

for the rate of increasing of the moments $\mu_{k,h}(t)$.

3.3 Representation of inverse families of operators

Using every operator $\mathscr{A}^h$ we can derive the family of the linear functionals $\mathscr{A}_x^h$ acting on the space of algebraic polynomials in the following way

$$\mathscr{A}_x^h p = (\mathscr{A}^h p)(x), \quad p \in \mathcal{P}.$$

The linearity of the functionals $\mathscr{A}_x^h$ is a direct consequence of the linearity of $\mathscr{A}^h$. Also, we can introduce the moments $\eta_{k,h}^x$, $k \in \mathbb{N}_0$, of the functionals $\mathscr{A}_x^h$, as well as $\mu_{k,h}^x = \mu_{k,h}(x)$ for $(\mathscr{A}^h)^{-1}$. The generating functions for the moments $\eta_{k,h}^x$ and $\mu_{k,h}^t$ of the functionals $\mathscr{A}_x^h$ and $\left(\mathscr{A}_t^h\right)^{-1}$, we denote by $f_{\mathscr{A}_x^h}(u)$ and $f_{(\mathscr{A}_t^h)^{-1}}$, respectively. There are the obvious connections

$$f_{\mathscr{A}_x^h}(u) = f_{\mathscr{A}^h}(u, x) \quad \text{and} \quad f_{\left(\mathscr{A}_t^h\right)^{-1}}(u) = f_{(\mathscr{A}^h)^{-1}}(u, t).$$

Definition 3.2 For the measure μ (possibly complex), we say it represents a linear functional $\mathcal{L} : \mathcal{P} \mapsto \mathbb{C}$, provided

$$\mathcal{L}\left(x^k\right) = \int_\Gamma x^k \, d\mu(x),$$

where Γ is a simple Jordan curve in the complex plane.

In a general case, according to Theorem of representation of the complex linear functionals (see [13, p. 74]), we can claim that every linear functional has an interpretation measure, which is even supported on the subset of the real line. We have just mention, that for positive definite linear functionals, if the representation measure is supported on a compact set of the real line, the representation measure is unique (see [13, p. 71], [26, p. 410]). If the supporting set is unbounded, the representing measure need not be unique (see [13, p. 73]), but there are some sufficient conditions for this measure to be unique (for example, see [3, 16]).

Since, we are interested to interpret the family of the linear functionals $(\mathscr{A}_t^h)^{-1}$, as a result we should get a family of the representing measures μ^t, with the property

$$(\mathscr{A}_t^h)^{-1}x^k = \int_\Gamma x^k \, d\mu_t^h(x).$$

However, when a family of operators satisfies the property (3.1), it is possible to get the representation using only one measure.

Theorem 3.6 *If a family of linear operators satisfies the property* (3.1) *we have the representation of* $(\mathscr{A}_t^h)^{-1}$ *in the form*

$$(\mathscr{A}_t^h)^{-1}x^k = \int_\Gamma (x+t)^k \, d\mu^h(x), \tag{3.8}$$

where Γ *is a simple Jordan curve in the complex plane.*

Proof According to Theorem 3.1, for some sequence B_k^h, $k \in \mathbb{N}_0$, we have

$$\frac{\mu_{n,h}(t)}{n!} = \sum_{k=0}^{n} \frac{B_{n-k}^h}{(n-k)!}\frac{t^k}{k!}, \qquad n \in \mathbb{N}_0.$$

Now, we suppose that the measure μ^h, supported on some curve Γ, has the moments B_k^h, $k \in \mathbb{N}_0$. Such a measure always exists according to [13, pp. 74–75]. Then, obviously we have

$$\begin{aligned}\int_\Gamma (x+t)^n \, d\mu^h(x) &= n!\sum_{k=0}^{n} \frac{t^k}{k!}\frac{1}{(n-k)!}\int_\Gamma x^{n-k}\, d\mu^h(x)\\ &= n!\sum_{k=0}^{n} \frac{B_{n-k}^h}{(n-k)!}\frac{t^k}{k!} = \mu_{n,h}(t), \qquad n \in \mathbb{N}_0.\end{aligned}$$

□

A direct consequence of the previous theorem is that the linear functional $\mathcal{L}^h$, defined in (2.7), can be represented as

$$\mathcal{L}^h(p) = \int d\mu(t) \int_\Gamma p(t+x) d\mu^h(x), \qquad p \in \mathcal{P}.$$

Lemma 3.1 *If the family of measures μ^h is a family of positive measures supported on the subset of the real line for $h \in [0, \varepsilon) \subset \mathbb{R}_0^+$, then $\mathcal{L}^h$ is a positive definite for $h \in [0, \varepsilon)$.*

Proof Assume that $p(x)$ is non-negative for any $x \in \mathbb{R}$. Then, for a fixed $t \in \mathbb{R}$ so is $p(t + x)$ for $x \in \mathbb{R}$. Since positive measures represent positive definite functionals, for given $t \in \mathbb{R}$ we have

$$q(t) = \int_\Gamma p(t + x) d\mu^h(x) > 0, \quad t \in \mathbb{R}.$$

Since q is a non-negative polynomial and the measure μ is positive we have also $\mathcal{L}^h(p) > 0$, which implies $\mathcal{L}^h$ is positive definite (see [13, p. 13]). □

We are going to see in the next section that such situations actually happen. Even more we are going to see that there are cases in which $\varepsilon = +\infty$. Positive definiteness of $\mathcal{L}^h$ guaranties the existence of the quadrature rule (2.3) with the real nodes and positive weights for any $h \in [0, \varepsilon)$, although it might happen nodes are not contained in the $\mathrm{Co}(\mathrm{supp}(\mu))$. Assuming the measure μ has a support which is unbounded towards $\pm\infty$, the positive definiteness of $\mathcal{L}^h$ will produce a quadrature rule with nodes inside of the convex hull of the supporting set.

Theorem 3.6 actually means that we have

$$\left(\mathscr{A}_t^h\right)^{-1} x^n = \left(\mathscr{A}_0^h\right)^{-1} (x + t)^n, \quad n \in \mathbb{N}_0,$$

where $\left(\mathscr{A}_0^h\right)^{-1}$ operates on x. Actually, we can prove that (3.1) is equivalent with the previous property.

Lemma 3.2 *The sequence of moments $\eta_{n,h}(t)$, $n \in \mathbb{N}_0$, of a linear operator $\mathscr{A}^h$ satisfies (3.1) if and only if for the associated linear functionals we have*

$$\mathscr{A}_t^h p(x) = \mathscr{A}_0^h p(x + t), \tag{3.9}$$

where the operators are acting on x.

Proof Due to linearity it is enough to give the proof only for the natural basis. Assume (3.1) holds, then putting $t = 0$, we get

$$\mathscr{A}_0^h x^n = C_n^h, \quad n \in \mathbb{N}_0,$$

so that

$$\frac{\mathscr{A}_t^h x^n}{n!} = \sum_{k=0}^n \frac{t^k}{k!} \frac{\mathscr{A}_0^h x^{n-k}}{(n-k)!} = \frac{\mathscr{A}_0^h (x + t)^n}{n!}, \quad n \in \mathbb{N}_0.$$

If (3.9) holds, we can use the linearity of $\mathscr{A}_0^h$ to obtain

$$\frac{1}{n!}\mathscr{A}_t^h x^n = \frac{1}{n!}\mathscr{A}_0^h (x+t)^n = \sum_{k=0}^{n} \frac{t^k}{k!}\frac{\mathscr{A}_0^h x^{n-k}}{(n-k)!}, \quad n \in \mathbb{N}_0.$$

Finally, choosing $C_n^h = \mathscr{A}_0^h x^n$, $n \in \mathbb{N}_0$, we get (3.1). □

We are going to illustrate this fact for the operators given by (2.4), (2.5) and (2.6). In order to give the results for the mentioned operators we need the following auxiliary result.

Lemma 3.3 *Let the function $f : \mathbb{R} \mapsto \mathbb{C}$ be infinitely continuously-differentiable, i.e. $f \in C^\infty(\mathbb{R})$, with all derivatives integrable on $\mathbb{R}$, and assume its Fourier transform is given by $2\pi w$. Then polynomials are integrable with respect to $\chi_\mathbb{R}(x)w(x)dx$, and we have*

$$f^{(n)}(0) = \int_\mathbb{R} (\mathrm{i}x)^n w(x)dx, \quad n \in \mathbb{N}_0.$$

Proof Since $f \in C^\infty$ it is the well-known that, for any $k \in \mathbb{N}_0$, there exist a positive constants C_k such that

$$|w(x)| \le \frac{C_k}{(1+|x|)^k}, \quad x \in \mathbb{R}.$$

This property guaranties that all polynomials are integrable with respect to the measure $\chi_\mathbb{R}(x)w(x)dx$. Using the continuity of f, we have the Fourier inversion formula

$$f(u) = \frac{1}{2\pi}\int_\mathbb{R} e^{\mathrm{i}ux} 2\pi w(x)dx, \quad u \in \mathbb{R}. \tag{3.10}$$

According to Lebesgue theorem of dominant convergence, we can differentiate (3.10), so that we have

$$f^{(n)}(0) = \int_\mathbb{R} (\mathrm{i}x)^n w(x)dx.$$

For the function which dominates $|x|^k|w(x)|$, $k \in \mathbb{N}_0$, we can take

$$\frac{C_{k+2}|x|^k}{(1+|x|)^{k+2}}, \quad k \in \mathbb{N}_0,$$

which is integrable on $\mathbb{R}$. □

This lemma together with Lemma 3.2, suggests that we can search for the representation of the inverse family $(\mathscr{A}_t^h)^{-1}$, in the following way: first take the generating function $f_{(\mathscr{A}_t^h)^{-1}}$ at $t = 0$, find the Fourier transform of it and then use Lemma 3.2 to recover $(\mathscr{A}_t^h)^{-1}$ for all t.

In the sequel we consider the representation of the inverse operators for the mentioned families of operators. In order to be able to present results in a simpler way we introduce the following definition.

Definition 3.3 With λ_ν, $\nu = 1, \ldots, M$, we denote distinct zeros of the characteristic polynomial Q of a family of linear operators $\mathscr{A}^h$, and with m_ν, $\nu = 1, \ldots, M$, their multiplicities, respectively, so that

$$Q(x) = A \prod_{\nu=1}^{M} (x - \lambda_\nu)^{m_\nu}, \tag{3.11}$$

where $\sum_{\nu=1}^{M} m_\nu = \deg(Q)$.

3.3.1 Representation of the inverse family for (2.6)

Since we work only with bijective operators, according to Theorem 3.4, we must have $Q(0) = 1$. We can obtain the moments of the linear functional $\left(\mathscr{A}_t^h\right)^{-1}$ by expanding the moment generating function $f_{\left(\mathscr{A}_t^h\right)^{-1}}(u)$ into a power series in the neighborhood of $u = 0$. Since, Q is a polynomial, at first we can expand the expression $1/Q$ into the partial fraction decomposition, and then expand every term into the power series at the point $u = 0$. According to (3.11) we have the following partial fraction decomposition and the series expansion

$$\frac{1}{Q(hu)} = \sum_{\nu=1}^{M} \sum_{j=1}^{m_\nu} \frac{Q_\nu^j}{(hu - \lambda_\nu)^j} = \sum_{k=0}^{+\infty} \frac{u^k}{k!} \sum_{\nu=1}^{M} \sum_{j=1}^{m_\nu} Q_\nu^j \mu_{k,h}^{\nu,j}, \tag{3.12}$$

where

$$\mu_{k,h}^{\nu,j} = \frac{(-1)^j h^k}{\lambda_\nu^{j+k}} (j)_k.$$

Symbol $(j)_k = \Gamma(j+k)/\Gamma(j)$ is Pochhammer's symbol.

