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Rational algorithm for quadratic Christoffel modification and applications to the constrained L^2-approximation

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In this paper, we consider a rational algorithm for modification of a positive measure by quadratic factor, $\mathrm{d}\hat{\sigma}(t) = (t-z)^2\,\mathrm{d}\sigma(t)$, where it is allowed z to be in $\mathrm{supp}(\mathrm{d}\sigma)$. Also, we present an application of modified algorithm to the measures $\mathrm{d}\hat{\sigma}(t) = T_2^2(t)\,\mathrm{d}\sigma(t)$ and $\mathrm{d}\sigma'(t) = t^2T_2^2(t)\,\mathrm{d}\sigma(t)$, where $T_2(t) = t^2 - \frac{1}{2}$ is the second degree monic Chebyshev polynomial of the first kind and $\mathrm{d}\sigma(t) = \sqrt{1-t^2}\,\mathrm{d}t$, $t \in [-1, 1]$, is the Chebyshev measure of the second kind. Also, we present an application to the constrained L^2-polynomial approximation.

Keywords: orthogonal polynomials; Chebyshev polynomials; positive measure; Christoffel algorithm; three-term recurrence relation; constrained L^2-approximation

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1. Introduction

Let $\mathrm{d}\sigma$ be a positive measure on $\mathbb{R}$ with an infinite support such that polynomials are integrable and let $\{p_n\}$, $n \in \mathbb{N}_0$, be a sequence of the corresponding monic orthogonal polynomials,

$$p_n(t) = p_n(\mathrm{d}\sigma; t), \quad n \in \mathbb{N}_0.$$

It is known that they satisfy a three-term recurrence relation of the form

$$\begin{aligned} p_{n+1}(t) &= (t-\alpha_n)p_n(t) - \beta_n p_{n-1}(t), \quad n \in \mathbb{N}_0, \\ p_0(t) &= 1, \quad p_{-1}(t) = 0, \end{aligned}$$

where $\alpha_n = \alpha_n(\mathrm{d}\sigma) \in \mathbb{R}$, $\beta_n = \beta_n(\mathrm{d}\sigma) > 0$, and by convention, $\beta_0 = \beta_0(\mathrm{d}\sigma) = \sigma(\mathbb{R})$.

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We define

$$d\hat{\sigma}(t) := (t - z)^2 \, d\sigma(t), \quad z \in \mathbb{C}.$$

The set of orthogonal polynomials we wish to study is

$$\hat{p}_n(t) = \hat{p}_n(d\hat{\sigma}; t), \quad n \in \mathbb{N}_0.$$

The modification by a quadratic factor can be achieved by two successively modifications by linear factors (see [4] and [5, pp. 121–124]). A problem with this approach appears when z is inside of the supp$(d\sigma)$. Actually, when z is a zero of the polynomial p_n, orthogonal with respect to the measure $d\sigma$, an application of the modification by linear factor crashes, due to the fact that we have a division by zero. An alternative approach is to apply one step of the QR algorithm to the Jacobi matrix $J = J(d\sigma)$ for the measure $d\sigma$, i.e. the infinite symmetric tridiagonal matrix

$$J = J(d\sigma) = \mathrm{tri}(\alpha_0, \alpha_1, \alpha_2, \ldots, \sqrt{\beta_1}, \sqrt{\beta_2}, \ldots),$$

with the recursion coefficients $\alpha_k = \alpha_k(d\sigma)$ on the main diagonal, and the $\sqrt{\beta_k} = \sqrt{\beta_k(d\sigma)}$ on two-side diagonals (see [5, pp. 127–128]). This algorithm needs the computation of the square roots, hence it is not rational. Here, we present an algorithm which can be applied regardless of the fact where the point z lies. In particular, the presented algorithm is rational, as the one presented in [2] which is dealing with a linear modification. For some other modifications see [6].

Our modification by the quadratic factor can be successfully applied in the constrained L^2-polynomial approximation, for constructing the so-called s- (or σ-) orthogonal polynomials and the corresponding quadratures of Turán type (see [7,11]), etc. For example, a typical application in the constrained L^2-approximation requires orthogonal polynomials with respect to the measure $q_m(t)^2 \, d\sigma(t)$, where q_m is a monic polynomial of degree m with the zeros $\tau_1, \ldots, \tau_m$, which belong to the support of the measure $d\sigma(t)$ (see [9, p. 388] and [10]). It can be done easily by repeating our modification m times by the quadratic factors $(t - \tau_k)^2$, $k = 1, \ldots, m$.

The paper is organized as follows. In Section 2 we present the modification of the measure by a quadratic factor and in Section 3 we give the corresponding algorithm and some of its interesting applications with analytic solutions. Finally, an application to the constrained L^2-approximation is given in Section 4.

2. The modification by a quadratic factor

DEFINITION 2.1 *Let* $d\sigma$ *be a positive measure and* $p_n(\cdot) = p_n(d\sigma; \cdot)$ *be the sequence of monic orthogonal polynomials with respect to* $d\sigma$. *Let* $z \in \mathbb{C}$ *and assume that* $p_n(z) \neq 0$ *for* $n \in \mathbb{N}$. *Then*

$$\tilde{p}_n(t; z) = \frac{1}{t - z} \left[p_{n+1}(t) - \frac{p_{n+1}(z)}{p_n(z)} p_n(t) \right], \tag{1}$$

is called the kernel polynomial for the measure $d\sigma$.

Evidently, $\tilde{p}_n(t, z) \in \mathcal{P}_n$ as a function of t.

DEFINITION 2.2 *We call the measure* $d\sigma$ *quasi-definite if and only if there exists a sequence of polynomials orthogonal with respect to* $d\sigma$ (*cf.* [3]).

THEOREM 2.3 *Let* $d\sigma$ *be a positive measure and* $z \in \mathbb{C}$ *be such that* $p_n(z) \neq 0$, $n \in \mathbb{N}$. *Let* $d\tilde{\sigma}(t) = (t - z) \, d\sigma(t)$. *Then* $d\tilde{\sigma}$ *is quasi-definite and the kernel polynomials* $\tilde{p}_k$, $k \in \mathbb{N}_0$, *are* (*monic*) *formal orthogonal polynomials with respect to* $d\tilde{\sigma}$.

Proof For the proof see, for example, [5, p. 38]. ■

LEMMA 2.4 *The three-term recurrence coefficients* $\hat{\alpha}_k$ *and* $\hat{\beta}_k$, $k \in \mathbb{N}_0$, *for the sequence of polynomials orthogonal with respect to the measure* $\mathrm{d}\hat{\sigma}$, *are continuous functions of* $z \in \mathbb{R}$.

