

NEW INTEGRAL FORMS OF GENERALIZED MATHIEU SERIES AND RELATED APPLICATIONS

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The main object of this article is to present a systematic study of integral representations for generalized MATHIEU series and its alternating variant, and to derive a new integral expression for these special functions by contour integration using rectangular integration path. By virtue of newly established integral form of generalized MATHIEU series, we obtain a new integral expression for the BESSEL function of the first kind of half integer order, solving a related FREDHOLM integral equation of the first kind with nondegenerate kernel.

1. INTRODUCTION AND PRELIMINARIES

The series

$$(1.1) \quad S(r) = \sum_{n \geq 1} \frac{2n}{(n^2 + r^2)^2}, \quad r \in \mathbb{R}^+,$$

was introduced and studied by ÉMILE LEONARD MATHIEU (1835–1890) in his book [9] devoted to the elasticity of solid bodies. We call $S(r)$ MATHIEU series. The alternative version of $S(r)$ is

$$\tilde{S}(r) = \sum_{n \geq 1} (-1)^{n-1} \frac{2n}{(n^2 + r^2)^2}, \quad r \in \mathbb{R}^+,$$

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which was introduced by POGÁNY *et al.* in [14]. Integral forms of $S(r)$, $\tilde{S}(r)$ are given by EMERSLEBEN [7] and by POGÁNY *et al.* [14] respectively as

$$(1.2) \quad \begin{aligned} S(r) &= \frac{1}{r} \int_0^\infty \frac{t \sin(rt)}{e^t - 1} dt, \\ \tilde{S}(r) &= \frac{1}{r} \int_0^\infty \frac{t \sin(rt)}{e^t + 1} dt. \end{aligned}$$

Several interesting problems and solutions dealing with integral representations and bounds for the following modest generalization of the MATHIEU series with a fractional power

$$S_\mu(r) = \sum_{n \geq 1} \frac{2n}{(n^2 + r^2)^{\mu+1}}, \quad r \in \mathbb{R}^+; \mu > 0,$$

can be found in the recent works by DIANANDA [5], TOMOVSKI and TRENČEVSKI [18] and CERONE and LENARD who studied the integral expression [3, Theorem 2.1]

$$(1.3) \quad S_\mu(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}} \Gamma(\mu+1)} \int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x - 1} J_{\mu-\frac{1}{2}}(rx) dx \quad \mu > 0,$$

where J_ν stands for the familiar BESSEL function of the first kind of order ν . The authors gave two proofs for this result, where the second one uses the GEGENBAUER'S formula from 1875 (in fact the LAPLACE–MELLIN transform of the BESSEL function J_ν) [19]:

$$(1.4) \quad \int_0^\infty e^{-px} x^{\nu+1} J_\nu(qx) dx = \frac{2^{\nu+1} p q^\nu \Gamma(\nu + \frac{3}{2})}{\sqrt{\pi} (q^2 + p^2)^{\nu+\frac{3}{2}}},$$

where $\Re\{\nu\} > -1$, $\Re\{p\} > |\Im\{q\}|$.

The generalized alternating MATHIEU series was introduced by POGÁNY *et al.* [14] in the form

$$\tilde{S}_\mu(r) = \sum_{n \geq 1} (-1)^{n-1} \frac{2n}{(n^2 + r^2)^{\mu+1}}, \quad r \in \mathbb{R}^+; 2\mu > -1,$$

which can be also expressed in the integral form

$$(1.5) \quad \tilde{S}_\mu(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}} \Gamma(\mu+1)} \int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x + 1} J_{\mu-\frac{1}{2}}(rx) dx \quad 2\mu + 1, r > 0,$$

say. We can derive (1.5) by the GEGENBAUER formula (1.4), putting $n \mapsto p$, $r \mapsto q$, $\nu \mapsto \mu - \frac{1}{2}$, and multiplying it by $(-1)^{n-1}$ and summing up both sides for $n \in \mathbb{N}$.