We can conclude that it is enough to give a representation of the linear functional $\mathfrak{L}_{t,h}^{\lambda,m}$, with a characteristic polynomial

$$Q := Q_\lambda^m(z) = \left(\frac{\lambda - z}{\lambda}\right)^m.$$

Then we obviously have

$$\left(\mathscr{A}_t^h\right)^{-1} = \sum_{\nu=1}^{M} \sum_{j=1}^{m_\nu} \frac{(-1)^j Q_\nu^j}{\lambda_\nu^j} \mathfrak{L}_{t,h}^{\lambda_\nu, j}. \tag{3.13}$$

For a representation of the previous linear functional we have the following result:

Theorem 3.7 *Let* $r(\lambda) = \mathrm{sgn}(\Re(\lambda))$ *and* $i(\lambda) = \mathrm{sgn}(\Im(\lambda))$ *and let a family of linear operators, defined by* (2.6)*, have the characteristic polynomial determined by* $Q := Q_\lambda^m(z)$*.*

1° *For* $\Re(\lambda) \neq 0$*, we have*

$$\left(\mathscr{A}_t^h\right)^{-1} p = \left(\frac{\lambda r(\lambda)}{h}\right)^m \frac{1}{\Gamma(m)} \int_{\mathbb{R}^+} p(t + yr(\lambda)) y^{m-1} e^{-\lambda y r(\lambda)/h}\, dy.$$

2° *For* $\Re(\lambda) = 0$*, we have*

$$\left(\mathscr{A}_t^h\right)^{-1} p = \left(\frac{-\mathrm{i}\lambda i(\lambda)}{h}\right)^m \frac{1}{\Gamma(m)} \int_{\mathbb{R}^+} p(t - \mathrm{i}yi(\lambda)) y^{m-1} e^{\mathrm{i}\lambda y i(\lambda)/h}\, dy.$$

Proof According to Lemma 3.2, it is enough to give a representation for $\left(\mathscr{A}_0^h\right)^{-1}$. We are not going to use method based on the Fourier transform, given in Lemma 3.3, since for $m = 1$, we clearly does not have integrability of the function $1/Q_\lambda^1$.

The moments of the functional $\left(\mathscr{A}_0^h\right)^{-1}$ can be obtained easily. Namely, we have

$$f_{\left(\mathscr{A}_0^h\right)^{-1}}(u) = \frac{1}{Q_\lambda^m(hu)} = \left(\frac{1}{1 - hu/\lambda}\right)^m = \sum_{k \in \mathbb{N}_0} \frac{\Gamma(m+k)}{k!\,\Gamma(m)} \left(\frac{hu}{\lambda}\right)^k,$$

so that the moments are $(\Gamma(m+k)/\Gamma(m))(h/\lambda)^k$, $k \in \mathbb{N}_0$.

We present the proof only in the case $\Re(\lambda) > 0$ and $\Im(\lambda) > 0$. For other cases the proof is almost the same. Let the contour C in the complex y-plane be the union of the following arcs

$$\gamma_1^R = \{y \mid 0 \le y \le R\}, \quad \gamma_2^R = \{y \mid |y| = R,\ -\arg(\lambda) \le \arg(y) \le 0\},$$
$$\gamma_3^R = \{y \mid 0 \le y \le R,\ \arg(y) = -\arg(\lambda)\}.$$

The function $y \mapsto G_k(y) = y^k y^{m-1} e^{-\lambda y/h}$, $k \in \mathbb{N}_0$, is analytic, except for the singularity at $y = \infty$. According to the Cauchy residue theorem, we have $\oint_C G_k(y)\, dy = 0$. For the integral over the arc γ_3^R, we have

$$\begin{aligned} \int_{\gamma_3^R} G_k(y)\, dy &= \int_0^R t^{m+k-1} e^{i(m+k)\arg(\lambda)} e^{-|\lambda| t/h}\, dt \\ &= \left(\frac{h}{\lambda}\right)^{m+k} \int_0^{|\lambda| R/h} t^{m+k-1} e^{-t}\, dt \to \left(\frac{h}{\lambda}\right)^{m+k} \Gamma(m+k), \end{aligned}$$

as $R \to +\infty$. It is simple to prove that $\int_{\gamma_2^R} \to 0$ as $R \to +\infty$, which implies $\int_{\gamma_1^{+\infty}} = \int_{\gamma_3^{+\infty}}$. Using the integral calculated over γ_3^R as the value of the integral over γ_1^R, after multiplication with the constant from the statement, we get exactly the moments we need. □

Note that the representation theorem recovers the well-known generalized Laguerre measure. Also, note that in the case $Q := Q_\lambda^m(z)$ and $\lambda > 0$, the representation measure is positive. We have the following result:

Theorem 3.8 *Suppose all the roots of the characteristic polynomial Q, for the family given by (2.6), are positive. The linear functional $\left(\mathscr{A}_t^h\right)^{-1}$ has a representation given by*

$$\left(\mathscr{A}_t^h\right)^{-1} p = \int_{\mathbb{R}^+} p(x+t)\, d\mu^h(x), \quad h > 0, \quad p \in \mathcal{P},$$

where the measure μ^h is positive.

Proof According to Theorem 3.7, we can always represent the measure μ^h from this theorem as a linear combination of the generalized Laguerre measures. So, we search for the measure μ^h in an absolutely continuous form $w^h_{m_1,\ldots,m_M}(x)dx$, where we assume the notation from Definition 3.3.

Let $\sum_{\nu=1}^M m_\nu > 2$. Then, using Lemma 3.3, we have

$$w'_{m_1,\ldots,m_M}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{-\mathrm{i}ue^{-\mathrm{i}ux}}{Q_{m_1,\ldots,m_M}(\mathrm{i}hu)}du$$

i.e.,

$$w'_{m_1,\ldots,m_M}(x) + \frac{\lambda_M}{h} w_{m_1,\ldots,m_M}(x) = \frac{\lambda_M}{2\pi h}\int_{\mathbb{R}} \frac{e^{-\mathrm{i}xu}}{Q_{m_1,\ldots,m_M-1}(\mathrm{i}hu)}du = \frac{\lambda_M}{h} w_{m_1,\ldots,m_M-1}(x).$$

According to the fact that all singularities are placed in the lower half plane and $Q_{m_1,\ldots,m_M}(\mathrm{i}hu)$ is of order at least u^{-3} at infinity, we have

$$w_{m_1,\ldots,m_M}(0) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{du}{Q_{m_1,\ldots,m_M}(\mathrm{i}hu)} = 0.$$

The previous means that our function $w_{m_1,\ldots,m_M}$ is a solution of the following differential equation

$$w'_{m_1,\ldots,m_M}(x) + \frac{\lambda_M}{h} w_{m_1,\ldots,m_M}(x) = \frac{\lambda_M}{h} w_{m_1,\ldots,m_M-1}(x), \quad w_{m_1,\ldots,m_M}(0) = 0,$$

for which we can give the explicit solution in the form

$$w_{m_1,\ldots,m_M}(x) = \frac{\lambda_M}{h} e^{-\lambda_M x/h} \int_0^x e^{\lambda_M t/h} w_{m_1,\ldots,m_M-1}(t)\, dt.$$

It is clear that if $w_{m_1,\ldots,m_M-1}(x) > 0$, $x > 0$, and $w_{m_1,\ldots,m_M-1}(x) = 0$, $x \le 0$, then also $w_{m_1,\ldots,m_M}$ has the same properties. Now, we can apply an inductive argument.

What is left to prove is that the inductive base is true. Thus, we need to prove the statement of theorem for

$$\sum_{\nu=1}^{M} m_\nu \le 2.$$

If the previous sum is one we have already proved it in the representation theorem for the characteristic polynomial $Q := Q_\lambda^1(z)$. If the previous sum equals two we distinguish two cases. The first case, when $m_1 = 2$, is also proved using the representation theorem for $Q := Q_\lambda^2(z)$, and the second one for which $\lambda_1 < \lambda_2$ and $m_1 = m_2 = 1$, $M = 2$.

For this case we calculate directly $w_{1,1}$ and we get

$$w_{1,1}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\lambda_1\lambda_2 e^{-\mathrm{i}xu}}{(\lambda_1 - \mathrm{i}hu)(\lambda_2 - \mathrm{i}hu)} du,$$

i.e.,

$$w_{1,1}(x) = \begin{cases} \dfrac{1}{h}\dfrac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 x/h} - e^{-\lambda_2 x/h}\right), & x > 0, \\ 0, & x \le 0. \end{cases}$$

Thus, we convince ourself that inductive base is satisfied. □

Using Lemma 2.6, we can interpret this result also in the following form:

Theorem 3.9 *Assume that all zeros of the characteristic polynomial Q are positive, then for $h \in \mathbb{R}^+$ and any positive measure μ, the linear functional*

$$\mathcal{L}^h(p) = \int \left((\mathscr{A}^h)^{-1} p\right)(x)\, d\mu(x), \qquad p \in \mathcal{P},$$

is positive definite.

3.3.2 Representation of the inverse family for (2.5)

For the family given by (2.5) we can use also the partial fraction decomposition to get

$$\frac{1}{Q(z)} = \sum_{\nu=1}^{M}\sum_{j=1}^{m_\nu} \frac{Q_\nu^j}{(z-\lambda_\nu)^j}.$$

Using series expansions for the functions $1/(e^{hu} - \lambda_\nu)^j$, $j \in \mathbb{N}_0$, $\nu = 1, \ldots, M$, in the form

$$\frac{1}{(e^{hu} - \lambda_\nu)^j} = \sum_{k\in\mathbb{N}_0} \frac{u^k}{k!}\mu_{k,h}^{\nu,j},$$

we obtain

$$\frac{1}{Q(e^{hu})} = \sum_{k\in\mathbb{N}_0} \frac{u^k}{k!} \sum_{\nu=1}^{M} \sum_{j=1}^{m_\nu} Q_\nu^j \mu_{k,h}^{\nu,j}.$$

Thus, our problem of the representation is reduced to a representation of the family of linear functionals having the characteristic polynomial of the form

$$Q := Q_\lambda^m(z) = \left(\frac{z-\lambda}{1-\lambda}\right)^m.$$

However, we must distinguish the cases $|\lambda| = 1$ and $|\lambda| \neq 1$. (In general, since $Q(1) = 1$, we know that $\lambda_\nu \neq 1$, $\nu = 1, \dots, \deg(Q)$.)