Proof Consider a sequence of moments $\hat{\mu}_n = \int x^n \, \mathrm{d}\hat{\sigma}(x)$, $n \in \mathbb{N}_0$. We obtain easily that every $\hat{\mu}_n$ is a polynomial of the second degree in z. Since $\hat{p}_n$ can be expressed as

$$\hat{p}_n(x) = \frac{1}{\hat{H}_n} \begin{vmatrix} \hat{\mu}_0 & \hat{\mu}_1 & \cdots & \hat{\mu}_n \\ \hat{\mu}_1 & \hat{\mu}_2 & \cdots & \hat{\mu}_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x & \cdots & x^n \end{vmatrix},$$

(see [3, p. 17]), where $\hat{H}_n$ is the main minor of rank n, which is also known as the Hankel determinant and which is not zero according to positivity of the measure $\mathrm{d}\hat{\sigma}$. Hence, the coefficients of $\hat{p}_n$ are continuous functions in $z \in \mathbb{R}$ and the three-term recurrence coefficients are too. ■

LEMMA 2.5 *If* $\Delta_n \equiv \Delta_n(z) = p_{n+1}(z)p_n'(z) - p_n(z)p_{n+1}'(z)$, $n \in \mathbb{N}_0$, *we have*

$$\Delta_{n+1} = \beta_{n+1}\Delta_n - p_{n+1}^2(z), \quad n \in \mathbb{N}_0. \tag{2}$$

Proof According to Christoffel–Darboux formulae (cf. [3, pp. 23–24] and [5, pp. 15–16]) we have

$$-\frac{\Delta_n}{\|p_n\|^2} = \sum_{k=0}^{n} \frac{p_k^2(z)}{\|p_k\|^2}, \quad -\frac{\Delta_{n+1}}{\|p_{n+1}\|^2} = \sum_{k=0}^{n+1} \frac{p_k^2(z)}{\|p_k\|^2},$$

wherefrom, by subtracting, we get

$$\frac{p_{n+1}^2(z)}{\|p_{n+1}\|^2} = -\frac{\Delta_{n+1}}{\|p_{n+1}\|^2} + \frac{\Delta_n}{\|p_n\|^2}.$$

Using the identity $\beta_{n+1} = \|p_{n+1}\|^2/\|p_n\|^2$, we get what is stated. ■

THEOREM 2.6 *Let* $z \in \mathbb{C}$ *be such that the measure* $\mathrm{d}\hat{\sigma}$ *is quasi-definite. The coefficients of the three-term recurrence relation for the polynomial sequence orthogonal with respect to the measure*

$$\mathrm{d}\hat{\sigma}(t) = (t-z)^2 \, \mathrm{d}\sigma(t), \quad z \in \mathbb{C},$$

are given by

$$\hat{\alpha}_n = \frac{-p_{n+1}^2(p_n p_{n+1} + z\Delta_n) + \beta_{n+1}(2p_n p_{n+1}\Delta_n + \alpha_{n+1}\Delta_n^2)}{\Delta_n \Delta_{n+1}}, \tag{3}$$

$$\hat{\beta}_0 = \beta_0[\beta_1 + (z-\alpha_0)^2], \quad \hat{\beta}_n = \beta_n \frac{\Delta_{n-1}\Delta_{n+1}}{\Delta_n^2}, \tag{4}$$

where we denote $p_n := p_n(\mathrm{d}\sigma; z)$, $n \in \mathbb{N}_0$, *and* $\{p_n\}$ *is a set of polynomials orthogonal with respect to the measure* $\mathrm{d}\sigma$.

Proof First, we prove that $\Delta_n(z) < 0$ for all $n \in \mathbb{N}_0$ and $z \in \mathbb{R}$. Using the Christoffel–Darboux formulae (cf. [3, pp. 23–24] and [5, pp. 15–16]), it follows

$$-\frac{\Delta_n}{\|p_n\|^2} = \frac{p'_{n+1}p_n - p'_n p_{n+1}}{\|p_n\|^2} = \sum_{k=0}^{n} \frac{p_k^2}{\|p_k\|^2} > 0,$$

wherefrom we get the previous statement.

Denote by $\mathcal{Z}_n$ the set of all zeros of the polynomial p_n. Obviously $\mathcal{Z}_n$, $n \in \mathbb{N}$, consists of n real numbers, hence, $\mathcal{Z} = \bigcup_{n\in\mathbb{N}} \mathcal{Z}_n$ is a set of zeros of polynomials p_n, $n \in \mathbb{N}$, and it is a countable set. Therefore, $\mathbb{R} \setminus \mathcal{Z}$ is not an empty set and has a continuum many elements.

Choose $z \in \mathbb{R} \setminus \mathcal{Z}$. For such z we have $p_n(z) \neq 0$, $n \in \mathbb{N}$, and also $\Delta_n(z) < 0$, $n \in \mathbb{N}$. The condition $p_n(z) \neq 0$, $n \in \mathbb{N}$, according to Theorem 2.3, assures that the measure $\mathrm{d}\tilde{\sigma}(t) = (t - z)\,\mathrm{d}\sigma(t)$ is quasi-definite and that the orthogonal polynomials with respect to it are given by Equation (1). Thus, we can express the polynomials $\tilde{p}_n$, $n \in \mathbb{N}$, orthogonal with respect to the measure $\mathrm{d}\tilde{\sigma}$, as

$$\tilde{p}_n(t, z) = \frac{1}{t - z}\left[p_{n+1}(t) - \frac{p_{n+1}}{p_n} p_n(t)\right], \quad n \in \mathbb{N}_0.$$

Our target measure is $\mathrm{d}\hat{\sigma}(t) = (t - z)\,\mathrm{d}\tilde{\sigma}(t) = (t - z)^2\,\mathrm{d}\sigma(t)$, hence, we want to apply once more Theorem 2.3, and we get

$$\hat{p}_n(t, z) = \frac{1}{t - z}\left[\tilde{p}_{n+1}(t, z) - \frac{\tilde{p}_{n+1}(z, z)}{\tilde{p}_n(z, z)} \tilde{p}_n(t, z)\right], \quad n \in \mathbb{N}_0.$$

As we can inspect we have

$$\tilde{p}_n(z, z) = -\frac{\Delta_n(z)}{p_n(z)}.$$

According to the condition $\Delta_n \neq 0$, $n \in \mathbb{N}_0$, an application of Theorem 2.3 is justified.

