Orthogonal polynomials for modified Gegenbauer weight and corresponding quadratures*

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Abstract—In this paper we consider polynomials orthogonal with respect to the linear functional $\mathcal{L} : \mathcal{P} \to \mathbb{C}$, defined by $\mathcal{L}[p] = \int_{-1}^{1} p(x)(1-x^2)^{\lambda-1/2} \exp(\mathrm{i}\zeta x)\,\mathrm{d}x$, where $\mathcal{P}$ is a linear space of all algebraic polynomials, $\lambda > -1/2$ and $\zeta \in \mathbb{R}$. We prove the existence of such polynomials for some pairs of λ and ζ, give some their properties, and finally give an application to numerical integration of highly oscillatory functions.

Keywords—Orthogonal polynomials; Modified Gegenbauer weight function; Moments; Three-term recurrence relation; Gaussian quadrature.

1. INTRODUCTION

Let $\mathcal{P}$ be a linear space of all algebraic polynomials, $\mathcal{P}_n$ its subspace of polynomials of degree at most n, and $\mathcal{L} : \mathcal{P} \to \mathbb{C}$ a linear functional defined by

$$\mathcal{L}[p] = \mathcal{L}[w;p] = \int_{-1}^{1} p(x)w(x)\exp(\mathrm{i}\zeta x)\,\mathrm{d}x, \quad \zeta \in \mathbb{R}\backslash\{0\}, \tag{1.1}$$

where w is a suitable "weight function."

Taking $\mathcal{L}[x^k] = \mu_k$, $k \in \mathbb{N}_0$, and using a concept of orthogonality with respect to the linear functional $\mathcal{L}$ (cf. Chihara [1, pp. 5–17]), the *necessary* and *sufficient conditions* for the existence of the corresponding orthogonal polynomials π_n $(n \in \mathbb{N}_0)$ can be expressed in terms of Hankel determinants,

$$(\forall n \in \mathbb{N}) \qquad \Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix} \neq 0. \tag{1.2}$$

Recently, we have considered orthogonal polynomials with respect to the functional $\mathcal{L}[x;p]$ with $\zeta = m\pi$ $(m \in \mathbb{N})$, as well as a case with a modified Chebyshev weight $w(x) = x(1-x^2)^{-1/2}$ and $\zeta > 0$ (cf. [2] and [3]). With a matrix Riemann-Hilbert problem formulation of the orthogonality relations, Aptekarev and Van Assche [4] have investigated the case $\mathcal{L}[p] = \int_{-1}^{1} p(x)\rho(x)(1 -$

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$x^2)^{-1/2}\,\mathrm{d}x$, where ρ is a complex valued non-vanishing function on $[-1,1]$, which is holomorphic in some domain containing the interval $[-1,1]$. In a special case ρ can be $\rho(x)=\mathrm{e}^{\mathrm{i}\zeta x}$.

Much earlier, orthogonal polynomials on the semicircle with respect to

$$\mathcal{L}[w;p] := \int_\Gamma p(z)w(z)(\mathrm{i}z)^{-1}\,\mathrm{d}z = \int_0^\pi p(\mathrm{e}^{\mathrm{i}\theta})w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta,$$

where $\Gamma=\{z\in\mathbb{C}\,|\,z=\mathrm{e}^{\mathrm{i}\theta},\ 0\le\theta\le\pi\}$ and w is a suitable "weight" function, as well as several applications in numerical integration and numerical differentiation, were investigated (see e.g. [5] and [6]). In all previous cases, the (quasi) inner-product $(p,q):=\mathcal{L}[w;p\cdot q]$ has the property $(zp,q)=(p,zq)$, and because of that the corresponding (monic) polynomials $\{\pi_n\}_{n\in\mathbb{N}_0}$ satisfy the fundamental three-term recurrence relation.

In this paper we study the existence of orthogonal polynomials π_n with respect to $\mathcal{L}$, given by (1.1), where $w(x)=(1-x^2)^{\lambda-1/2}$, $\lambda>-1/2$, and $\zeta\in\mathbb{R}\backslash\{0\}$. In Section 2, for certain combinations of λ and ζ, we prove that these polynomials exist, and in Section 3 we consider some possibilities (numerical and symbolic) for computing three-term recurrence coefficients. Finally, in Section 4 we give the corresponding Gaussian quadratures.

2. MOMENTS AND EXISTENCE OF ORTHOGONAL POLYNOMIALS

We consider the linear functional

$$\mathcal{L}[p] = \int_{-1}^{1} p(x)(1-x^2)^{\lambda-1/2}\exp(\mathrm{i}\zeta x)\,\mathrm{d}x, \tag{2.1}$$

with the restriction $\zeta>0$. The case $\zeta<0$ can obtain under substitution $x:=-x$.