Theorem 3.10 *For $|\lambda| \neq 1$, the linear functional $\left(\mathscr{A}_t^h\right)^{-1}$, with the characteristic polynomial $Q := Q_\lambda^m(z)$, has the following representation*

$$\left(\mathscr{A}_t^h\right)^{-1} p = \begin{cases} (1-\lambda)^m \displaystyle\sum_{j\in\mathbb{N}_0} \binom{m+j-1}{j} \lambda^j p(t-(j+m/2)h), & |\lambda| < 1, \\ \left(1-\dfrac{1}{\lambda}\right)^m \displaystyle\sum_{j\in\mathbb{N}_0} \binom{m+j-1}{j} \dfrac{p(t+(j+m/2)h)}{\lambda^j}, & |\lambda| > 1. \end{cases}$$

Proof If we expand $f_{\left(\mathscr{A}_t^h\right)^{-1}}$ at the point $u = 0$, for $|\lambda| > 1$ we have

$$\begin{aligned} f_{\left(\mathscr{A}_t^h\right)^{-1}}(u) &= (1-\lambda)^m \sum_{k=0}^{+\infty} \frac{u^k}{k!}(t+mh/2)^k \left(\frac{-1}{\lambda}\right)^m \sum_{j=0}^{+\infty} \binom{-m}{j} \frac{e^{jhu}}{(-\lambda)^j} \\ &= \left(1-\frac{1}{\lambda}\right)^m \sum_{k=0}^{+\infty} \frac{u^k}{k!}(t+mh/2)^k \sum_{k=0}^{+\infty} \frac{u^k}{k!} \sum_{j=0}^{+\infty} \binom{m+j-1}{j} \frac{(jh)^k}{\lambda^j} \\ &= \left(1-\frac{1}{\lambda}\right)^m \sum_{k=0}^{+\infty} \frac{u^k}{k!} \sum_{j=0}^{+\infty} \binom{m+j-1}{j} \frac{(t+(j+m/2)h)^k}{\lambda^j}. \end{aligned}$$

Here, we have used the series expansions for the geometric progression and the exponential function, as well as the intensive applications of Fubini's theorem and Cauchy theorem on the product of series. Then we conclude that

$$\mu_{k,h}^t = \left(1-\frac{1}{\lambda}\right)^m \sum_{j=0}^{+\infty} \binom{m+j-1}{j} \frac{(t+(j+m/2)h)^k}{\lambda^j}, \quad k \in \mathbb{N}_0,$$

which finishes the proof in the case $|\lambda| > 1$.

Using a completely similar argumentation we prove the result for $|\lambda| < 1$. □

As it is easily verified we see that in the case $\lambda > 1$ or $0 < \lambda < 1$, we have a positive measure for the representation. In fact in [31], we proved the following theorem.

Theorem 3.11

1° *Suppose all zeros of the characteristic polynomial Q are real and larger than 1. Then, the functionals $\left(\mathscr{A}_t^h\right)^{-1}$ admit a representation over a positive measure supported on the real line.*

2° *Suppose all zeros of the characteristic polynomial Q are positive and smaller than 1. Then, the functionals $\left(\mathscr{A}_t^h\right)^{-1}$ admit a representation over a positive measure supported on the real line.*

In the case when the characteristic polynomial is given by $Q := Q_\lambda^m(z)$, we recognize that the linear functional is well-known linear functional of the Meixner polynomials of the first kind (see [13, p. 176], [24, p. 45]).

For the case $|\lambda| = 1$, with $\lambda \neq 1$, we have the following theorem.

Theorem 3.12 *The linear functional $\left(\mathscr{A}_t^h\right)^{-1}$, given with the characteristic polynomial $Q := Q_\lambda^m(z)$, where $\lambda = e^{i\varphi}$, $\varphi \in (-\pi, \pi]\backslash\{0\}$, has a representation given by*

$$\left(\mathscr{A}_t^h\right)^{-1} p = (\sin|\varphi|/2)^m \int_{\mathbb{R}} p(t + ix) w^\varphi(x)\, dx,$$

where w^φ is a positive function given by

$$w^\varphi(x) = \frac{e^{-(\operatorname{sgn}(\varphi)\pi - \varphi)x/h}}{2\pi h} \int_{\mathbb{R}} \frac{e^{-ixu/h}}{\cosh^m u/2}\, du,$$

i.e.,

$$w^\varphi(x) = \frac{e^{-(\operatorname{sgn}(\varphi)\pi - \varphi)x/h}}{(m-1)!h} \begin{cases} \dfrac{\prod_{k=0}^{(m-3)/2} (4(x/h)^2 + (2k+1)^2)}{\cosh(\pi x/h)}, & m \in 2\mathbb{N}_0 + 1, \\[2ex] \dfrac{2x}{h} \dfrac{\prod_{k=1}^{m/2-1} (4(x/h)^2 + (2k)^2)}{\sinh(\pi x/h)}, & m \in 2\mathbb{N}. \end{cases}$$

Proof The generating function for the moments is given by

$$f_{(\mathscr{A}_t^h)^{-1}}(u) = \frac{\exp(u(t + mh/2))}{Q(e^{hu})},$$

so for $t = 0$, we have

$$f_{(\mathscr{A}_0^h)^{-1}}(u) = \exp(hum/2) \left(\frac{1 - e^{i\varphi}}{e^{hu} - e^{i\varphi}} \right)^m .$$

It is easily checked that $f_{(\mathscr{A}_0^h)^{-1}}(u)$ is infinitely continuously-differentiable, with all derivatives being integrable on the real line. Hence, we can apply Lemma 3.3 to recover the weight function. We have

$$
\begin{aligned}
w(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mathrm{i}ux} f_{(\mathscr{A}_0^h)^{-1}}(u)du = \frac{(1-\lambda)^m}{2\pi}\int_{\mathbb{R}} \frac{e^{-\mathrm{i}ux}e^{hum/2}}{(e^{hu}-e^{\mathrm{i}\varphi})^m}du \\
&= \frac{e^{-\mathrm{i}\varphi m/2}}{2\pi}\left(\frac{1-\lambda}{2}\right)^m \int_{\mathbb{R}} \frac{e^{-\mathrm{i}ux}}{\sinh^m(hu-\mathrm{i}\varphi)/2}du \\
&= \frac{e^{-\mathrm{i}\varphi m/2}}{2\pi h}\left(\frac{1-\lambda}{2}\right)^m \int_{C} \frac{e^{-\mathrm{i}(z+\mathrm{i}\varphi)x/h}}{\sinh^m z/2}dz,
\end{aligned}
$$

where the contour of integration is given by $C = \{z \mid \Im(z) = -\varphi\}$. The function in the integrand is analytic except for the points $z_k = 2k\pi, k \in \mathbb{Z}$. For $\varphi > 0$, we can consider the integral over the contour $\gamma^R = \gamma_1^R \cup \gamma_2^R \cup \gamma_3^R \cup \gamma_4^R$, where

$$
\begin{aligned}
&\gamma_1^R = \{z \mid \Im(z) = -\varphi,\ |\Re(z)| \le R\}, \quad \gamma_2^R = \{z \mid \Im(z) = -\pi,\ |\Re(z)| \le R\}, \\
&\gamma_3^R = \{z \mid \Re(z) = -R,\ -\pi < \Im(z) < -\varphi\}, \\
&\gamma_4^R = \{z \mid \Re(z) = R,\ -\pi < \Im(z) < -\varphi\}.
\end{aligned}
$$

Here, γ^R is a closed contour free of singularities, so that, according to Cauchy residue theorem, $\int_{\gamma^R} = 0$. Taking the limit as $R \to +\infty$, integrals over γ_3^R and γ_4^R are vanish, and we are left with integrals over $\gamma_1^{+\infty}$ and $\gamma_2^{+\infty}$ and these are the same. The limit of the integral over $\gamma_1^{+\infty}$ is equal to the integral over C. We choose to express the result as an integral over $\gamma_2^{+\infty}$, so that for w we have

$$
w(y) = \frac{e^{-\mathrm{i}m\varphi/2}}{2\pi h}\left(\frac{1-\lambda}{-2\mathrm{i}}\right)^m e^{-(\pi-\varphi)x/h}\int_{\mathbb{R}} \frac{e^{-\mathrm{i}ux/h}}{\cosh^m u/2}\,du.
$$

Similar result we get for $\varphi < 0$, in which case the integration should be applied over the contour $\gamma^R = \gamma_1^R \cup \gamma_2^R \cup \gamma_3^R \cup \gamma_4^R$, where

$$
\begin{aligned}
&\gamma_1^R = \{z \mid \Im(z) = -\varphi,\ |\Re(z)| \le R\}, \quad \gamma_2^R = \{z \mid \Im(z) = \pi,\ |\Re(z)| \le R\}, \\
&\gamma_3^R = \{z \mid \Re(z) = -R,\ -\varphi < \Im(z) < \pi\}, \\
&\gamma_4^R = \{z \mid \Re(z) = R,\ -\varphi < \Im(z) < \pi\}.
\end{aligned}
$$

In that case the result is

$$
w(x) = \frac{e^{-\mathrm{i}m\varphi/2}}{2\pi h}\left(\frac{1-\lambda}{2\mathrm{i}}\right)^m e^{(\pi+\varphi)x/h}\int_{\mathbb{R}} \frac{e^{-\mathrm{i}ux/h}}{\cosh^m u/2}\,du.
$$

Now, it is easy to identify w^{φ} from these two expressions.

What is left to prove is to calculate the integral, using two integration by parts, so that we have

$$\int_{\mathbb{R}} \frac{e^{-\mathrm{i}xu/h}}{\cosh^{m+2} u/2}\, du = \frac{4(x/h)^2 + m^2}{m(m+1)} \int_{\mathbb{R}} \frac{e^{-\mathrm{i}xu/h}}{\cosh^m u/2}\, du, \quad m \geq 1.$$

This gives a recurrence relation for the integrals, which reduces to our expression for the weight function w^φ. Thus, we need to calculate only integrals for $m = 1$ and $m = 2$, and they are known to be (see [16])

$$\int_{\mathbb{R}} \frac{e^{-\mathrm{i}xu/h}}{\cosh u/2}\, du = \frac{2\pi}{\cosh \pi x/h}, \qquad \int_{\mathbb{R}} \frac{e^{-\mathrm{i}xu/h}}{\cosh^2 u/2}\, du = \frac{4\pi x/h}{\sinh \pi x/h}.$$

□

Remark 3.2 The weight functions from the previous theorem, for $m = 1$ and $m = 2$ are known as Lindelöf and Abel weight function, respectively (see [14–16, 37]).

The previous theorem gives us an opportunity to find the representation of a general family of linear operators (2.5) in the form

$$\left(\mathscr{A}_t^h\right)^{-1} = \sum_{\nu=1}^{M} \sum_{j=1}^{m_\nu} \frac{Q_\nu^j}{(1-\lambda_\nu)^j} \mathcal{L}^{\nu,j}_{t+(\deg(Q)-j)h/2,h}, \tag{3.14}$$

where $\mathcal{L}^{\nu,j}_{t,h}$ is a representational functional for the family with the characteristic polynomial $Q := Q^j_{\lambda_\nu}(z)$.

There are also some other special cases of the family (2.5), for which we can find an analytic representation of the inverse family. We present those results in the next two theorems.

Theorem 3.13 *Let the family of linear operators be given by* (2.5), *with the characteristic polynomial*

$$Q(z) = \sum_{k=-m}^{m} (-1)^{k+m} z^{k+m} = \frac{z^M + 1}{z + 1}, \qquad M = 2m + 1.$$

The linear functional $\left(\mathscr{A}_t^h\right)^{-1}$ *has the following representation*

$$\left(\mathscr{A}_t^h\right)^{-1} p = \frac{2\cos\frac{\pi}{2M}}{Mh} \int_{\mathbb{R}} \frac{p(t+\mathrm{i}x)\cosh\frac{\pi x}{Mh}}{\cosh\frac{2\pi x}{Mh} + \cos\frac{\pi}{M}}\, dx. \tag{3.15}$$

Proof Direct calculation. □

The measure which appears in this theorem can be connected to the dual Hahn polynomials (see [13, p. 159], [24, p. 34]), although this connection is not so evident. We are going to present that connection in the next subsection.

Theorem 3.14 *Let the family of linear operators be given by* (2.5)*, with the characteristic polynomial*

$$Q(z) = \frac{1}{M}\sum_{k=0}^{M-1} z^k = \frac{1}{M}\frac{z^M - 1}{z - 1}.$$

The linear functional $\left(\mathscr{A}_t^h\right)^{-1}$ *has the following representation*

$$\left(\mathscr{A}_t^h\right)^{-1} p = \frac{\sin\frac{\pi}{M}}{h}\int_{\mathbb{R}} \frac{p(t+\mathrm{i}x)}{\cosh\frac{2\pi x}{Mh} + \cos\frac{\pi}{M}}\,dx. \tag{3.16}$$

Proof Direct calculation. □

The measure which appears in this theorem is a special case of the continuous Hahn polynomials (see [24, p. 31]). We will give this connection in the next subsection.