Motivated essentially by the work of CERONE and LENARD [3], a family of so-called MATHIEU \mathbf{a} -series was introduced by POGÁNY *et al.* in [14]:

$$(1.6) \quad S_{\mu,\alpha}(r; \mathbf{a}) = \sum_{n \geq 1} \frac{a_n}{(a_n^\alpha + r^2)^\mu}, \quad r, \alpha, \mu \in \mathbb{R}^+, \beta \geq 0,$$

where it is tacitly assumed that the monotone increasing, divergent sequence of positive real numbers

$$\mathbf{a} = (a_n)_{n \geq 1}, \quad \lim_{n \rightarrow \infty} a_n = +\infty,$$

is so chosen that series (1.6) converges, that is, the auxiliary series $\sum_{n \geq 1} a_n^{-\mu\alpha}$ is convergent. Comparing the definitions (1.1), (1.3) and (1.6), we see that $S_2(r) = S(r)$ and $S_\mu(r) = S_{\mu+1,2}(r; \mathbb{N})$. Related integral expression reads as follows [13, 14]

$$(1.7) \quad S_{\mu,\alpha}(r; \mathbf{a}) = \mu \int_{a_1^\alpha}^\infty \int_0^{[a^{-1}(t^{1/\alpha})]} \frac{a(u) + a'(u)\{u\}}{(t + r^2)^{\mu+1}} dt du, \quad r, \alpha, \mu, \mathbf{a} > 0.$$

For \mathbf{a} assume $a_x = a(x) \in C^1[0, \infty)$, $a'(x) > 0$ and a^{-1} denotes the inverse of a . Here $[z]$ and $\{z\}$ stand for the integer and fractional part of $z \in \mathbb{R}$.

Similar integral expressions of another kind (derived not only for $S_{\mu,\alpha}(r; \mathbf{a})$ but for its alternating variant $\tilde{S}_{\mu,\alpha}(r; \mathbf{a})$ as well) are discussed in detail in [14].

POGÁNY [13] considered, as a further generalization, the so-called MATHIEU $(\mathbf{a}, \boldsymbol{\lambda})$ -series defined by

$$(1.8) \quad S_\mu(\varrho; \mathbf{a}, \boldsymbol{\lambda}) = \sum_{n \geq 0} \frac{a_n}{(\lambda(n) + \varrho)^\mu} \quad \mu, r > 0.$$

Here the series $(\lambda(n))_{n \in \mathbb{N}}$ monotonously diverges, that is

$$0 \leq \lambda_0 < \lambda_1 < \dots < \lambda_n < \dots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty.$$

The related integral expression, derived by the DIRICHLET-series technique can be found in [13, Theorem 1]:

$$S_\mu(\varrho; \mathbf{a}, \boldsymbol{\lambda}) = \frac{a_0}{\varrho^\mu} + \mu \int_0^\infty \int_0^{[\lambda^{-1}(t)]} \frac{a(u) + a'(u)\{u\}}{(\varrho + t)^{\mu+1}} dt du,$$

where $a \in C^1(\mathbb{R}^+)$, $a|_{\mathbb{N}} = (a_n)_{n \in \mathbb{N}}$; (1.8) completely solves the Open problem on the integral form of $S_\mu^\alpha(r; \mathbf{a})$ posed by FENG QI.

Finally, let us recall in short that DRAŠČIĆ and POGÁNY [6] established a first kind FREDHOLM integral equation with non-degenerated kernel which connects two types of integral representations of generalized MATHIEU series $S_\mu(r)$. The benefit was a new integral representation for the BESSEL function of the first kind with general positive order $\nu > 0$ [6, Theorem 1, Eq. (15)]. However, we point out that

integral representations for $S_{\mu,\alpha}(r; \mathbf{a})$ and $S_{\mu}(\varrho; \mathbf{a}, \boldsymbol{\lambda})$ are of highly complicated structure, so there was a need to simplify the associated FREDHOLM type integral equation to obtain an easily handlable formula for the BESSEL function of the first kind J_{ν} .