Now, we have

$$\begin{aligned}
\hat{p}_n(t, z) &= \frac{1}{t - z}\left[\tilde{p}_{n+1}(t) - \frac{\tilde{p}_{n+1}(z)}{\tilde{p}_n(z)} \tilde{p}_n(t)\right] \\
&= \frac{1}{t - z}\left\{\frac{p_{n+2}(t) - (p_{n+2}/p_{n+1})p_{n+1}(t)}{t - z}\right. \\
&\quad \left. - \frac{-(\Delta_{n+1}/p_{n+1})}{-(\Delta_n/p_n)}\left(\frac{p_{n+1}(t) - (p_{n+1}/p_n)p_n(t)}{t - z}\right)\right\} \\
&= \frac{1}{(t - z)^2}\left\{\frac{p_{n+1}p_{n+2}(t) - p_{n+2}p_{n+1}(t)}{p_{n+1}} - \frac{\Delta_{n+1}p_n}{\Delta_n p_{n+1}} \frac{p_{n+1}(t)p_n - p_{n+1}p_n(t)}{p_n}\right\} \\
&= \frac{1}{(t - z)^2}\left\{p_{n+2}(t) - \frac{p_{n+2}}{p_{n+1}} p_{n+1}(t) - \frac{\Delta_{n+1}}{\Delta_n}\left(\frac{p_n}{p_{n+1}} p_{n+1}(t) - p_n(t)\right)\right\} \\
&= \frac{1}{(t - z)^2\Delta_n}\left\{p_{n+2}(t)\Delta_n - p_{n+1}(t)\left(\Delta_n \frac{p_{n+2}}{p_{n+1}} + \Delta_{n+1}\frac{p_n}{p_{n+1}}\right) + p_n(t)\Delta_{n+1}\right\}.
\end{aligned}$$

This gives

$$\hat{p}_n(t, z) = \frac{p_{n+2}(t)\Delta_n - p_{n+1}(t)(p_{n+2}p'_n - p_n p'_{n+2}) + p_n(t)\Delta_{n+1}}{(t - z)^2\Delta_n}. \tag{5}$$

Putting Equation (5) in the three-term recurrence relation for polynomials $\{\hat{p}_n\}$,

$$\hat{p}_{n+1}(t) = (t - \hat{\alpha}_n)\hat{p}_n(t) - \hat{\beta}_n \hat{p}_{n-1}(t), \quad n \in \mathbb{N}_0,$$

it follows

$$\begin{aligned}
&\frac{p_{n+3}(t)\Delta_{n+1} - p_{n+2}(t)(p_{n+3}p'_{n+1} - p_{n+1}p'_{n+3}) + p_{n+1}(t)\Delta_{n+2}}{(t-z)^2\Delta_{n+1}} \\
&= (t - \hat{\alpha}_n)\frac{p_{n+2}(t)\Delta_n - p_{n+1}(t)(p_{n+2}p'_n - p_n p'_{n+2}) + p_n(t)\Delta_{n+1}}{(t-z)^2\Delta_n} \\
&\quad - \hat{\beta}_n \frac{p_{n+1}(t)\Delta_{n-1} - p_n(t)(p_{n+1}p'_{n-1} - p_{n-1}p'_{n+1}) + p_{n-1}(t)\Delta_n}{(t-z)^2\Delta_{n-1}}.
\end{aligned}$$

Adjusting the previous term and using the three-term recurrence relation for polynomials $\{p_n\}$ it follows

$$\begin{aligned}
&p_{n+3}(t) - \frac{p_{n+2}(t)}{\Delta_{n+1}}[(z - \alpha_{n+2})\Delta_{n+1} - p_{n+1}p_{n+2}] + p_{n+1}(t)\frac{\Delta_{n+2}}{\Delta_{n+1}} \\
&= (t - \hat{\alpha}_n)\left(p_{n+2}(t) - \frac{p_{n+1}(t)}{\Delta_n}[(z - \alpha_{n+1})\Delta_n - p_n p_{n+1}] + p_n(t)\frac{\Delta_{n+1}}{\Delta_n}\right) \\
&\quad - \hat{\beta}_n\left(p_{n+1}(t) - \frac{p_n(t)}{\Delta_{n-1}}[(z - \alpha_n)\Delta_{n-1} - p_{n-1}p_n] + p_{n-1}(t)\frac{\Delta_n}{\Delta_{n-1}}\right).
\end{aligned}$$

Putting $p_{n+3}(t) = (t - \alpha_{n+2})p_{n+2}(t) - \beta_{n+2}p_{n+1}(t)$ in the previous equality, we get

$$\begin{aligned}
&-\beta_{n+2}p_{n+1}(t) + \frac{\Delta_{n+2}p_{n+1}(t)}{\Delta_{n+1}} + \hat{\beta}_n \\
&\quad \times \left(\frac{\Delta_n p_{n-1}(t)}{\Delta_{n-1}} - \frac{p_n(t)((z - \alpha_n)\Delta_{n-1} - p_{n-1}p_n)}{\Delta_{n-1}} + p_{n+1}(t)\right) \\
&\quad + (t - \alpha_{n+2})p_{n+2}(t) - (t - \hat{\alpha}_n) \\
&\quad \times \left(\frac{\Delta_{n+1}p_n(t)}{\Delta_n} - \frac{p_{n+1}(t)((z - \alpha_{n+1})\Delta_n - p_n p_{n+1})}{\Delta_n} + p_{n+2}(t)\right) \\
&\quad - \frac{p_{n+2}(t)((z - \alpha_{n+2})\Delta_{n+1} - p_{n+1}p_{n+2})}{\Delta_{n+1}} = 0.
\end{aligned}$$

Since $\Delta_{n+2} = \beta_{n+2}\Delta_{n+1} - p_{n+2}^2$, $p_{n+2}(t) = (t - \alpha_{n+1})p_{n+1}(t) - \beta_{n+1}p_n(t)$, and $p_{n+1}(t) = (t - \alpha_n)p_n(t) - \beta_n p_{n-1}(t)$, solving the system of two equations, we obtain

$$\begin{aligned}
\hat{\alpha}_n &= \frac{\alpha_{n+1}\Delta_n\Delta_{n+1} + \Delta_{n+1}p_n p_{n+1} - \Delta_n p_{n+1}p_{n+2}}{\Delta_n\Delta_{n+1}}, \\
\hat{\beta}_n &= \frac{(\alpha_{n+1} - z)\beta_n\Delta_{n-1}\Delta_n\Delta_{n+1}p_n p_{n+1} + \beta_n\Delta_{n-1}\Delta_{n+1}p_n^2 p_{n+1}^2}{\Delta_n^2\Delta_{n+1}(\Delta_n - \beta_n\Delta_{n-1})} \\
&\quad - \frac{(\alpha_{n+1} - z)\beta_n\Delta_{n-1}\Delta_n^2 p_{n+1}p_{n+2} + \beta_n\Delta_{n-1}\Delta_n p_{n+2}(\Delta_n p_{n+2} + p_n p_{n+1}^2)}{\Delta_n^2\Delta_{n+1}(\Delta_n - \beta_n\Delta_{n-1})}.
\end{aligned} \tag{6}$$

Now, putting $\Delta_{n+1} = \beta_{n+1}\Delta_n - p_{n+1}^2$ and $p_{n+2} = (z - \alpha_{n+1})p_{n+1} - \beta_{n+1}p_n$ in Equation (6) we get the expressions (3) and (4).