As it can be seen in the previous two theorems, the measure which appears in the representation is positive. Actually we have the following result:

Theorem 3.15 *Let the family of linear operators be given by* (2.5)*, with a characteristic polynomial Q with all zeros lying on the unit circle. Then the linear functional* $\left(\mathscr{A}_t^h\right)^{-1}$ *has the representation in the form*

$$\left(\mathscr{A}_t^h\right)^{-1} p = \int_{\mathbb{R}} p(t+\mathrm{i}x) w^h(x)\,dx,$$

where the weight function w^h is nonnegative on $\mathbb{R}$.

Proof We are searching for the representation given by

$$\mathscr{A}_t^h p = \int_{\mathbb{R}} p(t+\mathrm{i}x) w^h(x) dx.$$

Using Lemma 3.3, we known we can represent our weight function in the following form

$$w^h(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-\mathrm{i}xu} e^{hu \deg(Q)/2}}{Q(e^{hu})} du,$$

or, we can recast in the following form

$$w^h(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mathrm{i}xu} du \prod_{\nu=1}^{M} \left(\frac{e^{hu/2}}{e^{hu} - \lambda_\nu}\right)^{M_\nu},$$

where $|\lambda_\nu| = 1$, $\lambda_\nu \neq 1$, $\nu = 1, \ldots, n$.

Now, denote $\widehat{w}^h$, a weight function associated with the characteristic polynomial $\widehat{Q}(z) = (1-\lambda_M)Q(z)/(z-\lambda_M)$. Finally, denote by w_1^h a weight function associated with the characteristic polynomial $Q_1(z) = (z-\lambda_M)/(1-\lambda_M)$, which is already given explicitly in Theorem 3.12. According to the mentioned theorem we know that w_1^h is positive everywhere on $\mathbb{R}$.

It is straightforward to see that $Q = \widehat{Q} Q_1$, but then according to the convolution Theorem ([4, p. 6]), we have that the weight functions are connected through the convolution, i.e.,

$$w^h(x) = \int_{\mathbb{R}} w_1^h(t) \widehat{w}^h(x-t) dt.$$

Since w_1^h is positive, using an induction argument, we conclude that w^h is positive for any λ_ν, $\nu = 1, \ldots, n$, which have modulus one, except $\lambda_\nu = 1$.

Thus, we have a representation for $\left(\mathscr{A}_0^h\right)^{-1}$. Using Theorem 3.6 we complete the proof. □

3.3.3 Representation of the inverse family for (2.4)

For the family given by (2.4) we have the following representation theorem.

Theorem 3.16 *Let the family of linear operators be given by*

$$\mathscr{A}^h p = \frac{1}{2h} \int_{x-h}^{x+h} p(u)\, du.$$

Then the inverse family has the following representation

$$\left(\mathscr{A}_t^h\right)^{-1} p = \frac{\pi}{2h} \int_{\mathbb{R}} p(t + \mathrm{i}x) \frac{\exp(-\pi x/h)}{(1 + \exp(-\pi x/h))^2}\, dx.$$

Proof We are searching for the representation given by

$$\left(\mathscr{A}_t^h\right)^{-1} = \int_{\mathbb{R}} p(t + \mathrm{i}x) w(x)\, dx.$$

Using Lemma 3.3, the weight function w can be found as a Fourier transform of the function $f_{\left(\mathscr{A}_0^h\right)^{-1}}$, i.e.,

$$w(x) = \frac{1}{2\pi h} \int_{\mathbb{R}} \frac{u e^{-\mathrm{i}xu/h}}{\sinh u}\, du = \frac{\pi}{h} \frac{e^{-\pi x/h}}{(1 + e^{-\pi x/h})^2}.$$

□

The weight function which appears in this theorem is the well-known *logistic weight function* (see [18]). This result, for example, means that in order to construct quadrature formula of the form

$$\int p(x)\, d\mu(x) = \sum_{k=1}^{n} \frac{w_k}{2h} \int_{x_k-h}^{x_k+h} p(x)\, dx, \qquad p \in \mathcal{P}_{2n-1},$$

we need the n-th polynomial in the sequence of polynomials orthogonal with respect to the linear functional

$$\mathcal{L}^h(p) = \frac{\pi}{2h}\int d\mu(x)\int_{\mathbb{R}} p(x+iy)\frac{\exp(-\pi y/h)}{(1+\exp(-\pi y/h))^2}\,dy, \quad p \in \mathcal{P}.$$

3.4 Polynomials orthogonal with respect to $\left(\mathscr{A}_0^h\right)^{-1}$

In the next section, concerned with the numerical construction, we need to calculate integrals of polynomials with respect to the representation measures from the previous subsection. In order to calculate integrals we are going to apply Gaussian quadrature rules, for whose construction we need polynomials orthogonal with respect to the families of linear functionals $\left(\mathscr{A}_t^h\right)^{-1}$. We give this orthogonal polynomials in this subsection. We continue to examine the case of the degree preserving, continuous families $\mathscr{A}^h$, satisfying the property (3.1).

We summarize some basic facts on Gaussian quadrature rules in the next lemma.

Lemma 3.4 *Let $\mathcal{L} : \mathcal{P} \mapsto \mathbb{C}$ be a regular linear functional and $\{\pi_n\}_{n\in\mathbb{N}_0}$ be a sequence of monic polynomials orthogonal with respect to $\mathcal{L}$. Then, the polynomials π_n satisfy the three-term recurrence relation*

$$\pi_{n+1}(x) = (x-\alpha_n)\pi_n(x) - \beta_n\pi_{n-1}(x), \quad n \in \mathbb{N}_0, \tag{3.17}$$

with $\pi_0(x) = 1$ and $\pi_{-1}(x) = 0$. The zeros of π_n are eigenvalues of the following tridiagonal matrix

$$J_n(\mathcal{L}) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & O \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ O & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix},$$

known as Jacobi matrix. The sequence of monic numerator polynomials $\{\rho_n\}_{n\in\mathbb{N}_0}$, associated to the sequence $\{\pi_n\}_{n\in\mathbb{N}_0}$, satisfies the same three-term recurrence relation (3.17), but with the initial conditions $\rho_{-1}(x) = -1/\beta_0$ and $\rho_0(x) = 0$, where $\beta_0 = \mathcal{L}(1)$. Then, the linear functional $G_n : \mathcal{P}_{2n-1} \mapsto \mathbb{C}$, defined by

$$G_n(p) = \int_C \frac{\beta_0\rho_n(z)}{\pi_n(z)}p(z)\,dz,$$

where C is a simple closed Jordan curve in $\mathbb{C}$ with all zeros of π_n in its interior, is known as the Gaussian quadrature rule for $\mathcal{L}$, and it has the property

$$\mathcal{L}(p) = G_n(p), \quad p \in \mathcal{P}_{2n-1}.$$

In the case all zeros of π_n are simple, using Cauchy residue theorem, G_n can be represented in the standard form

$$G_n(p(x)) = \sum_{k=1}^{n} \omega_k p(x_k), \quad p \in \mathcal{P}_{2n-1},$$

where the weights ω_k, $k = 1, \ldots, n$, can be calculated as multiples by β_0, of the squared first components of the eigenvectors of $J_n(\mathcal{L})$, normalized to have the unit Euclidian norm, and where the nodes x_k, $k = 1, \ldots, n$, are zeros of π_n.

Note that this lemma deals with the general case of regular functionals, i.e., the functional $\mathcal{L}$ need not be positive definite. Unlike the positive definite case where all zeros of π_n (nodes x_k, $k = 1, \ldots, n$, of G_n) are distinct, in the case of general regular functionals this property is not guarantied. This is the reason we choose to present the Gaussian quadrature rule using the contour integral. An application of Cauchy residue theorem in the case of multiple zeros of π_n produces terms with derivatives of p. Finally, we mention that an efficient construction of the Gaussian quadrature rule for a positive definite functional $\mathcal{L}$ can be achieved using QR-algorithm (see [22]).

For a general family of linear functionals $\left(\mathscr{A}_t^h\right)^{-1}$, we cannot claim the regularity, i.e., we cannot claim the existence of the sequence of orthogonal polynomials. For example, for the family $\mathscr{A}^h$, given by (2.5), with the characteristic polynomial

$$Q(z) = \frac{z^2 + 4z + 1}{6},$$

when operators $\mathscr{A}^h$ are basically Simpson quadrature rules, for the moments of $\left(\mathscr{A}_0^h\right)^{-1}$, we get the following sequence of Hankel determinants

$$\Delta_1 = 1, \quad \Delta_2 = -\frac{h^2}{3}, \quad \Delta_3 = -\frac{2h^2}{27}, \quad \Delta_4 = 0.$$

Hence, the linear functional $\left(\mathscr{A}_0^h\right)^{-1}$ is not regular.

We have the following simple result:

Lemma 3.5 *A linear functional $\left(\mathscr{A}_t^h\right)^{-1}$ is regular if and only if $\left(\mathscr{A}_0^h\right)^{-1}$ is regular. Furthermore, if we denote the monic polynomials orthogonal with respect to $\left(\mathscr{A}_t^h\right)^{-1}$ and $\left(\mathscr{A}_0^h\right)^{-1}$, with $\pi_n^{t,h}$ and $\pi_n^{0,h}$, respectively, we have*

$$\pi_n^{0,h}(-t + x) = \pi_n^{t,h}(x), \quad n \in \mathbb{N}_0.$$

Proof Assume that $\left(\mathscr{A}_0^h\right)^{-1}$ is regular. According to Lemma 3.2, we have

$$\left(\mathscr{A}_t^h\right)^{-1} x^k = \left(\mathscr{A}_0^h\right)^{-1} (t + x)^k, \quad k \in \mathbb{N}_0. \tag{3.18}$$

This can be considered also as a triangular system of equations for the moments $\left(\mathscr{A}_0^h\right)^{-1}x^k$, $k \in \mathbb{N}_0$. Since the corresponding determinant equals one, the system has a unique solution given by

$$\left(\mathscr{A}_t^h\right)^{-1}(-t+x)^k = \left(\mathscr{A}_0^h\right)x^k, \quad k \in \mathbb{N}_0. \tag{3.19}$$

Now assume $\left(\mathscr{A}_0^h\right)^{-1}$ is regular with the monic orthogonal polynomial sequence $\{\pi_n^{0,h}\}_{n\in\mathbb{N}_0}$, according to (3.18) and [13, p. 25], we can obtain the monic orthogonal polynomials with respect to $\left(\mathscr{A}_t^h\right)^{-1}$ by merely shifting the arguments as given in the statement of theorem.

For the rest of the proof we use the same arguments with equation (3.19).

Now suppose $n \neq k$. Then we have $\left(\mathscr{A}_0^h\right)^{-1}\pi_n^{0,h}(x)\pi_k^{0,h}(x) = 0$, but also

$$\begin{aligned}
\left(\mathscr{A}_t^h\right)^{-1}\pi_n^{t,h}(x)\pi_k^{t,h}(x) &= \left(\mathscr{A}_0^h\right)^{-1}\pi_n^{t,h}(t+x)\pi_n^{t,h}(t+x)\\
&= \left(\mathscr{A}_0^h\right)^{-1}\pi_n^{0,h}(-t+t+x)\pi_n^{0,h}(-t+t+x)\\
&= \left(\mathscr{A}_0^h\right)^{-1}\pi_n^{0,h}(x)\pi_k^{0,h}(x) = 0.
\end{aligned}$$

□

The order in which we present results about orthogonal polynomials, completely follows the previous subsection.