Consequently, our main goal is twofold: **(i)** to establish another type of integral formula for the generalized MATHIEU series by virtue of contour integration on a suitable rectangular integration path, and **(ii)** to apply the derived integral expression in getting new integral expression for the BESSEL function $J_{m-\frac{1}{2}}$, $m \in \mathbb{N}$ being a particular solution of the related first kind FREDHOLM type integral equation.

2. INTEGRAL FORMULAE FOR $S_m(r)$ AND $\tilde{S}_m(r)$

In this section we discuss the generalized MATHIEU possessing positive integer parameter $\mu = m \in \mathbb{N}$, that is

$$(2.1) \quad S_m(r) = \sum_{n \geq 1} \frac{2n}{(n^2 + r^2)^{m+1}} \quad m \in \mathbb{N}, r > 0,$$

and its alternating variant

$$\tilde{S}_m(r) = \sum_{n \geq 1} (-1)^{n-1} \frac{2n}{(n^2 + r^2)^{m+1}},$$

having the same parameter space. To derive integral form of these special functions we use contour integration technique on a suitable rectangular contour. The whole integration procedure is explained in detail by MILOVANOVIĆ [11, Section 2.2] (see also [10] and [12, Section 6.4.1]), so we only recall it in short during our first main result's proving procedure.

Theorem 2.1. *The following integral representation formulae hold true*

$$(2.2) \quad S_m(r) = \frac{\pi}{m} \int_0^{\infty} \frac{\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^j \binom{m}{2j} (r^2 - x^2 + \frac{1}{4})^{m-2j} x^{2j}}{\left[(x^2 - r^2 + \frac{1}{4})^2 + r^2 \right]^m} \cdot \frac{dx}{\cosh^2 \pi x},$$

$$\tilde{S}_m(r) = \frac{\pi}{m} \int_0^{\infty} \frac{\sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^j \binom{m}{2j+1} (r^2 - x^2 + \frac{1}{4})^{m-2j-1} x^{2j+1}}{\left[(x^2 - r^2 + \frac{1}{4})^2 + r^2 \right]^m} \cdot \frac{\sinh \pi x dx}{\cosh^2 \pi x}.$$

Proof. The general contour integration procedure reads as follows. Let the function f be analytic in $\Delta(\beta) = \{z \in \mathbb{C} : \Re\{z\} > \beta, \beta \in (k-1, k)\}$. Actually, we consider the series

$$A_k = \sum_{n \geq k} f(n), \quad \tilde{A}_k = \sum_{n \geq k} (-1)^n f(n).$$

Applying contour integration over a rectangle $\partial\Delta(\beta, \gamma, \delta)$ where

$$\Delta(\beta, \gamma, \delta) = \left\{ z \in \mathbb{C} : \beta \leq \Re\{z\} \leq \gamma, |\Im\{z\}| \leq \frac{\delta}{\pi} \right\} \subseteq \Delta(\beta, \infty, \infty) \equiv \Delta(\beta).$$

and $\beta \in (k-1, k)$, $\gamma \in (\ell, \ell+1)$, $k, \ell \in \mathbb{Z}$, $k \leq \ell$. By the CAUCHY residue theorem series A_k, \tilde{A}_k are expressed *via* appropriate BROMWICH–WAGNER type contour integrals along the line $z = \beta + iy$, $y \in \mathbb{R}$:

$$A_k = -\frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \left(\frac{\pi}{\sin \pi z} \right)^2 F(z) dz,$$

$$\tilde{A}_k = -\frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \left(\frac{\pi}{\sin \pi z} \right)^2 \cos \pi z F(z) dz,$$

which can be reduced to real integrals, that is, to Gaussian quadrature rules on \mathbb{R}^+ with respect to the hyperbolic weights [11, Section 2.2]

$$w_1(t) = \frac{1}{\cosh^2 t} \quad \text{and} \quad w_2(t) = \frac{\sinh t}{\cosh^2 t},$$

say, respectively. Precisely, let F be the indefinite integral of f which satisfies the conditions [12, Teorema 3]:

(C1) F is holomorphic in $\Delta(\beta)$;

(C2) $\lim_{|t| \rightarrow \infty} e^{-c|t|} F\left(x + i\frac{t}{\pi}\right) = 0$, uniformly in $x \geq \beta$;

(C3) $\lim_{x \rightarrow \infty} \int_{\mathbb{R}} e^{-c|t|} \left| F\left(x + i\frac{t}{\pi}\right) \right| dt = 0$, choosing $c = 2$ or $c = 1$, when we consider A_k, \tilde{A}_k , respectively.

The associated integration constant we calculate by (C3). Having such f, F , we deduce [11, Eqs. (2.14-15)]

$$A_k = \int_0^\infty w_1(t) \Phi\left(k - \frac{1}{2}, \frac{t}{\pi}\right) dt, \quad \tilde{A}_k = \int_0^\infty w_2(t) \Psi\left(k - \frac{1}{2}, \frac{t}{\pi}\right) dt,$$

where

$$\Phi(x, y) = -\frac{1}{2} (F(x + iy) + F(x - iy)) = -\Re\{F(z)\},$$

$$\Psi(x, y) = \frac{(-1)^k}{2i} (F(x + iy) - F(x - iy)) = (-1)^k \Im\{F(z)\}.$$

Now, we are looking for the integral forms of A_1 and \tilde{A}_1 , because of

$$(2.3) \quad S_m(r) = A_1 = \pi \int_0^\infty w_1(\pi x) \Phi(1/2, x) dx$$

and

$$(2.4) \quad \tilde{S}_m(r) = -\tilde{A}_1 = -\pi \int_0^\infty w_2(\pi x) \Psi(1/2, x) dx.$$

So, $f(z) = 2z(z^2 + r^2)^{-m-1}$, and $F(z) = -m^{-1}(z^2 + r^2)^{-m}$ and the integration constant vanishes on account of (C3). For $z = \xi + i\eta$ we have

$$[\xi^2 - \eta^2 + r^2 + i2\xi\eta]^m F(\xi + i\eta) = -\frac{1}{m}.$$

Putting $\xi = 1/2$, $\eta = x$, and $k = 1$, it reduces to

$$\left(r^2 - x^2 + \frac{1}{4} + ix\right)^m (\Phi(1/2, x) + i\Psi(1/2, x)) = \frac{1}{m}.$$

Since

$$\left(r^2 - x^2 + \frac{1}{4} + ix\right)^m = U_m(r; x) + iV_m(r; x),$$

where

$$U_m(r; x) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^j \binom{m}{2j} \left(r^2 - x^2 + \frac{1}{4}\right)^{m-2j} x^{2j},$$

$$V_m(r; x) = \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^j \binom{m}{2j+1} \left(r^2 - x^2 + \frac{1}{4}\right)^{m-2j-1} x^{2j+1},$$

and

$$U_m(r; x)^2 + V_m(r; x)^2 = \left[\left(r^2 - x^2 + \frac{1}{4}\right)^2 + x^2\right]^m = \left[\left(x^2 - r^2 + \frac{1}{4}\right)^2 + r^2\right]^m,$$

we obtain

$$\Phi(1/2, x) + i\Psi(1/2, x) = \frac{1}{m} \cdot \frac{U_m(r; x) - iV_m(r; x)}{\left[\left(x^2 - r^2 + \frac{1}{4}\right)^2 + r^2\right]^m}.$$

Substituting these expressions for $\Phi(1/2, x)$ and $\Psi(1/2, x)$ in (2.3) and (2.4), respectively, we get the assertion of this theorem. \square

The adequate formulae for the MATHIEU series $S(r) = S_1(r)$ and $\tilde{S}(r) = \tilde{S}_1(r)$ we present in the following