The obtained expressions for the three-term recurrence coefficients $\hat{\alpha}_n$ and $\hat{\beta}_n$, $n \in \mathbb{N}_0$, are valid for $z \in \mathbb{R} \setminus \mathcal{Z}$. Since the measure $\mathrm{d}\hat{\sigma}$ is positive for $z \in \mathbb{R}$, the orthogonal polynomials with respect to $\mathrm{d}\hat{\sigma}$ exist. Now, we prove that the same relations are valid for any $z \in \mathbb{R}$.

First, note that we have proved that $\Delta_n(z) < 0$, $z \in \mathbb{R}$, such that the right-hand side expressions in Equations (3) and (4) are defined for any $z \in \mathbb{R}$. Consider now a sequence of open sets $O_n = \mathbb{R} \setminus \mathcal{Z}_n$, $n \in \mathbb{N}$. Every O_n, $n \in \mathbb{N}$, is dense in $\mathbb{R}$. Since $\mathbb{R}$ is a complete metric space, it is a Baire space (see [1, p. 31]). According to the Baire category theorem every residual set, i.e. a countable intersection of open sets which are dense in $\mathbb{R}$, is dense in $\mathbb{R}$. Thus, $\bigcap_{n\in\mathbb{N}} O_n = \mathbb{R} \setminus \mathcal{Z}$ is dense in $\mathbb{R}$. This means that Equations (3) and (4) are valid on the set dense in $\mathbb{R}$. Now, $\hat{\alpha}_n$ and $\hat{\beta}_n$, $n \in \mathbb{N}_0$, are continuous functions of $z \in \mathbb{R}$, according to Lemma 2.4, and also the right-hand sides in Equations (3) and (4) are continuous in $z \in \mathbb{R}$, since $\Delta_n(z) < 0$, $n \in \mathbb{N}$, $z \in \mathbb{R}$. For any $z \in \mathcal{Z}$, we can construct a sequence of points $z_n \in \mathbb{R} \setminus \mathcal{Z}$, $n \in \mathbb{N}$, such that $\lim z_n = z$, due to the fact that $\mathbb{R} \setminus \mathcal{Z}$ is dense in $\mathbb{R}$. According to the continuity, the equalities (3) and (4) are valid in z as well. Since $z \in \mathcal{Z}$ was arbitrary, the equalities are valid for every $z \in \mathbb{R}$.

Now, let us consider $z \in \mathbb{C} \setminus \mathbb{R}$. Denote

$$\tilde{\mathcal{Z}}_n = \{z_\ell^n \in \mathbb{C} \mid \hat{H}_n(z_\ell^n) = 0,\ \ell = 1, \ldots, N(n)\},$$

where $N(n)$ is the number of zeros of the polynomial $\hat{H}_n$. So, $\bigcup_{n\in\mathbb{N}} \tilde{\mathcal{Z}}_n$ is a countable set of all complex points where the measure $\mathrm{d}\hat{\sigma}$ is not quasi-definite. Let $z_0 \in \mathbb{C} \setminus \mathbb{R}$ be such that the measure $\mathrm{d}\hat{\sigma}$ is quasi-definite. Suppose that we have for some $n \in \mathbb{N}$, $\Delta_n(z_0) = 0$. Define

$$n_0 = \min\{n \in \mathbb{N} \mid \Delta_n(z_0) = 0\}.$$

As $\hat{H}_n$ is a continuous function in z and since $\hat{\beta}_n = \hat{H}_{n-2}\hat{H}_n/\hat{H}_{n-1}^2$, it follows that $\hat{\beta}_n$, $n \in \mathbb{N}_0$, is also a continuous function in z on $\mathbb{C}\setminus\tilde{\mathcal{Z}}_n$. Therefore, there is an open neighbourhood of z_0, $\mathcal{O}_1(z_0)$, such that for each $z \in \mathcal{O}_1(z_0)$ we have $\hat{H}_n \neq 0$, $\hat{\beta}_n \neq 0$, for $n = 0, 1, \ldots, 4n_0$.

As $\Delta_{n_0}(z_0) = 0$, there is an open neighbourhood of z_0, $\mathcal{O}_2(z_0)$, such that for each $z \in \mathcal{O}_2(z_0) \setminus \{z_0\}$ we have $\Delta_{n_0}(z) \neq 0$. If not, there exist a point z in every open neighbourhood of z_0, different from z_0, such that $\Delta_{n_0}(z) = 0$. Since, Δ_{n_0} is a entire function of z it would imply that $\Delta_{n_0}(z) = 0$, $z \in \mathbb{C}$ (see [8, p. 168]), which is impossible according to the fact that $\Delta_{n_0}(z) < 0$, $z \in \mathbb{R}$.

Define $\mathcal{O}(z_0) = \mathcal{O}_1(z_0) \cap \mathcal{O}_2(z_0)$, then $\mathcal{O}(z_0)$ is an open neighbourhood of z_0 on which $\Delta_{n_0}(z) \neq 0$, $z \neq z_0$, and on which $\hat{\beta}_{n_0}$ is continuous. Now, we have

$$\{0, +\infty\} \not\ni \lim_{z\to z_0} \hat{\beta}_{n_0} = \lim_{z\to z_0} \beta_{n_0} \frac{\Delta_{n_0-1}\Delta_{n_0+1}}{\Delta_{n_0}^2},$$

from which $\Delta_{n_0+1}(z_0) = 0$.

Since

$$\Delta_{n_0} = -\|p_{n_0+1}\|^2 \sum_{k=0}^{n_0} \frac{p_k^2(z_0)}{\|p_k\|^2} = 0, \quad \Delta_{n_0+1} = -\|p_{n_0+2}\|^2 \sum_{k=0}^{n_0+1} \frac{p_k^2(z_0)}{\|p_k\|^2} = 0,$$

we get $p_{n_0+1}^2(z_0) = 0$, which is a contradiction, since p_{n_0+1} cannot have a complex zero, as a member of the sequence of polynomials orthogonal with respect to positive measure $\mathrm{d}\sigma$ supported on the real line.

Accordingly, Equations (3) and (4) are valid for all $z \in \mathbb{C}$, such that $\mathrm{d}\hat{\sigma}$ is quasi-definite at z.