3.4.1 Calculation of $\left(\mathscr{A}_t^h\right)^{-1}$ *for the family* (2.6)

Using (3.13), it is enough to be able to calculate the values of the functionals $\left(\mathscr{A}_t^h\right)^{-1}$ for which the corresponding characteristic polynomial is given by $Q := Q_\lambda^m(z)$. A representation of such functionals is given in Theorem 3.7.

Theorem 3.17 *Let x_k, ω_k, $k = 1, \dots, n$, be nodes and weights for the generalized Gauss–Laguerre quadrature rule with respect to the measure $x^{m-1}e^{-x}\chi_{\mathbb{R}+}(x)dx$. Then, for the linear functionals from Theorem* 3.7, *we have*

$$\left(\mathscr{A}_t^h\right)^{-1}p = \frac{1}{\Gamma(m)}G_n\left(p\left(t+\frac{hx}{\lambda}\right)\right) = \frac{1}{\Gamma(m)}\sum_{k=1}^{n}\omega_k p\left(t+\frac{x_k h}{\lambda}\right), \quad p \in \mathcal{P}_{2n-1}. \tag{3.20}$$

Proof Using nearly the same arguments as in the proof of representation Theorem 3.7, we can prove that the linear functionals $\left(\mathscr{A}_t^h\right)^{-1}$ can be expressed as

$$\left(\mathscr{A}_t^h\right)^{-1}p = \frac{1}{\Gamma(m)}\int_{\mathbb{R}^+} p\left(t+\frac{xh}{\lambda}\right)x^{m-1}e^{-x}\,dx.$$

Since the well-known Gaussian quadrature rule with respect to the generalized Laguerre measure $x^{m-1}e^{-x}\chi_{\mathbb{R}^+}(x)dx$, with nodes x_k and weights ω_k, $k = 1, \dots, n$, integrates exactly all polynomials of degree at most $2n-1$ (see

[13, p. 31], [48, p. 47], [19, p. 22]), we find that the statement of this theorem holds true. □

We remark that the three-term recurrence coefficients for the generalized Laguerre polynomials are

$$\alpha_k = 2k + m, \quad \beta_k = k(k + m - 1),$$

with $\beta_0 = \Gamma(m)$ (cf. [19, p. 29]).

Since we can identify the monic polynomials orthogonal with respect to $\left(\mathscr{A}_0^h\right)^{-1}$ as

$$\pi_n^{0,h}(x) = \left(\frac{h}{\lambda}\right)^n L_n^{m-1}\left(\frac{\lambda x}{h}\right), \quad n \in \mathbb{N}_0,$$

where L_n^{m-1}, $n \in \mathbb{N}_0$, are monic generalized Laguerre polynomials orthogonal with respect to $x^{m-1}e^{-x}\chi_{\mathbb{R}^+}(x)dx$, the previous consideration also grants the regularity of $\left(\mathscr{A}_t^h\right)^{-1}$.

3.4.2 Calculation of $\left(\mathscr{A}_t^h\right)^{-1}$ for the family (2.5)

Again using (3.14), it is enough to know the way how to calculate $\left(\mathscr{A}_0^h\right)^{-1}$ for the characteristic polynomial $Q := Q_\lambda^m(z)$. The representation of such functionals is given in Theorems 3.12 and 3.10.

Theorem 3.18 *Let a family of operators (2.5) be given by the characteristic polynomial $Q := Q_\lambda^m(z)$, $\lambda \neq 0$. Then the linear functionals $\left(\mathscr{A}_t^h\right)^{-1}$ are regular and the monic polynomials $\{\pi_n^{0,h}\}_{n\in\mathbb{N}_0}$ orthogonal with respect to $\left(\mathscr{A}_0^h\right)^{-1}$ satisfy the following three-term recurrence relation*

$$\begin{aligned}\pi_{n+1}^{0,h}(x) = &\left(x - \frac{\lambda+1}{\lambda-1}\left(n + \frac{1}{2}m\right)h\right)\pi_n^{0,h}(x)\\ &- \frac{\lambda}{(\lambda-1)^2}n(n+m-1)h^2\pi_{n-1}^{0,h}(x), \quad n \in \mathbb{N}_0.\end{aligned} \tag{3.21}$$

In the case $\lambda > 1$ or $0 < \lambda < 1$, the linear functional $\left(\mathscr{A}_t^h\right)^{-1}$ can be expressed in the following form

$$\left(\mathscr{A}_t^h\right)^{-1}p = G_n(p(t + xh)) = \sum_{k=0}^{n} \omega_k p(t + x_k h), \quad p \in \mathcal{P}_{2n-1}. \tag{3.22}$$

The nodes x_k and weights ω_k, $k = 1, \ldots, n$, in the Gaussian quadrature rule are constructed for the three-term recurrence coefficients

$$\alpha_n = \frac{\lambda+1}{\lambda-1}\left(n + \frac{1}{2}m\right), \quad \beta_n = \frac{\lambda}{(\lambda-1)^2}n(n+m-1),$$

with $\beta_0 = 1$.

In the case $\lambda = e^{i\varphi}$, $\lambda \neq 1$, $\left(\mathscr{A}_t^h\right)^{-1}$ *can be calculated using the following quadrature rule*

$$\left(\mathscr{A}_t^h\right)^{-1} p = G_n(p(t + ixh)) = \sum_{k=1}^{n} \omega_k p(t + ix_k h), \tag{3.23}$$

where the nodes x_k *and the weights* ω_k, $k = 1, \ldots, n$, *are determined for the Gaussian quadrature rule for three-term recurrence coefficients*

$$\alpha_n = \frac{n + m/2}{\tan \varphi/2}, \qquad \beta_n = \frac{n(n + m - 1)}{4 \sin^2 \varphi/2},$$

with $\beta_0 = 1$.

Proof This result can be found in parts in the various references. For example, the special case $|\lambda| = 1$ is connected with Meixner–Pollaczek polynomials or Meixner polynomials of the second kind (see [24, p. 37], [13, p. 179]). The case $|\lambda| \neq 1$ is treated originally by Meixner and related polynomials are known as the Meixner polynomials of the first kind (see [24, p. 45], [13, p. 175]). For $\lambda = -1$, the result for three-term recurrence coefficients has been proved by Stieltjes (see [50, p. 53,93], [16]). Especially, the case for $|\lambda| > 1$ and $m = 1$ has been considered by Carlitz (see [13, p. 177]).

The second part of this theorem connected with calculations is obvious in the cases $\lambda > 1$ and $0 < \lambda < 1$. It needs some comment only for $\lambda = e^{i\varphi}$, $\lambda \neq 1$. If we put $\lambda = e^{i\varphi}$ into the recurrence relation, we get

$$\alpha_n = -i\frac{n + m/2}{\tan \varphi/2}, \qquad \beta_n = -\frac{n(n + m - 1)}{4 \sin^2 \varphi/2}.$$

If we substitute $x := -ix$ into the three-term recurrence relation we obtain what is stated. □

If we allow usage of the generalized Gaussian quadrature rules for the general regular linear functionals, we can state further.

Theorem 3.19 *Suppose* $|\lambda| > 1$ *or* $0 < |\lambda| < 1$, *then we calculate* $\left(\mathscr{A}_t^h\right)^{-1}$ *in the following form*

$$\left(\mathscr{A}_t^h\right) p = G_n(p(t + xh)), \qquad p \in \mathcal{P}_{2n-1}, \tag{3.24}$$

where G_n *acts on* x *and is constructed for the three-term recurrence coefficients*

$$\alpha_n = \frac{\lambda + 1}{\lambda - 1}\left(n + \frac{1}{2}m\right), \qquad \beta_n = \frac{\lambda}{(\lambda - 1)^2} n(n + m - 1),$$

with $\beta_0 = 1$.

For the special cases of families given in Theorems 3.13 and 3.14, we present results in the next two theorems.

Theorem 3.20 *Let the family (2.5) be given by the characteristic polynomial*

$$Q(z) = \sum_{k=-m}^{m} (-1)^{k+m} z^k = \frac{z^M + 1}{z + 1}.$$

Then, the linear functionals $\left(\mathscr{A}_t^h\right)^{-1}$ are regular. The monic polynomials $\{\pi_n^{0,h}\}_{n\in\mathbb{N}_0}$ orthogonal with respect to the $\left(\mathscr{A}_0^h\right)^{-1}$ satisfy the following three-term recurrence relation

$$\pi_{n+1}^{0,h}(-\mathrm{i}x) = -\mathrm{i}x\pi_n^{0,h}(-\mathrm{i}x) - h^2\beta_n\pi_{n-1}^{0,h}(-\mathrm{i}x),$$

where

$$\beta_0 = 1, \quad \beta_n = \begin{cases} \dfrac{1}{4}(M^2n^2 - 1), & n\,(odd\,) \geq 1, \\[2ex] \dfrac{1}{4}\,M^2n^2, & n\,(even\,) \geq 2. \end{cases} \tag{3.25}$$

If x_k and ω_k, $k = 1, \dots, n$, are Gaussian nodes and weights, respectively, constructed for three-term recurrence coefficients (3.25), then

$$\left(\mathscr{A}_t^h\right)^{-1} p = G_n(p(t + \mathrm{i}xh)) = \sum_{k=1}^{n} \omega_k p(t + \mathrm{i}x_k h), \quad p \in \mathcal{P}_{2n-1}.$$

Proof This is a special case connected with continuous dual Hahn polynomials, but this connection is not quite obvious. Denote the weight in the representation integral (3.15) by w. At first, note that the measure which appears in the representation integral is symmetric. Hence, we can construct two sequences of orthogonal polynomials by the well-known process explained, for example, in [13, pp. 45–47]. Denote the monic polynomials orthogonal with respect to the representation measure by π_n, $n \in \mathbb{N}_0$, then by the symmetry argument (see [13, pp. 45–47]), we can choose

$$q_n(t) = \pi_{2n}(\sqrt{t}), \quad \sqrt{t}\, r_n(t) = \pi_{2n+1}(\sqrt{t}), \quad n \in \mathbb{N}_0,$$

to be two polynomial sequences orthogonal with respect to the weights $w(\sqrt{t})/\sqrt{t}$ and $\sqrt{t}\, w(\sqrt{t})$ on $\mathbb{R}^+$, respectively. The corresponding recurrences for these polynomials are

$$q_{n+1}(x) = (x - \beta_{2n+1} - \beta_{2n})q_n(x) - \beta_{2n}\beta_{2n-1}q_{n-1}(x),$$
$$r_{n+1}(x) = (x - \beta_{2n+2} - \beta_{2n+1})r_n(x) - \beta_{2n+1}\beta_{2n}r_{n-1}(x).$$

It is known that the continuous dual Hahn polynomials have three-term recurrence coefficients given by

$$\alpha_k = 2k^2 + k(2a + 2b + 2c - 1) + ab + ac + bc,$$
$$\beta_k = k(k + a + b - 1)(k + a + c - 1)(k + b + c - 1).$$

Choosing

$$a = \frac{1}{2}\left(1 - \frac{1}{M}\right), \quad b = \frac{1}{2}\left(1 + \frac{1}{M}\right), \quad c = 0,$$

and substituting $t := t/(Mh)$, we get the q-sequence. For r-sequence, we make a normalization again with $t := t/(Mh)$ and we choose

$$a = \frac{1}{2}\left(1 - \frac{1}{M}\right), \quad b = \frac{1}{2}\left(1 + \frac{1}{M}\right), \quad c = 1.$$

The orthogonality relation for continuous dual Hahn polynomials is given by (see [24, p. 29])

$$\int_{\mathbb{R}^+} \left| \frac{\Gamma(a + \mathrm{i}x)\Gamma(b + \mathrm{i}x)\Gamma(c + ix)}{\Gamma(2\mathrm{i}x)} \right|^2 p_n(x^2) p_m(x^2)\, dx = \delta_{m,n},$$

for the choices we made for a, b, and c, we get our weight functions $w(\sqrt{t})/\sqrt{t}$ and $\sqrt{t}\, w(\sqrt{t})$, respectively.