Corollary 2.2. *We have*

$$(2.5) \quad S(r) = \pi \int_0^\infty \frac{r^2 - x^2 + \frac{1}{4}}{\left(x^2 - r^2 + \frac{1}{4}\right)^2 + r^2} \cdot \frac{dx}{\cosh^2 \pi x},$$

$$\tilde{S}(r) = \pi \int_0^\infty \frac{x}{\left(x^2 - r^2 + \frac{1}{4}\right)^2 + r^2} \cdot \frac{\sinh \pi x dx}{\cosh^2 \pi x}.$$

Using integral representations from Theorem 2.1 and Gaussian quadratures developed in [10] we are able to calculate the functions $S_m(r)$ and $\tilde{S}_m(r)$ with a very high precision. Plots of $S_m(r)$ and $\tilde{S}_m(r)$ for $m = 1, 2, 3, 4$ are presented in Figure 1.

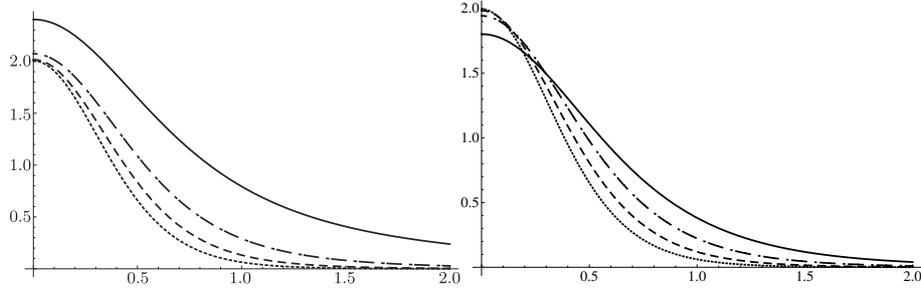


Figure 1: Plots of $S_m(r)$ (left) and $\tilde{S}_m(r)$ (right) for $m = 1$ (solid line), 2 (dash-dot line), 3 (dashed line), 4 (dotted line) in both cases

3. FREDHOLM TYPE INTEGRAL EQUATION FOR $J_{m-1/2}$

The generalized MATHIEU series $S_{\mu,\alpha}(r)$ has several closed form representations involving definite integrals. The recent ones are listed in the Introduction section of this paper, consult the ones by CERONE and LENARD (1.5), by POGÁNY (1.7) (suitable for $\alpha = 2$, $\mathbf{a} = \mathbb{N}$, see [13]) and the here obtained formula (2.2). So, the heart of the matter are integral expressions of the same subject, of the generalized MATHIEU series $S_{\mu,2}(r)$. Since its complicated structure integral form (1.7) is hard to use, we concentrate on the CERONE and LENARD integral (1.5), which contains BESSEL function of the first kind J and the newly established formula (2.2).

We say that functions h_1 and h_2 are *orthogonal a.e. with respect to the ordinary LEBESGUE measure* $\lambda(x) = x$ on the set of positive reals when

$$\int_{\mathbb{R}^+} h_1(x)h_2(x)dx = 0,$$

in notation $h_1 \perp h_2$.

Theorem 3.1. *The first kind FREDHOLM type convolutional integral equation*

$$(3.1) \quad \int_0^\infty \frac{x^{m+\frac{1}{2}}}{e^x - 1} f(rx) dx = \frac{(2r)^{m-\frac{1}{2}} m!}{\sqrt{\pi}} \cdot S_m(r), \quad r \in \mathbb{R}^+$$

has a particular solution $f(x) = J_{m-\frac{1}{2}}(x) + \mathfrak{h}_m(x)$, where

$$\mathfrak{h}_m(x) \perp \frac{x^{m+\frac{1}{2}}}{e^x - 1} \quad m \in \mathbb{N}, x \in \mathbb{R}^+.$$

Here $S_m(r)$ stands for the generalized MATHIEU series of positive integer order, expressed by (2.1).

The proof of (3.1) is obvious, therefore it is omitted.