Finally, for some $z \in \mathbb{C}$, for which $d\hat{\sigma}$ is quasi-definite, we compute $\hat{\beta}_0$,

$$\begin{aligned}\hat{\beta}_0 &= \int (t-z)^2 \, d\sigma = \int (t - \alpha_0 + \alpha_0 - z)^2 \, d\sigma \\ &= \int (t-\alpha_0)^2 \, d\sigma + 2 \int (t-\alpha_0)(\alpha_0 - z) \, d\sigma + \int (\alpha_0 - z)^2 \, d\sigma \\ &= \beta_0 \beta_1 + (\alpha_0 - z)^2 \beta_0 = \beta_0 [\beta_1 + (z - \alpha_0)^2].\end{aligned}$$

■

3. Algorithm and its application

In this section, we present a rational algorithm for a modification by quadratic factor, $d\hat{\sigma}(t) = (t-z)^2 \, d\sigma(t)$, where $z \in \mathbb{C}$ is such that $d\hat{\sigma}$ is quasi-definite, as well as its application to some modified Chebyshev measure of the second kind.

THEOREM 3.1 *The coefficients $\hat{\alpha}_n$ and $\hat{\beta}_n$ in the three-term recurrence relation for the polynomial sequence orthogonal with respect to the quasi-definite measure*

$$d\hat{\sigma}(t) = (t-z)^2 \, d\sigma(t),$$

can be computed in the following way:
Initialization:

$$f_0 = 0, \quad e_0 = 1.$$

Continuation: for $i = 0, 1, 2, \ldots, n$

$$\begin{aligned} a &= \alpha_i - z - f_i, \\ b &= \begin{cases} \dfrac{a^2}{e_i}, & if\ e_i \neq 0, \\ e_{i-1}\beta_i, & if\ e_i = 0, \end{cases} \\ \hat{\beta}_i &= (1 - e_i)(b + \beta_{i+1}), \\ e_{i+1} &= \frac{b}{b + \beta_{i+1}}, \\ f_{i+1} &= (1 - e_{i+1})(a + \alpha_{i+1} - z), \\ \hat{\alpha}_i &= a + f_{i+1} + z. \end{aligned}$$

Proof First, we suppose that $p_i(z) \neq 0$ for all $i \in \mathbb{N}_0$. The proof goes using an inductive argument. We prove that for, all $n \in \mathbb{N}_0$,

$$e_n = -\frac{p_n^2}{\Delta_n}, \quad f_n = \alpha_n - z - \frac{p_n p_{n+1}}{\Delta_n}.$$

For $i = 0$ we have:

$$\begin{aligned} a &= \alpha_0 - z - f_0 = \alpha_0 - z = \frac{p_0 p_1}{\Delta_0}, \\ b &= \frac{a^2}{e_0} = (\alpha_0 - z)^2, \end{aligned}$$

$$
\begin{aligned}
&\hat{\beta}_0 = 0,\\
&e_1 = \frac{b}{b+\beta_1} = \frac{(\alpha_0 - z)^2}{(\alpha_0 - z)^2 + \beta_1} = -\frac{p_1^2}{\Delta_1},\\
&f_1 = (1-e_1)(a+\alpha_1 - z) = \frac{\beta_1(\alpha_0 + \alpha_1 - 2z)}{(\alpha_0 - z)^2 + \beta_1} = \alpha_1 - z - \frac{p_1 p_2}{\Delta_1},\\
&\hat{\alpha}_0 = a + f_1 + z = \frac{-p_1^2(p_0 p_1 + z\Delta_0) + \beta_1(2p_0 p_1 \Delta_0 + \alpha_1 \Delta_0^2)}{\Delta_0 \Delta_1}.
\end{aligned}
$$

Let the statement be true for n. According to Algorithm for $i = n+1$ it follows:

$$
\begin{aligned}
a &= \alpha_{n+1} - z - f_{n+1} = \alpha_{n+1} - z - (\hat{\alpha}_n - a - z)\\
&= \alpha_{n+1} - z - \hat{\alpha}_n + a + z = \alpha_{n+1} + \frac{p_n p_{n+1}}{\Delta_n} - \hat{\alpha}_n = \frac{p_{n+1} p_{n+2}}{\Delta_{n+1}},\\
b &= \frac{a^2}{e_{n+1}} = -\frac{p_{n+2}^2}{\Delta_{n+1}}, \quad \hat{\beta}_{n+1} = (1-e_{n+1})(b+\beta_{n+2}) = \beta_{n+1}\frac{\Delta_n \Delta_{n+2}}{\Delta_{n+1}^2},
\end{aligned}
$$

$$
\begin{aligned}
e_{n+2} &= \frac{p_{n+2}^2}{p_{n+2}^2 - \beta_{n+2}\Delta_{n+1}} = -\frac{p_{n+2}^2}{\Delta_{n+2}},\\
f_{n+2} &= (1-e_{n+2})(a+\alpha_{n+2} - z) = \left(1 + \frac{p_{n+2}^2}{\Delta_{n+2}}\right)\left(\frac{p_{n+1}p_{n+2}}{\Delta_{n+1}} + \alpha_{n+2} - z\right)\\
&= \frac{\beta_{n+2}(p_{n+1}p_{n+2} + \alpha_{n+2}\Delta_{n+1} - z\Delta_{n+1})}{\Delta_{n+2}},\\
\hat{\alpha}_{n+1} &= a + f_{n+2} + z = \frac{p_{n+1}p_{n+2}}{\Delta_{n+1}} + \frac{\beta_{n+2}(p_{n+1}p_{n+2} + \alpha_{n+2}\Delta_{n+1} - z\Delta_{n+1})}{\Delta_{n+2}} + z\\
&= \frac{p_{n+1}p_{n+2}\Delta_{n+2} + \Delta_{n+1}\beta_{n+2}p_{n+1}p_{n+2} + \Delta_{n+1}^2\beta_{n+2}\alpha_{n+2} - z\Delta_{n+1}^2\beta_{n+2} + z\Delta_{n+1}\Delta_{n+2}}{\Delta_{n+1}\Delta_{n+2}}\\
&= \frac{p_{n+1}p_{n+2}(\beta_{n+2}\Delta_{n+1} - p_{n+2}^2) + \Delta_{n+1}\beta_{n+2}p_{n+1}p_{n+2} + \Delta_{n+1}^2\beta_{n+2}\alpha_{n+2} - z\Delta_{n+1}p_{n+2}^2}{\Delta_{n+1}\Delta_{n+2}}\\
&= \frac{-p_{n+2}^2(p_{n+1}p_{n+2} + z\Delta_{n+1}) + \beta_{n+2}(2p_{n+1}p_{n+2}\Delta_{n+1} + \alpha_{n+2}\Delta_{n+1}^2)}{\Delta_{n+1}\Delta_{n+2}}.
\end{aligned}
$$