The last statement in this theorem holds due to the general theory of orthogonal polynomials (see [13, p. 31], [19, p. 22], [48, p. 47]). □

Theorem 3.21 *Let the family* (2.5) *be given by the characteristic polynomial*

$$Q(z) = \frac{1}{M} \sum_{k=0}^{M-1} z^k = \frac{z^M - 1}{z - 1}.$$

Then, the linear functionals $\left(\mathscr{A}_t^h\right)^{-1}$ *are regular. The monic polynomials* $\{\pi_n^{0,h}\}_{n\in\mathbb{N}_0}$ *orthogonal with respect to* $\left(\mathscr{A}_0^h\right)^{-1}$ *satisfy the following three-term recurrence relation*

$$\pi_{n+1}^{0,h}(-\mathrm{i}x) = -\mathrm{i}x\pi_n^{0,h}(-\mathrm{i}x) - \beta_n h^2 \pi_{n-1}^{0,h}(-\mathrm{i}x), \quad n \in \mathbb{N}_0,$$

where

$$\beta_n = \frac{n^2(M^2n^2 - 1)}{4(4n^2 - 1)}, \quad n \in \mathbb{N}_0, \quad \beta_0 = 1. \tag{3.26}$$

If x_k and ω_k, $k = 1, \ldots, n$, are Gaussian nodes and weights, respectively, constructed for three-term recurrence coefficients (3.26)*, then*

$$\left(\mathscr{A}_t^h\right)^{-1} p = G_n(p(t + \mathrm{i}xh)) = \sum_{k=1}^{n} \omega_k p(t + \mathrm{i}x_k h), \quad p \in \mathcal{P}_{2n-1}.$$

Proof This is a special case of the continuous Hahn polynomials, which satisfy the following orthogonality relation (see [24, p. 31])

$$\int_{\mathbb{R}} \Gamma(a+\mathrm{i}x)\Gamma(b+\mathrm{i}x)\Gamma(c-\mathrm{i}x)\Gamma(d-\mathrm{i}x)p_n(x)p_m(x)\,dx = \delta_{n,m}.$$

Choosing

$$a = d = \frac{1}{2}\left(1+\frac{1}{M}\right) \quad \text{and} \quad b = c = \frac{1}{2}\left(1-\frac{1}{M}\right),$$

and substituting $x := x/(Mh)$, we get the statement.

The last statement in this theorem holds due to the general theory of orthogonal polynomials (see [13, p. 31], [19, p. 22], [48, p. 47]). □

3.4.3 Calculation of the $\left(\mathscr{A}_t^h\right)^{-1}$ for the family (2.4)

In this case the problem of the calculation can be reduced into one simple statement.

Theorem 3.22 *Let x_k and ω_k, $k=1,\ldots,n$, be nodes and weights of the Gaussian quadrature rule for the logistic measure $\pi\exp(-\pi x)/(1+\exp(-\pi x))^2dx$, and let the family $\mathscr{A}^h$ be given by (2.4). Then, we have*

$$\left(\mathscr{A}_t^h\right)^{-1} p = G_n(p(t+\mathrm{i}xh)) = \sum_{k=1}^{n} \omega_k p(t+\mathrm{i}x_k h), \quad p \in \mathcal{P}_{2n-1}.$$

The three-term recurrence coefficients for the monic polynomials orthogonal with respect to the logistic weight function are (cf. [14–16, 37])

$$\alpha_k = 0, \quad \beta_k = \frac{k^4\pi^2}{4k^2-1}, \quad k \in \mathbb{N},$$

with $\beta_0 = 1$.

Also it is obvious that the functionals $\left(\mathscr{A}_t^h\right)^{-1}$ are regular.

4 Numerical construction

In this section we consider the numerical construction of the quadrature rule (2.3). As it is proven in Theorem 2.3, in order to find the nodes for the quadrature rule

$$\int p(x)\,d\mu(x) = \sum_{k=1}^{n} w_k(\mathscr{A}^h p)(x_k), \quad p \in \mathcal{P}_{2n-1}, \tag{4.1}$$

we need to construct n-th polynomial orthogonal with respect to the linear functional

$$\mathcal{L}^h(p) = \int \big((\mathscr{A}^h)^{-1} p\big)(x)\, d\mu(x), \qquad p \in \mathcal{P}. \tag{4.2}$$

In the rest of this paper we denote by $\{p_k^h\}_{k\in\mathbb{N}_0}$ the sequence of monic polynomials orthogonal with respect to $\mathcal{L}^h$ and we assume they satisfy the following three-term recurrence relation

$$p_{k+1}^h(x) = \big(x - \alpha_k^h\big) p_k^h(x) - \beta_k^h p_{k-1}^h(x), \qquad p_0^h(x) = 1,\ p_{-1}^h(x) = 0. \tag{4.3}$$

Although β_0^h can be arbitrary, since it multiples $p_{-1}^h(x) = 0$, it is convenient for later purposes to put $\beta_0^h = \mathcal{L}^h(1)$.

The next theorem, which is quite similar to the well-known theorem of Golub and Welsch [22], shows that what we really need for the construction of the quadrature rule (2.3) are three-term recurrence coefficients α_k^h, β_k^h, $k = 0, 1, \ldots, n-1$.

Theorem 4.1 *The nodes x_k, $k = 1, \ldots, n$, of the quadrature rule* (2.3) *are eigenvalues of the three-diagonal Jacobi matrix*

$$J_n^h = \begin{bmatrix} \alpha_0^h & \sqrt{\beta_1^h} & & & O \\ \sqrt{\beta_1^h} & \alpha_1^h & \sqrt{\beta_2^h} & & \\ & \sqrt{\beta_2^h} & \alpha_2^h & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}^h} \\ O & & & \sqrt{\beta_{n-1}^h} & \alpha_{n-1}^h \end{bmatrix}.$$

Let $\boldsymbol{v}_k$, $k = 1, \ldots, n$, be the eigenvectors of the matrix J_n^h, with the Euclidian norm one, corresponding to the eigenvalues x_k, $k = 1, \ldots, n$. Then for weights we have $w_k = \beta_0^h \mathrm{v}_{k,1}^2$, $k = 1, \ldots, n$, where $\mathrm{v}_{k,1}$ is the first components of the vector $\boldsymbol{v}_k$, $k = 1, \ldots, n$.

Proof According to Lemma 3.4, the statement for nodes is correct. Using the same lemma we know that expressions $\widehat{w}_k = \beta_0^h v_{k,1}^2$, $k = 1, \ldots, n$, represent the weights for the Gaussian quadrature rule constructed for the linear functional $\mathcal{L}^h$, defined by (4.2). What we need to prove only is that weights for the quadrature rule for the linear functional $\mathcal{L}^h$ are the same as for our quadrature rule (2.3).

In the quadrature rule (2.3), we choose $f = p$, where p is a polynomial such that

$$p(t) = \left(\mathscr{A}_t^h\right)^{-1} \frac{p_n^h(x)}{(x-x_\ell)(p_n^h)'(x_\ell)} \frac{p_n^h(x)}{(x-x_\nu)(p_n^h)'(x_\nu)} \in \mathcal{P}_{2n-1}.$$

Then we obtain easily

$$w_\nu = \int \left(\left(\mathscr{A}_t^h\right)^{-1} \left(\frac{p_n^h(x)}{(x-x_\nu)(p_n^h)'(x_\nu)} \right)^2 \right) d\mu(t), \quad \nu = 1, \ldots, n,$$

or we can recast it in the following form

$$w_\nu = \mathcal{L}^h \left(\frac{p_n^h(\cdot)}{(\cdot - x_\nu)(p_n^h)'(x_\nu)} \right)^2, \quad \nu = 1, \ldots, n.$$

This means that our weights w_k, $k = 1, \ldots, n$, are the same as the weights in the Gaussian quadrature rule for the linear functional $\mathcal{L}^h$. □

In [39], it was proven that if we choose μ to be the Legendre measure and $\mathscr{A}^h$ to be the following family

$$\mathscr{A}^h p = \frac{1}{2h} \int_{x-h}^{x+h} p(u)\, du, \quad p \in \mathcal{P},$$

then the sequence of monic polynomials orthogonal with respect to the linear functional $\mathcal{L}^h$, defined in (4.2), satisfies the following three-term recurrence relation

$$p_{k+1}^h(x) = x p_k^h(x) - \frac{\left(1 - h^2k^2\right)k^2}{4k^2 - 1} p_{k-1}^h(x), \quad k \in \mathbb{N}_0, \quad p_0^h(x) = 1,\ p_{-1}^h(x) = 0.$$

This result completely solves the question of the construction of the quadrature rule of the form

$$\int_{-1}^{1} p(x)\, dx = \sum_{k=1}^{n} \frac{w_k}{2h} \int_{x_k-h}^{x_k+h} p(u)\, du, \quad p \in \mathcal{P}_{2n-1}.$$

However, for a general combination of a measure μ and a family $\mathscr{A}^h$ in (4.2), we do not know three-term recurrence coefficients for the sequence of orthogonal polynomials, but we present few more partial results in the next subsection. Furthermore, there exists a general numerical procedure based on an old Stieltjes' idea from 1884.

Using the orthogonality of the sequence $\{p_k^h\}_{k\in\mathbb{N}_0}$, from (4.3) we can get the recursion coefficients α_k^h and β_k^h in the following (Darboux's) form

$$\alpha_k^h = \frac{\mathcal{L}^h\left(x\left(p_k^h\right)^2\right)}{\mathcal{L}^h\left(\left(p_k^h\right)^2\right)}, \quad \beta_k^h = \frac{\mathcal{L}^h\left(\left(p_k^h\right)^2\right)}{\mathcal{L}^h\left(\left(p_{k-1}^h\right)^2\right)}. \tag{4.4}$$

The *Stieltjes procedure* is a combination of Darboux's formulae (4.4) with the basic three-term recurrence relation (4.3). Thus, assuming we are able to calculate the values of the functional $\mathcal{L}^h$ on $\mathcal{P}_{2n-1}$, we can apply Darboux's formulae (4.4) in tandem with the basic linear relation (4.3) in order to construct the recursion coefficients α_k^h and β_k^h for $k \le n-1$.

Since $p_0^h(x) = 1$, we can compute α_0^h from (4.4) for $k = 0$, and $\beta_0^h = \mathcal{L}^h(1)$. Having obtained α_0^h, we then use (4.3) with $k = 0$ to compute $p_1^h(x)$. Now, we reapply Darboux's formulae (4.4), with $k = 1$, in order to obtain α_1^h and β_1^h. With these coefficients, using (4.3) for $k = 1$, we calculate $p_2^h(x)$. Thus, in this way, alternating between Darboux's formulae and the three-term recurrence relation (4.3), we can determine all desired coefficients α_k^h, β_k^h, $k \le n-1$.

A crucial assumption is that we are able to compute the values of the linear functional $\mathcal{L}^h$ defined by (4.2). This task can be accomplished provided a computation of the inverse operators $(\mathscr{A}^h)^{-1}$ or equivalently a computation of the associated linear functionals $\left(\mathscr{A}_t^h\right)^{-1}$, can be done using appropriate quadrature rules. In that cases we call such a procedure as the *Stieltjes–Gautschi procedure* (see [18], [19, p. 95]).