DRAŠČIĆ and POGÁNY constructed a function \mathfrak{h} valid for the case, when the right-hand-side integral expression is built by $S_{\mu,\alpha}(r; \mathbb{N})$, consult [6, Example]. This example does work in our situation as well putting $\alpha = 2$ and $\nu = m + \frac{1}{2}$. So, assuming that a positive r.v. ζ_m , defined on a fixed standard probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ has a probability density function (PDF)

$$g_m(x) = \begin{cases} \frac{2^{m+1}x^{m+\frac{1}{2}}}{\sqrt{\pi}(2m+1)!!\zeta(m+\frac{3}{2})(e^x-1)} & x > 0, \\ 0 & x \leq 0, \end{cases}$$

where $\zeta(\cdot)$ denotes the RIEMANN Zeta function. Now, there exists the median $x_{0.5}$, i.e. the solution of the equation

$$\mathbb{P}\{\xi_{\nu,\alpha} \leq x_{0.5}\} = \int_0^{x_{0.5}} g_m(x) dx = \frac{1}{2}.$$

Then

$$\mathfrak{h}(x) = \chi_{[0,x_{0.5})}(x) - \chi_{[x_{0.5},\infty)}(x).$$

is the solution of the homogeneous variant of the equation (3.1). Let us remark that ζ_m has the so-called PLANCK distribution.

Corollary 3.2. *The first kind FREDHOLM type integral equation associated with the MATHIEU series $S(r), r > 0$ reads*

$$\begin{aligned} \int_0^\infty \frac{x^{\frac{3}{2}}}{e^x-1} f(rx) dx &= \sqrt{\frac{2}{r\pi}} \int_0^\infty \frac{x \sin(rx)}{e^x-1} dx \\ &= \sqrt{2r\pi} \int_0^\infty \frac{r^2 - x^2 + \frac{1}{4}}{(x^2 - r^2 + \frac{1}{4})^2 + r^2} \cdot \frac{dx}{\cosh^2 \pi x}. \end{aligned}$$

In both cases there exists the same particular solution $f(x) = J_{\frac{1}{2}}(x) + \mathfrak{h}_{\frac{1}{2}}(x)$, where $\mathfrak{h}_{\frac{1}{2}}(x)$ is not necessarily the same in above equations.

Remark 3.3. It is interesting to remark that EMERSLEBEN's formula (1.2) is equivalent to CERONE-LENARD formula (1.3) for $\mu = 1$, see [3, Remark 2.2].

4. SOLVING INTEGRAL EQUATION (3.1) IN $J_{m-1/2}(x)$ VIA MELLIN TRANSFORM

The MELLIN transform pair of a suitable function f one defines as

$$\mathfrak{M}_p[f] := \int_0^\infty r^{p-1} f(r) dr, \quad \mathfrak{M}_r^{-1}[f] := \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} r^{-p} \mathfrak{M}_p[f] dp.$$

Here the real γ belongs to the so-called *fundamental strip* $u_1 < \Re\{p\} < u_2$ of the inverse MELLIN transform \mathfrak{M}^{-1} (see [4, 17]).

Making use of integral representation (2.2), we transform the integral equation (3.1) into

$$(4.1) \quad \int_0^\infty \frac{x^{m+\frac{1}{2}}}{e^x - 1} J_{m-\frac{1}{2}}(rx) dx = (2r)^{m-\frac{1}{2}} (m-1)! \sqrt{\pi} \int_0^\infty R_m(x; r^2) \frac{dx}{\cosh^2 \pi x}$$

with the rational function in the integrand

$$R_m(x; r^2) = \frac{\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^j \binom{m}{2j} (r^2 - x^2 + \frac{1}{4})^{m-2j} x^{2j}}{\left[(x^2 - r^2 + \frac{1}{4})^2 + r^2 \right]^m},$$

which consists of a polynomial of degree m in numerator, and of a polynomial of degree $2m$ in denominator, both in variable r^2 . By applying MELLIN transform to the convolutional equation (4.1) as given in [6], we conclude