Now, let $p_i(z) = 0$ for some i. Then from the three-term recurrence relation we have $p_{i+1} = (z-\alpha_i)p_i - \beta_i p_{i-1} = -\beta_i p_{i-1}$. The previous part of the proof implies

$$
a = \frac{p_i p_{i+1}}{\Delta_i}, \quad e_i = -\frac{p_i^2}{\Delta_i},
$$

which gives

$$
\frac{a^2}{e_i} = -\frac{p_i^2 p_{i+1}^2}{p_i^2 \Delta_i} = -\frac{p_{i+1}^2}{p_{i+1}p_i' - p_i p_{i+1}'}.
$$

Since $p_{i+1}p_i' - p_i p_{i+1}' = -\beta_i p_{i-1} p_i' = \beta_i(p_i p_{i-1}' - p_{i-1}p_i')$, we get

$$
\frac{a^2}{e_i} = -\frac{\beta_i p_{i-1}^2}{p_i p_{i-1}' - p_{i-1}p_i'} = \beta_i e_{i-1}.
$$

This completes the proof. ■

In the sequel we apply Theorem 3.1 twice to compute the coefficients of three-term recurrence relation for polynomials orthogonal with respect to the modified Chebyshev measure of the second kind,

$$\mathrm{d}\hat{\sigma}(t) = \left(t^2 - \frac{1}{2}\right)^2 \sqrt{1-t^2}\,\mathrm{d}t, \quad t \in [-1, 1],$$

and

$$\mathrm{d}\sigma'(t) = t^2\left(t^2 - \frac{1}{2}\right)^2 \sqrt{1-t^2}\,\mathrm{d}t, \quad t \in [-1, 1].$$

In the first step, we compute the coefficients $\tilde{\alpha}_k$, $\tilde{\beta}_k$, $k \in \mathbb{N}_0$, for polynomials orthogonal with respect to the measure $\mathrm{d}\tilde{\sigma}(t) = (t - 1/\sqrt{2})^2\sqrt{1-t^2}\,\mathrm{d}t$, where $t \in [-1, 1]$. Finally, using already computed coefficients $\tilde{\alpha}_k$ and $\tilde{\beta}_k$ (Theorem 3.1) we get desired coefficients $\hat{\alpha}_k$ and $\hat{\beta}_k$, $k \in \mathbb{N}_0$, by using $\mathrm{d}\hat{\sigma}(t) = (t - (-1/\sqrt{2}))^2\,\mathrm{d}\tilde{\sigma}(t)$.

THEOREM 3.2 *The coefficients of the three-term recurrence relation for the sequence of polynomials orthogonal with respect to the measure*

$$\mathrm{d}\tilde{\sigma}(t) = \left(t - \frac{1}{\sqrt{2}}\right)^2 \sqrt{1-t^2}\,\mathrm{d}t, \quad t \in [-1, 1],$$

are

$$(\tilde{\alpha}_k, \tilde{\beta}_k) = \begin{cases} \left(-\dfrac{\sqrt{2}}{(k+1)(k+3)}, \dfrac{k(k+3)}{4(k+1)^2}\right), & k \equiv 0 \bmod 4, \\ \left(-\dfrac{1}{\sqrt{2}(k+2)}, \dfrac{k(k+3)}{4(k+2)^2}\right), & k \equiv 1 \bmod 4, \\ \left(0, \dfrac{k+1}{4(k+2)}\right), & k \equiv 2 \bmod 4, \\ \left(\dfrac{1}{\sqrt{2}(k+2)}, \dfrac{k+2}{4(k+1)}\right), & k \equiv 3 \bmod 4, \end{cases}$$

where $k \in \mathbb{N}_0$.

Proof As this is a modification of the Chebyshev measure of the second kind, the coefficients α_k and β_k from Theorem 3.1 are $\alpha_k = 0$, $k \in \mathbb{N}_0$, $\beta_0 = \pi/2$ and $\beta_k = 1/4$, $k \in \mathbb{N}$, and $z = 1/\sqrt{2}$. In order to prove the statement of this theorem we show that for all $k \in \mathbb{N}_0$, f_k and e_k from Theorem 3.1 are given by

$$(f_k, e_k) = \begin{cases} \left(-\dfrac{k\sqrt{2}}{2(k+1)}, \dfrac{1}{k+1}\right), & k \equiv 0 \bmod 4, \\ \left(-\dfrac{(k+1)}{\sqrt{2}(k+2)}, \dfrac{2}{k+2}\right), & k \equiv 1 \bmod 4, \\ \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{k+2}\right), & k \equiv 2 \bmod 4, \\ \left(-\dfrac{1}{\sqrt{2}}, 0\right), & k \equiv 3 \bmod 4. \end{cases}$$

For the proof we apply the principle of mathematical induction. To start, we prove that the statement is valid for $k = 0, 1, 2, 3$.

By direct computation, we have

$$\begin{aligned}
&k=0, \quad \tilde{\beta}_0=0, \quad e_1=\frac{2}{3}, \quad f_1=-\frac{\sqrt{2}}{3}, \quad \tilde{\alpha}_0=-\frac{\sqrt{2}}{3};\\
&k=1, \quad \tilde{\beta}_1=\frac{1}{9}, \quad e_2=\frac{1}{4}, \quad f_2=-\frac{1}{\sqrt{2}}, \quad \tilde{\alpha}_1=-\frac{1}{3\sqrt{2}};\\
&k=2, \quad \tilde{\beta}_2=\frac{3}{16}, \quad e_3=0, \quad f_3=-\frac{1}{\sqrt{2}}, \quad \tilde{\alpha}_2=0;\\
&k=3, \quad \tilde{\beta}_3=\frac{5}{16}, \quad e_4=\frac{1}{5}, \quad f_4=-\frac{2\sqrt{2}}{5}, \quad \tilde{\alpha}_3=\frac{1}{5\sqrt{2}}.
\end{aligned}$$

Let the statement be true for $k \in \mathbb{N}$. Applying Theorem 3.1 it follows:

$$\begin{aligned}
&a=-\frac{1}{\sqrt{2}}+\frac{k\sqrt{2}}{2(k+1)}=-\frac{1}{\sqrt{2}(k+1)}, \quad b=\frac{1}{2(k+1)}, \quad \hat{\beta}_k=\frac{k(k+3)}{4(k+1)^2},\\
&e_{k+1}=\frac{2}{k+3}, \quad f_{k+1}=-\frac{k+2}{\sqrt{2}(k+3)}, \quad \hat{\alpha}_{k+1}=-\frac{\sqrt{2}}{(k+1)(k+3)},\\
&a=-\frac{1}{\sqrt{2}(k+3)}, \quad b=\frac{1}{4(k+3)}, \quad \hat{\beta}_{k+1}=\frac{(k+1)(k+4)}{4(k+3)^2},\\
&e_{k+2}=\frac{1}{k+4}, \quad f_{k+2}=-\frac{1}{\sqrt{2}}, \quad \hat{\alpha}_{k+2}=-\frac{1}{\sqrt{2}(k+3)},\\
&a=0, \quad b=0, \quad \hat{\beta}_{k+2}=\frac{k+3}{4(k+4)},\\
&e_{k+3}=0, \quad f_{k+3}=-\frac{1}{\sqrt{2}}, \quad \hat{\alpha}_{k+3}=0,\\
&a=0, \quad b=\frac{1}{4(k+4)}, \quad \hat{\beta}_{k+3}=\frac{k+5}{4(k+4)},\\
&e_{k+4}=\frac{1}{k+5}, \quad f_{k+4}=-\frac{k+4}{\sqrt{2}(k+5)}, \quad \hat{\alpha}_{k+4}=\frac{1}{\sqrt{2}(k+5)}.
\end{aligned}$$