For the special families (2.6), (2.5) and (2.4), we discussed the computation question of the linear functionals $\left(\mathscr{A}_t^h\right)^{-1}$ in Subsection 3.4. This allows us to formulate the following results:

Theorem 4.2 *Let $\mathscr{A}^h$ be the family of linear operators* (2.6)*, given by the characteristic polynomial Q, with mutually distinct zeros λ_ν of the corresponding multiplicities m_ν, $\nu = 1, \ldots, M$. Let the measure μ be given and let G_n^μ be a Gaussian quadrature rule for the measure μ. Finally, let $\mathcal{L}_{t,h}^{\nu,j}$ be the linear functional with the characteristic polynomial $Q := Q_{\lambda_\nu}^j(z)$. Then, for the linear functional*

$$\mathcal{L}^h(p) = \int \big((\mathscr{A}^h)^{-1} p\big)(x) d\mu(x), \quad p \in \mathcal{P},$$

with (3.13)*, we have*

$$\mathcal{L}^h(p) = \sum_{\nu=1}^{M} \sum_{j=1}^{m_\nu} \frac{(-1)^j Q_\nu^j}{\lambda_\nu^j} G_n^\mu\big(\mathcal{L}_{0,h}^{\nu,j}(p(x+t))\big), \quad p \in \mathcal{P}_{2n-1},$$

where G_n^μ is assumed to act on x and $\mathcal{L}_{0,h}^{\nu,j}$ acts on t. The computation of the linear functionals $\mathcal{L}_{0,h}^{\nu,j}$ on $\mathcal{P}_{2n-1}$ can be done using quadrature rules given in (3.20)*. Applying the n-point Gaussian quadrature rules for the measure μ and linear functionals $\mathcal{L}_{0,h}^{\nu,j}$, $j = 1, \ldots, m_\nu$, $\nu = 1, \ldots, M$, using the Stieltjes–Gautschi procedure we calculate exactly the coefficients α_k^h, β_k^h, $k = 0, 1, \ldots, n-1$.*

Proof It is just enough to note that in (4.4), we need to compute the value of the functional $\mathcal{L}^h$ for a polynomial with the highest degree $2n-1$ in order to calculate mentioned three-term recurrence coefficients. □

A similar result with the same argumentation can be stated for the other family of operators.

Theorem 4.3 *Let $\mathscr{A}^h$ be the family of linear operators (2.5), given by the characteristic polynomial Q, with mutually distinct zeros are λ_ν of the corresponding multiplicities m_ν, $\nu = 1, \ldots, M$. Let the measure μ be given and let G_n^μ be the Gaussian quadrature rule for the measure μ. Finally, let $\mathcal{L}_{t,h}^{\nu,j}$ be the linear functional with characteristic polynomial $Q := Q_{\lambda_\nu}^j(z)$. Then, for the linear functional*

$$\mathcal{L}^h(p) = \int \big((\mathscr{A}^h)^{-1} p\big)(x)\, d\mu(x), \quad p \in \mathcal{P},$$

with (3.14), we have

$$\mathcal{L}^h(p) = \sum_{\nu=1}^{M} \sum_{j=1}^{m_\nu} \frac{Q_\nu^j}{(1-\lambda_\nu)^j} G_n^\mu\big(\mathcal{L}_{0,h}^{\nu,j}(p(x + t + (\deg(Q) - j)h/2))\big), \quad p \in \mathcal{P}_{2n-1}, \tag{4.5}$$

where G_n is assumed to act on x and $\mathcal{L}_{0,h}^{\nu,j}$ acts on t. The computation of the linear functionals $\mathcal{L}_{0,h}^{\nu,j}$ on $\mathcal{P}_{2n-1}$ can be done using quadrature rules given in (3.22) for $0 < \lambda_\nu < 1$ and $\lambda_\nu > 1$, in (3.23) for $|\lambda_\nu| = 1$, $\lambda_\nu \neq 1$, or in (3.24) for the general complex $|\lambda_\nu| \neq 1$. Applying the n-point Gaussian quadrature rules for the measure μ and the linear functionals $\mathcal{L}_{0,h}^{\nu,j}$, $j = 1, \ldots, m_\nu$, $\nu = 1, \ldots, M$, using the Stieltjes–Gautschi procedure we calculate exactly the coefficients α_k^h, β_k^h, $k = 0, 1, \ldots, n-1$.

Finally, the Stieltjes–Gautschi procedure for the linear operators (2.4) has a more explicit form.

Theorem 4.4 *Let the measure μ be given and G_n^μ be a Gaussian quadrature rule for the measure μ. Let a family of linear operators $\mathscr{A}^h$ be given by (2.4) and let G_n be the Gaussian quadrature rule from Theorem 3.7. Then*

$$\mathcal{L}^h(p) = \int \big((\mathscr{A}^h)^{-1} p\big)(x)\, d\mu(x) = G_n^\mu(G_n(p(t + \mathrm{i}xh))), \quad p \in \mathcal{P}_{2n-1},$$

where G_n^μ acts on t and G_n on x. Using the last expression for $\mathcal{L}^h$, with the Stieltjes–Gautschi algorithm (4.4), we calculate exactly the coefficients α_k^h, β_k^h, $k = 0, 1, \ldots, n-1$.

We illustrate a numerical construction of (2.3) in the following example.

Example 4.1 We are going to construct a Gaussian quadrature of the following form

$$\int_{-1}^{1} p(x) \frac{dx}{\sqrt{1-x^2}} = \sum_{k=1}^{n} w_k \left(\frac{1}{3} p(x_k - h) - 2p(x_k) + \frac{8}{3} p(x_k + h)\right), \tag{4.6}$$

which is exact for each $p \in \mathcal{P}_{2n-1}$.

At first, we recognize the family of the linear functionals to be

$$(\mathscr{A}^h p)(x) = \frac{1}{3}p(x-h) - 2p(x) + \frac{8}{3}p(x+h), \quad p \in \mathcal{P}. \tag{4.7}$$

According to Definition 3.1, the characteristic polynomial is given by

$$Q(z) = \frac{1}{3} - 2z + \frac{8}{3}z^2.$$

As it can easily be seen we have $Q(1) = 1$, so that, according to Theorem 3.4, our family is a family of continuous isomorphisms.

In order to be able to apply the Stieltjes–Gautschi procedure we need a representation of the inverse family $(\mathscr{A}^h)^{-1}$. Since

$$Q(z) = \frac{8}{3}(z-1/2)(z-1/4),$$

we have

$$\frac{1}{Q(z)} = \frac{3}{2}\frac{1}{z-1/2} - \frac{3}{2}\frac{1}{z-1/4}.$$

According to Theorem 3.10 and relation (3.14), we find the representation

$$\left(\mathscr{A}_0^h\right)^{-1} p = \frac{3}{2}\sum_{j\in\mathbb{N}_0}\left(\frac{1}{2^j} - \frac{1}{4^j}\right)p(-jh), \quad p \in \mathcal{P},$$

which means we have to construct orthogonal polynomials with respect to

$$\mathcal{L}^h(p) = \frac{3}{2}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}\sum_{j\in\mathbb{N}_0}\left(\frac{1}{2^j} - \frac{1}{4^j}\right)p(x-jh), \quad p \in \mathcal{P}. \tag{4.8}$$

According to Theorem 3.11 (and we can check also directly), we even know that the functional $\left(\mathscr{A}_0^h\right)^{-1}$ is positive definite. It gives the positive definiteness of $\mathcal{L}^h$, $h \in \mathbb{R}^+$, which means that for every $h \in \mathbb{R}^+$, the orthogonal polynomial sequence with respect to $\mathcal{L}^h$ exists. Thus, for every $h \in \mathbb{R}^+$, our quadrature rule exists with real nodes and positive weights, except that we have to expect that for some h, sufficiently large, the nodes of our quadrature rule will not be inside $(-1, 1)$.

An application of Theorem 4.3 gives us a possibility for the construction of the term recurrence coefficients for polynomials orthogonal with respect to $\mathcal{L}^h$. In relation (4.5) the quadrature rule G_n^μ is simply a Gaussian quadrature rule constructed for the Chebyshev measure of the first kind, while $\mathcal{L}_{0,h}^{1,1}$ and $\mathcal{L}_{0,h}^{2,1}$ can be calculated using quadrature rules given in Theorem 3.18, with $\lambda_1 = 1/2$, $m_1 = 1$, and $\lambda_2 = 1/4$, $m_2 = 1$, respectively.

Table 1 Three term recurrence coefficients for polynomials orthogonal with respect to $\mathcal{L}^h$, given by (4.8), where $h = 10^{-3}$

k	α_k^h	β_k^h
0	−0.2333333333333333(−2)	3.1415926535897932
1	−0.2333346814748906(−2)	0.5000024444444444
2	−0.2333522068061735(−2)	0.2500085555928885
3	−0.2334263463378997(−2)	0.2500232229374217
4	−0.2336244631775208(−2)	0.2500452273213437
5	−0.2340407535576782(−2)	0.2500745779815562
6	−0.2347960262858544(−2)	0.2501112958475398
7	−0.2360372635016554(−2)	0.2501554207696042
8	−0.2379368438009344(−2)	0.2502070202245272
9	−0.2406912783912424(−2)	0.2502661990701115
10	−0.2445192763647380(−2)	0.2503331096336867
11	−0.2496589147931819(−2)	0.2504079610384240
12	−0.2563636423862751(−2)	0.2504910261996318
13	−0.2648967934907220(−2)	0.2505826443826311
14	−0.2755242385414072(−2)	0.2506832166526108
15	−0.2885047619168657(−2)	0.2507931910511669
16	−0.3040777642745741(−2)	0.2509130340413305
17	−0.3224479742663954(−2)	0.2510431848731469
18	−0.3437670795589471(−2)	0.2511839903032346
19	−0.3681126143692742(−2)	0.2513356188744236
20	−0.3954651291094443(−2)	0.2514979570449739
21	−0.4256856397685117(−2)	0.2516704940572230
22	−0.4584965517489564(−2)	0.2518522084595881
23	−0.4934704835976961(−2)	0.2520414760042613
24	−0.5300323198099807(−2)	0.2522360248127240
25	−0.5674798852007299(−2)	0.2524329669477871
26	−0.6050273014464188(−2)	0.2526289330027622
27	−0.6418719885855738(−2)	0.2528203254485985
28	−0.6772815181787746(−2)	0.2530036863619178
29	−0.7106909253558416(−2)	0.2531761480017368

The first 30 three-term recurrence coefficients for polynomials orthogonal with respect to $\mathcal{L}^h$, $h = 10^{-3}$, given by (4.8), are presented in Table 1. Numbers in parentheses denote decimal exponents.

Using the recursion coefficients we find that the nodes $(x_1, \ldots, x_n)$ and the weights $(w_1, \ldots, x_n)$ in the quadrature formula (4.1), for $n = 20$ and $h = 10^{-3}$, are given in Table 2. We report that the corresponding Gaussian quadrature rule with 21 nodes already has one zero smaller than -1.

4.1 Special cases

In this subsection we consider some special families of linear functionals $\mathcal{L}^h$, for which the three-term recurrence coefficients can be given analytically.