$$(4.2) \quad \begin{aligned} X(p) &= \frac{2^{m-\frac{1}{2}} (m-1)! \sqrt{\pi}}{\Gamma(m-p+\frac{3}{2}) \zeta(m-p+\frac{3}{2})} \mathfrak{M}_p \left[r^{m-\frac{1}{2}} \int_0^\infty \frac{R_m(x; r^2)}{\cosh^2 \pi x} dx \right] \\ &= \frac{2^{m-\frac{1}{2}} (m-1)! \sqrt{\pi}}{\Gamma(m-p+\frac{3}{2}) \zeta(m-p+\frac{3}{2})} \int_0^\infty \frac{\mathfrak{M}_{p+m-\frac{1}{2}}[R_m(x; r^2)]}{\cosh^2 \pi x} dx. \end{aligned}$$

where $X(p)$ is the MELLIN transform of the BESSEL function $J_{m-\frac{1}{2}}(x)$. To express the BESSEL function explicitly, first we have to calculate $\mathfrak{M}_{p+m-\frac{1}{2}}[R_m(x; r^2)]$ which requires integration of an irrational function in the general case ($\Re\{p\} \in (u_1, u_2) \setminus \mathbb{Q}$) and then to apply the inverse MELLIN transform operator \mathfrak{M}^{-1} to (4.2). Finally, we arrive at

$$J_{m-\frac{1}{2}}(r) = \frac{2^{m-1} (m-1)!}{\sqrt{2\pi} i} \int_{\gamma-i\infty}^{\gamma+i\infty} r^{-p} \frac{\int_0^\infty \frac{\mathfrak{M}_{p+m-\frac{1}{2}}[R_m(x; r^2)]}{\cosh^2 \pi x} dx}{\Gamma(m-p+\frac{3}{2}) \zeta(m-p+\frac{3}{2})} dp,$$

where $\gamma \in (u_1, u_2)$, and for

$$\phi(r) = r^{m-\frac{1}{2}} \int_0^\infty \frac{R_m(x; r^2)}{\cosh^2 \pi x} dx = \frac{m}{\pi} r^{m-\frac{1}{2}} S_m(r)$$

we have to obtain

$$(4.3) \quad \phi(r) \sim \begin{cases} \mathcal{O}(r^{m-\frac{1}{2}}) & r \rightarrow 0_+, \\ \mathcal{O}(r^{-m-\frac{1}{2}}) & r \rightarrow \infty. \end{cases}$$

Indeed, since $S_m(0) = 2\zeta(2m+1)$, in the case $r \rightarrow 0_+$ it is true, while, for $r \rightarrow \infty$ we have

$$\phi(r) \sim \frac{r^{m-\frac{1}{2}}}{r^{2m}} \int_0^\infty \frac{dx}{\cosh^2 \pi x} = \frac{1}{\pi} r^{-m-\frac{1}{2}},$$

hence $(u_1, u_2) \supseteq (-m + \frac{1}{2}, m + \frac{1}{2})$, $m \in \mathbb{N}$, and so we deduce (4.3). Therefore we can clearly choose the value $\gamma = 0$ for the BROMWICH–WAGNER type integration contour in deriving the inverse MELLIN transform.

Theorem 4.1. *For all $m \in \mathbb{N}$ we have*

$$J_{m-\frac{1}{2}}(r) = \frac{2^{m-1} (m-1)!}{\sqrt{2\pi} i} \int_{-i\infty}^{i\infty} r^{-p} \frac{\int_0^\infty \mathfrak{M}_{p+m-\frac{1}{2}}[R_m(x; r^2)] \frac{dx}{\cosh^2 \pi x}}{\Gamma(m-p+\frac{3}{2})\zeta(m-p+\frac{3}{2})} dp.$$

Putting $m = 1$ in this theorem we get the following, not so obvious formula.