This completes the proof. ■

THEOREM 3.3 *The coefficients of the three-term recurrence relation for polynomials orthogonal with respect to for the measure*

$$d\hat{\sigma}(t)=\left(t^2-\frac{1}{2}\right)^2\sqrt{1-t^2}\,dt, \quad t\in[-1,1], \tag{7}$$

are

$$\hat{\alpha}_k=0, \quad k\in\mathbb{N}_0, \quad \hat{\beta}_0=\frac{\pi}{16},$$

$$\hat{\beta}_k = \begin{cases} \dfrac{k}{4(k+2)}, & k \equiv 0 \bmod 4, \\ \dfrac{1+k}{4(k+3)}, & k \equiv 1 \bmod 4, \\ \dfrac{k+4}{4(k+2)}, & k \equiv 2 \bmod 4, \\ \dfrac{5+k}{4(k+3)}, & k \equiv 3 \bmod 4, \end{cases}$$

for all $k \in \mathbb{N}_0$.

Proof To prove this theorem we show that for all $k \in \mathbb{N}_0$, f_k and e_k from Theorem 3.1 are given by

$$(\hat{f}_k, \hat{e}_k) = \begin{cases} \left(\dfrac{k}{\sqrt{2}(k+1)}, \dfrac{1}{k+2}\right), & k \equiv 0 \bmod 4, \\ \left(\dfrac{(k+1)}{\sqrt{2}(k+2)}, \dfrac{2(k+1)}{k(k+3)}\right), & k \equiv 1 \bmod 4, \\ \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{k+1}\right), & k \equiv 2 \bmod 4, \\ \left(\dfrac{1}{\sqrt{2}}, 0\right), & k \equiv 3 \bmod 4. \end{cases}$$

The rest of the proof is the same as in Theorem 3.2. ∎

Example 3.4 A few first polynomials p_k, orthogonal with respect to the measure (7), are:

$$\begin{aligned} p_0(x) &= 1, \quad p_1(x) = x, \quad p_2(x) = x^2 - \frac{1}{8}, \\ p_3(x) &= x^3 - \frac{1}{2}x, \quad p_4(x) = x^4 - \frac{5}{6}x^2 + \frac{1}{24}, \\ p_5(x) &= x^5 - x^3 + \frac{1}{8}x, \quad p_6(x) = x^6 - \frac{19}{16}x^4 + \frac{9}{32}x^2 - \frac{1}{128}, \\ p_7(x) &= x^7 - \frac{3}{2}x^5 + \frac{19}{32}x^3 - \frac{3}{64}x, \\ p_8(x) &= x^8 - \frac{9}{5}x^6 + \frac{19}{20}x^4 - \frac{21}{160}x^2 + \frac{3}{1280}, \\ p_9(x) &= x^9 - 2x^7 + \frac{5}{4}x^5 - \frac{1}{4}x^3 + \frac{3}{256}x, \\ p_{10}(x) &= x^{10} - \frac{53}{24}x^8 + \frac{13}{8}x^6 - \frac{43}{96}x^4 + \frac{5}{128}x^2 - \frac{1}{2048}, \ldots . \end{aligned}$$

Similarly, we get the following theorem.

THEOREM 3.5 *The coefficients of the three-term recurrence relation for polynomials orthogonal with respect to for the measure*

$$\mathrm{d}\sigma'(t) = t^2\left(t^2 - \frac{1}{2}\right)^2\sqrt{1-t^2}\,\mathrm{d}t, \quad t \in [-1, 1], \tag{8}$$

are

$$\hat{\alpha}_k = 0, \quad k \in \mathbb{N}_0, \quad \hat{\beta}_0 = \frac{\pi}{128},$$

$$\hat{\beta}_k = \begin{cases} \dfrac{k}{4(k+4)}, & k \equiv 0 \bmod 4, \\ \dfrac{k+7}{4(k+3)}, & k \equiv 1 \bmod 4, \\ \dfrac{1}{4}, & k \equiv 2 \bmod 4, \\ \dfrac{1}{4}, & k \equiv 3 \bmod 4, \end{cases}$$

for all $k \in \mathbb{N}_0$.

Example 3.6 A few first polynomials r_k, orthogonal with respect to the measure (8), are:

$$\begin{aligned}
&r_0(x) = 1, \quad r_1(x) = x, \quad r_2(x) = x^2 - \frac{1}{2}, \\
&r_3(x) = x^3 - \frac{3}{4}x, \quad r_4(x) = x^4 - x^2 + \frac{1}{8}, \\
&r_5(x) = x^5 - \frac{9}{8}x^3 + \frac{7}{32}x, \quad r_6(x) = x^6 - \frac{3}{2}x^4 + \frac{19}{32}x^2 - \frac{3}{64}, \\
&r_7(x) = x^7 - \frac{7}{4}x^5 + \frac{7}{8}x^3 - \frac{13}{128}x, \\
&r_8(x) = x^8 - 2x^6 + \frac{5}{4}x^4 - \frac{1}{4}x^2 + \frac{3}{256}, \\
&r_9(x) = x^9 - \frac{13}{6}x^7 + \frac{37}{24}x^5 - \frac{19}{48}x^3 + \frac{11}{384}x, \\
&r_{10}(x) = x^{10} - \frac{5}{2}x^8 + \frac{53}{24}x^6 - \frac{13}{16}x^4 + \frac{43}{384}x^2 - \frac{1}{256}, \ldots.
\end{aligned}$$