Theorem 4.5 *Let* $d\mu(x) = (1/B(\alpha+1, \beta+1))x^\beta(1-x)^\alpha \chi_{[0,1]}(x)\,dx$, *with* $\beta > -1$, $\alpha > 0$, *and let the family of operators be given by*

$$(\mathscr{A}^h p)(x) = \frac{x^{1/h}}{h}\int_0^x t^{1/h-1} p(t)\,dt, \qquad p \in \mathcal{P}. \tag{4.9}$$

Table 2 Nodes and weights in the quadrature rule (2.3), with $n = 20$ and $h = 10^{-3}$, constructed for the Chebyshev measure and family given by (4.7)

	Nodes	Weights
1	−0.9997235124159081	0.1501087585925222
2	−0.9758866812785312	0.1600973512659813
3	−0.9274358917353179	0.1583801394204367
4	−0.8561506841653389	0.1577962521579559
5	−0.7638340946313748	0.1575409750131710
6	−0.6527693506885036	0.1574093617304991
7	−0.5256944123315116	0.1573341428262217
8	−0.3857395525554556	0.1572885026329246
9	−0.2363515157019908	0.1572602665560158
10	−0.8120904146661547(−1)	0.1572434875215256
11	0.7586756781300102(−1)	0.1572353005158831
12	0.2310104499642163	0.1572347047834823
13	0.3803993892023086	0.1572422585606669
14	0.5203558607335555	0.1572605066682948
15	0.6474335852647777	0.1572956196643657
16	0.7585033443483269	0.1573622581569451
17	0.8508299058641808	0.1574998698561811
18	0.9221388582992601	0.1578447054103358
19	0.9706693255745008	0.1591870532247374
20	0.9949019873401245	0.1529711390316471

Then the monic polynomials, orthogonal with respect to the linear functional

$$\mathcal{L}^h(p) = \int \big((\mathscr{A}^h)^{-1} p\big)(x)\, d\mu(x), \qquad p \in \mathcal{P},$$

have the three-term recurrence coefficients given by

$$\alpha_n^h = y - \frac{P_n^{\alpha-1,\beta}(2y-1)}{2P_{n+1}^{\alpha-1,\beta}(2y-1)}\beta_{n+1}^J - \frac{P_{n+1}^{\alpha-1,\beta}(2y-1)}{2P_n^{\alpha-1,\beta}(2y-1)}, \qquad n \in \mathbb{N}_0,$$

$$\beta_0^h = 1, \quad \beta_n^h = \frac{P_{n+1}^{\alpha-1,\beta}(2y-1)\,P_{n-1}^{\alpha-1,\beta}(2y-1)}{(2P_n^{\alpha-1,\beta}(2y-1))^2}\beta_n^J, \qquad n \in \mathbb{N},$$

where $y = [h(1+\beta)-1]/[h(1+\alpha+\beta)-1]$ *and* $P_n^{\alpha-1,\beta}$, $n \in \mathbb{N}_0$, *are monic Jacobi polynomials with parameters* $\alpha - 1$ *and* β, *and* α_n^J *and* β_n^J, $n \in \mathbb{N}$, *are the three-term recurrence coefficients for the monic Jacobi polynomials* $P_n^{\alpha-1,\beta}$. *The functional* $\mathcal{L}^h$ *is positive definite provided* $h < 1/(1+\alpha+\beta)$.

Proof According to Theorem 2.4, we know that

$$\mathcal{L}^h(p) = \int (p(x) + xh\mathcal{D}p(x))d\mu(x).$$

If we apply an integration by parts we get

$$
\begin{aligned}
\mathcal{L}^h(p) &= \int p(x)d\mu(x) + \frac{h}{B(\alpha+1,\beta+1)} \\
&\quad \times \left(p(x)x^{\beta+1}(1-x)^{\alpha} \Big|_0^1 - \int_0^1 p(x)x^{\beta}(1-x)^{\alpha-1} \right. \\
&\qquad \left. \times \big[1+\beta - x(1+\alpha+\beta))\big]dx \right) \\
&= \frac{1-h(1+\alpha+\beta)}{B(\alpha+1,\beta+1)} \int_0^1 p(x)x^{\beta}(1-x)^{\alpha-1} \left(\frac{1-h(1+\beta)}{1-h(1+\alpha+\beta)} - x \right) dx,
\end{aligned}
$$

we recognize that our functional $\mathcal{L}^h$ is an integration with respect to the Jacobi measure, which is modified by a linear factor. Such a modification is known also as the Christoffel modification (see [19, pp. 124–125]). Our functional is positive definite in the case

$$
\frac{1-h(1+\beta)}{1-h(1+\alpha+\beta)} \geq 1 \quad \text{and} \quad 1-h(1+\alpha+\beta) > 0,
$$

i.e., if $h < 1/(1+\alpha+\beta)$.

The formulae for three-term recurrence coefficients are known theoretically for the linear Christofell modification and they can be obtained by a simple substitution (see [19, pp. 124–125]). □

This theorem completely solves the problem of the construction of nodes in the following quadrature rule

$$
\int x^{\alpha}(1-x)^{\beta} p(x)dx = \sum_{k=1}^{n} w_k \frac{x_k^{1/h-1}}{h} \int_0^{x_k} x^{1/h-1} p(x)dx, \qquad p \in \mathcal{P}_{2n-1}.
$$

A similar theorem with the same argumentation can be given when the measure μ is a generalized Laguerre measure. We state it without proof.

Theorem 4.6 *Let $d\mu(x) = x^{\alpha}e^{-x}\chi_{\mathbb{R}^+}(x)\,dx$, $\alpha > -1$, and let the family of linear functionals be given by (4.9). Then the monic polynomials orthogonal with respect to*

$$
\mathcal{L}^h(p) = \int \big((\mathscr{A}^h)^{-1}p\big)(x)\,d\mu(x), \qquad p \in \mathcal{P},
$$

have the three-term recurrence coefficients given by

$$
\alpha_n^h = y - \frac{L_n^{\alpha}(y)}{L_{n+1}^{\alpha}(y)}\beta_n^L - \frac{L_{n+1}^{\alpha}(y)}{L_n^{\alpha}(y)}, \qquad \beta_n^h = \frac{L_{n+1}^{\alpha}(y)L_{n-1}^{\alpha}(y)}{\big(L_n^{\alpha}(y)\big)^2}\beta_n^L,
$$

where $y = -[1-(1+\alpha)h]/h$ and L_n^{α}, $n \in \mathbb{N}_0$, are monic generalized Laguerre polynomials with parameter α, and with the three-term recurrence coefficients α_n^L and β_n^L, $n \in \mathbb{N}_0$. The functional $\mathcal{L}^h$ is positive definite provided $h \leq 1/(1+\alpha)$.

Especially interesting case appears for $h = 1/(1+\alpha)$, i.e., when $y = 0$. Namely, using an integration by parts, we can represent $\mathcal{L}^h$ as

$$\mathcal{L}^h(p) = [1-(1+\alpha)h] \int_{\mathbb{R}^+} p(x)x^\alpha e^{-x}dx + h \int_{\mathbb{R}^+} p(x)x^{\alpha+1}e^{-x}dx, \quad p \in \mathcal{P}.,$$

If $h = 1/(1+\alpha)$, we see that $\mathcal{L}^h$ is an integration with respect to the generalized Laguerre measure with the parameter $1+\alpha$. Of course, in this case, we have $\alpha_n^h = 2+\alpha+2n$, $n \in \mathbb{N}_0$, and $\beta_n^h = n(1+\alpha+n)$, $n \in \mathbb{N}$.

Theorem 4.7 *Let $d\mu(x) = (1/\ell)x^{1/\ell-1}\chi_{[0,1]}(x)\,dx$ and let the family be given by (4.9). Then polynomials, orthogonal with respect to the linear functional*

$$\mathcal{L}^h(p) = \int \big((\mathscr{A}^h)^{-1}p\big)(x)\,d\mu(x), \quad p \in \mathcal{P},$$

have the three-term recurrence coefficients given by

$$\alpha_k^h = \frac{1}{2} + \frac{\alpha^2\beta^2 + \beta A_k^h(\alpha) + B_k^h(\alpha)}{C_k^h(\alpha,\beta)}, \quad k \geq 0,$$

$$\beta_0^h = 1, \ \beta_k^h = \frac{k^2(k+\alpha)^2\big[(k-1)^2+(k-2)\alpha+\beta\big]\big[(k+1)^2+k\alpha+\beta\big]}{(2k-1+\alpha)(2k+\alpha)^2(2k+1+\alpha)\big[k^2+(k-1)\alpha+\beta\big]^2}, \ k \geq 1,$$

where $\alpha = 1/\ell - 1$, $\beta = 1/h - 1$, and

$$\begin{aligned} A_k^h(\alpha) &= 4k(k+1) + 2(2k+1)\alpha + \big(2k^2+2k+3\big)\alpha^2 + (2k-1)\alpha^3, \\ B_k^h(\alpha) &= k(k-1)\alpha^4 + \big(2k^3+k^2-k-3\big)\alpha^3 + \big[(k^2-4)(k+1)^2+2\big]\alpha^2 \\ &\quad - 4k(k+1)^2(k+2\alpha), \\ C_k^h(\alpha,\beta) &= 2(2k+\alpha)\big[2(k+1)+\alpha\big]\big[k^2+(k-1)\alpha+\beta\big]\big[(k+1)^2+k\alpha+\beta\big]. \end{aligned}$$

Proof First we note that, according to Theorem 2.4, we need to construct polynomials orthogonal with respect to the linear functional

$$\mathcal{L}^h(p) = \int (p(x) + xh\mathcal{D}p(x))\,d\mu(x), \quad p \in \mathcal{P}.$$

Using an integration by parts, we get

$$\mathcal{L}^h(p) = \left(1 - \frac{h}{\ell}\right) \int_0^1 x^{1/\ell-1}p(x)dx + hp(1),$$

and our linear functional is given as a sum of a special case of the Jacobi measure and a punctual measure supported at one with the mass h. For this measure the three-term recurrence coefficients are known explicitly (see [36]). □

This result, according to Theorem 2.4, completely solves the construction of a quadrature rule of the form

$$\frac{1}{\ell}\int_0^1 x^{1/\ell-1}p(x)\,dx = \sum_{k=1}^n w_k \frac{x_k^{1/h}}{h}\int_0^{x_k} x^{1/h-1}p(x)\,dx$$
$$= \sum_{k=1}^n w_k' \int_{x_{k-1}}^{x_k} x^{1/h-1}p(x)\,dx, \quad p \in \mathcal{P}_{2n-1},$$

where $x_0 = 0$ and $w_k' = h^{-1}\sum_{i=1}^k w_{n-i+1}x_{n-i+1}^{1/h}$. Actually, since the three-term recurrence coefficients are given explicitly, it can be checked that $\beta_n^h > 0$, whenever $\beta_1^h > 0$. From the expression for β_1^h it can be checked that $\beta_1^h > 0$ is equivalent with $\beta < \alpha$ or $h < \ell$.

Finally, we mention an interesting result for which we believe that it is true.

Conjecture 4.1 Suppose the measure μ is purely atomic measure, with $\mu(2hk) = 1$, $k = -M, \ldots, M$, and we are given family

$$(\mathscr{A}^h p)(x) = \frac{1}{2}\left(p(x-h) + p(x+h)\right), \quad p \in \mathcal{P}.$$

Then polynomials orthogonal with respect to the linear functional

$$\mathcal{L}^h(p) = \int \big((\mathscr{A}^h)^{-1}p\big)(x)\,d\mu(x) = \sum_{k=-M}^{M} \big((\mathscr{A}^h)^{-1}p\big)(2hk), \quad p \in \mathcal{P},$$

have the three-term recurrence coefficients given by

$$\alpha_k^h = 0, \quad \beta_k^h = \frac{(2M+1)^2 - 4k^2}{4k^2-1}k^2h^2, \quad k \in \mathbb{N},$$

with $\alpha_0^h = 0$ and $\beta_0^h = 2M+1$.

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