Corollary 4.2.

$$\sin r = \frac{1}{2i} \int_{-i\infty}^{i\infty} r^{-p+\frac{1}{2}} \frac{\int_0^\infty \mathfrak{M}_{p+\frac{1}{2}} \left[\frac{r^2 - x^2 + \frac{1}{4}}{(x^2 - r^2 + \frac{1}{4})^2 + r^2} \right] \frac{dx}{\cosh^2 \pi x}}{\Gamma(\frac{5}{2}-p)\zeta(\frac{5}{2}-p)} dp.$$

5. NEW SIMPLE UPPER BOUND FOR $S(r)$

The bounds for Mathieu series $S(x)$ attracted many mathematicians like SCHRÖDER [16], EMERSLEBEN [7], BERG [2], MAKAI [8] and DIANANDA [5]. More recently we have the works by ALZER, GUO, LAMPRET, MORTICI, POGÁNY, QI, SRIVASTAVA, TOMOVSKI and coauthors (consult the exhaustive exposition concerning this question in [15, Section 2] and the adequate references therein), while MATHIEU himself conjectured [9, Ch. X, pp. 256–258] only the upper bound $S(r) < r^{-2}$, $r > 0$, proved first by BERG [2]. The bilateral bounding inequality of the same type like BERG's:

$$\frac{1}{r^2 + \frac{1}{2}} < S(r) < \frac{1}{r^2 + \frac{1}{6}},$$

has been given by MAKAI [8] who proved it in a highly elegant manner (compared to (5.1)). A result by ALZER *et al.* [1] states that

$$(5.1) \quad \frac{1}{r^2 + \frac{1}{2\zeta(3)}} < S(r) < \frac{1}{r^2 + \frac{1}{6}} =: \mathcal{A}(r) \quad r > 0,$$

where the constants $1/(2\zeta(3))$ and $1/6$ are sharp. Here $\zeta(3) \approx 1.2020569$ stands for the APÈRY's constant. The main advantage of ALZER's bound is its simple structure.

Here we establish a composite upper bound of simple structure such that is superior to (5.1) in interval $[0, 1.18772)$. Our main tool will be the formula (2.5).

Theorem 5.1. *For all positive $r > 0$ there holds true*

$$(5.2) \quad S(r) \leq \mathcal{B}(r) = \begin{cases} \frac{1}{r^2 + \frac{1}{4}} & 0 \leq r \leq \frac{\sqrt{3}}{2}, \\ \frac{1}{\sqrt{1 + 4r^2} - 1} & r > \frac{\sqrt{3}}{2}. \end{cases}$$

Proof. Considering the integrand of (2.5) it is not hard to obtain the global maximum of its rational term $R_1(x; r^2)$:

$$\mathcal{B}(r) := \max_{x>0} \frac{r^2 - x^2 + \frac{1}{4}}{(x^2 - r^2 + \frac{1}{4})^2 + r^2} = \begin{cases} \frac{4}{1 + 4r^2} & 0 \leq r \leq \frac{\sqrt{3}}{2}, \\ \frac{\sqrt{1 + 4r^2} + 1}{4r^2} & r > \frac{\sqrt{3}}{2}. \end{cases}$$

Thus, we get

$$S(r) \leq \pi \mathcal{B}(r) \int_0^\infty \frac{dx}{\cosh^2 \pi x} = \mathcal{B}(r).$$

The proof is complete. \square

Comparing now ALZER's bound with our bound (5.2) it is not hard to see that the newly established bound is superior to ALZER's in the following manner:

$$\mathcal{B}(r) < \mathcal{A}(r), \quad 0 < r \leq r_1 = \sqrt{\frac{1}{6}(5 + 2\sqrt{3})} \approx 1.18772.$$

However, for greater r -values ALZER's upper bound remains the better one.

Open Problem. *Let $\mu > 0$ be a fixed number. Determine the best possible values $\mathcal{C}(\mu)$ and $\mathcal{D}(\mu)$ such that the two-sided inequality*

$$\frac{1}{r^2 + \mathcal{C}(\mu)} < S_\mu(r) < \frac{1}{r^2 + \mathcal{D}(\mu)}$$

is valid for all $r > 0$.

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