4. Application to the constrained L^2-approximation

The previous results can be applied to a problem of least square approximation. The problem of a constrained L^2-approximation can be stated in the following form. Given set of points $t_i \in \mathrm{supp}(\mathrm{d}\mu)$, $i = 1, \ldots, m$, and the function f defined on $\mathrm{supp}(\mathrm{d}\mu)$, we want to find a polynomial $p \in \mathcal{P}^C_{n+m}$, where

$$\mathcal{P}^C_{n+m} = \{p \in \mathcal{P}_{n+m} | p(t_i) = f(t_i),\ i = 1, \ldots, m\},$$

which is the solution of the following constrained extremal problem

$$\min_{p \in \mathcal{P}^C_{n+m}} \left| \int_{\mathbb{R}} (f(x) - p(x))^2 \,\mathrm{d}\mu \right|^{1/2}.$$

A solution of this problem of a constrained L^2-approximation (see [9, p. 388]) can be given in a rather elegant form. The solution is a polynomial

$$p(x) = L_m(x) + \omega_m(x) S_n(x), \quad p \in \mathcal{P}_{n+m}^C, \tag{9}$$

where L_m is an interpolation polynomial for the set of data $(t_i, f(t_i))$, $i = 1, \ldots, m$, and $\omega_m(x) = \prod_{i=1}^m (x - t_i)$, and $S_n \in \mathcal{P}_n$ is the solution of the following unconstrained L^2-approximation problem

$$\min_{s \in \mathcal{P}_n} \left| \int_{\mathbb{R}} \left(\frac{f(x) - L_m(x)}{\omega_m(x)} - s(x) \right)^2 \omega_m(x)^2 \, \mathrm{d}\mu \right|^{1/2}.$$

We recognize that the polynomial S_n is actually L^2-approximation of the function $(f - L_m)/\omega_m$ with respect to the transformed measure $\omega_m^2 \, \mathrm{d}\mu$.

Example 4.1 We consider the problem of the constrained L^2-approximation with respect to the Chebyshev measure of the second kind with constraints given at the points $\pm 1/\sqrt{2}$ for the function $f(x) = \cos(\pi x/2)$. Here, $m = 2$ and the polynomial L_2 is an interpolation polynomial for the set of points $(-1/\sqrt{2}, a)$, $(1/\sqrt{2}, a)$, where $a = \cos(\pi/(2\sqrt{2})) = 0.4440158403262133$, i.e.

$$L_2(x) = a \left(\frac{x + 1/\sqrt{2}}{2/\sqrt{2}} + \frac{x - 1/\sqrt{2}}{-2/\sqrt{2}} \right) = a,$$

and $w_2(x) = x^2 - 1/2$. In order to solve the constrained L^2-approximation problem we need to solve the following unconstrained L^2-approximation problem

$$\min_{s \in \mathcal{P}_n} \left| \int_{-1}^{1} \left(\frac{f(x) - a}{x^2 - 1/2} - s(x) \right)^2 \left(x^2 - \frac{1}{2} \right)^2 \sqrt{1 - x^2} \, \mathrm{d}x \right|^{1/2}.$$

We recognize the measure $\mathrm{d}\tilde{\sigma}$ from Theorem 3.3. In turn we can use standard techniques for the construction of the polynomial S_n. The best approach is to get S_n as a linear combination of the polynomials p_n, $n \in \mathbb{N}_0$, orthogonal with respect to $\mathrm{d}\tilde{\sigma}$ (see Example 3.4). Then, the solution can be given in the following form

$$S_n(x) = \sum_{k=0}^{n} q_k p_k(x), \quad q_k = \frac{1}{\|p_k\|^2} \int_{-1}^{1} \frac{f(x) - a}{x^2 - 1/2} p_k(x) \, \mathrm{d}\tilde{\sigma}(x).$$

Evidently, in this case $q_{2k+1} = 0$, $k \geq 0$. The coefficients q_{2k} for $0 \leq k \leq 8$ are given in Table 1. All calculations are performed in double precision arithmetic (with machine precision m.p. $\approx 2.22 \times 10^{-16}$). Numbers in parentheses indicate decimal exponents.

The corresponding absolute error of the constrained L^2-approximation p (see (9)), given by

$$e_{n+m} = \max_{-1 \leq x \leq 1} \left| \cos \frac{\pi x}{2} - p(x) \right|, \quad p \in \mathcal{P}_{n+m}^C,$$

is presented in the same table. For example, the absolute error of the corresponding approximation for $n = 6$ and $m = 2$,

$$p(x) = a + \left(x^2 - \frac{1}{2} \right) (q_0 p_0(x) + q_2 p_2(x) + q_4 p_4(x) + q_6 p_6(x)),$$

is 3.68×10^{-5}.

Table 1. Numerical results for Examples 4.1 and 4.2.

k	q_{2k}	e_{2k+2}	q'_{2k+1}	e'_{2k+4}
0	-1.082769347042405	$4.44(-1)$	$2.287103863997141(-1)$	$2.39(-3)$
1	$2.270832557191690(-1)$	$9.74(-2)$	$-1.941680177208599(-2)$	$4.02(-5)$
2	$-1.936241857485842(-2)$	$1.98(-3)$	$8.641187093109698(-4)$	$2.79(-7)$
3	$8.629162200449605(-4)$	$3.68(-5)$	$-2.397440322975305(-5)$	$1.75(-9)$
4	$-2.395555770714487(-5)$	$2.51(-7)$	$4.507523722168420(-7)$	$5.99(-12)$
5	$4.505323478769008(-7)$	$1.64(-8)$	$-6.148761353874306(-9)$	$1.95(-14)$
6	$-6.146774659794560(-10)$	$5.56(-12)$	$6.344566432377592(-11)$	m.p.
7	$6.343138140954393(-11)$	$1.84(-14)$	$-5.135154904236841(-13)$	m.p.
8	$-5.134318126826883(-13)$	m.p.	$3.342631175921556(-15)$	m.p.

Example 4.2 Let again $f(x) = \cos(\pi x/2)$, $-1 \le x \le 1$. Similarly as in the previous example for a set of interpolation constraints at the points $0, \pm 1/\sqrt{2}$ and the constrained L^2-approximation with respect to the Chebyshev measure of the second kind, we can use the exposed technique and Theorem 3.5. In this case we have $L_3(x) = 1 - 4x^2 \sin^2(\pi/(4\sqrt{2})$ and $\omega_3(x) = x(x^2 - 1/2)$. If we denote the sequence of polynomials orthogonal with respect to $\mathrm{d}\sigma'$ with r_k, $k \in \mathbb{N}_0$ (see Example 3.6), we get

$$R_n(x) = \sum_{k=0}^{n} q'_k r_k(x), \quad q'_k = \frac{1}{\|r_k\|^2} \int_{-1}^{1} \frac{f(x) - L_3(x)}{x(x^2 - 1/2)} r_k(x)\, \mathrm{d}\sigma'(x).$$

Here, $q'_{2k} = 0$, $k \ge 0$. The coefficients $q'_{2k+1} = 0$, $0 \le k \le 8$, and the corresponding errors e'_{2k+4} are presented also in Table 1.

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