In our case the moments μ_k can be expressed in terms of Bessel functions J_ν of the order ν defined by (cf. [7, p. 40])

$$J_\nu(z)=\sum_{m=0}^{+\infty}\frac{(-1)^m(z/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)}.$$

Theorem 2.1. *The moments μ_k can be expressed in the form*

$$\mu_k=\frac{A}{(\mathrm{i}\zeta)^k}\left(P_k^\lambda(\zeta)J_\lambda(\zeta)+Q_k^\lambda(\zeta)J_{\lambda-1}(\zeta)\right),\quad k\in\mathbb{N}_0, \tag{2.2}$$

where $A=(2/\zeta)^\lambda\sqrt{\pi}\,\Gamma(\lambda+1/2)$, and P_k^λ and Q_k^λ are polynomials in ζ, which satisfy the following four-term recurrence relation

$$y_{k+2}=S[y_k]:=-(k+2\lambda+1)y_{k+1}-\zeta^2y_k-k\zeta^2y_{k-1}, \tag{2.3}$$

with the initial conditions

$$P_0^\lambda(\zeta)=1,\ P_1^\lambda(\zeta)=-2\lambda,\ P_2^\lambda(\zeta)=2\lambda(2\lambda+1)-\zeta^2\quad\text{and}\quad Q_0^\lambda(\zeta)=0,\ Q_1^\lambda(\zeta)=\zeta,\ Q_2^\lambda(\zeta)=-(2\lambda+1)\zeta,$$

respectively.

Proof. According to (2.1) and using the recurrence relation for Bessel functions, $J_{\lambda+1}(\zeta)=(2\lambda/\zeta)J_\lambda(\zeta)-J_{\lambda-1}(\zeta)$, for $\mu_k=\mathcal{L}[x^k]$, $k\in\mathbb{N}_0$, we get

$$\mu_0=AJ_\lambda(\zeta),\quad \mu_1=\mathrm{i}A\Big[\frac{2\lambda}{\zeta}J_\lambda(\zeta)-J_{\lambda-1}(\zeta)\Big],\quad \mu_2=A\Big[\Big(1-\frac{2\lambda(2\lambda+1)}{\zeta^2}\Big)J_\lambda(\zeta)+\frac{2\lambda+1}{\zeta}J_{\lambda-1}(\zeta)\Big], \tag{2.4}$$

where $A=(2/\zeta)^\lambda\sqrt{\pi}\,\Gamma(\lambda+1/2)$, from which we can identify the initial values for P_k^λ and Q_k^λ, $k=0,1,2$. Applying an integration by parts to the integral

$$\mu_k-\mu_{k+2}=\int_{-1}^{1}x^k(1-x^2)^{\lambda+1/2}\,\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x$$

we obtain the following four-term recurrence relation

$$\mu_{k+2} = -\frac{k+2\lambda+1}{\mathrm{i}\zeta}\mu_{k+1} + \mu_k + \frac{k}{\mathrm{i}\zeta}\mu_{k-1}. \tag{2.5}$$

In order to prove (2.2) we apply the induction. According to (2.4), it is clear that (2.2) is true for $k = 0, 1, 2$. Let suppose that it is true for some three consecutive nonnegative integers $k-1$, k and $k+1$, $k \in \mathbb{N}$. Then, using (2.5), the induction assumptions and (2.3), we get

$$\begin{aligned}
\mu_{k+2} &= -\frac{k+2\lambda+1}{\mathrm{i}\zeta}\frac{A}{(\mathrm{i}\zeta)^{k+1}}\left(P_{k+1}^{\lambda}(\zeta)J_\lambda(\zeta) + Q_{k+1}^{\lambda}(\zeta)J_{\lambda-1}(\zeta)\right)\\
&\quad+\frac{A}{(\mathrm{i}\zeta)^{k}}\left(P_{k}^{\lambda}(\zeta)J_\lambda(\zeta) + Q_{k}^{\lambda}(\zeta)J_{\lambda-1}(\zeta)\right) + \frac{k}{\mathrm{i}\zeta}\frac{A}{(\mathrm{i}\zeta)^{k-1}}\left(P_{k-1}^{\lambda}(\zeta)J_\lambda(\zeta) + Q_{k-1}^{\lambda}(\zeta)J_{\lambda-1}(\zeta)\right)\\
&= \frac{A}{(\mathrm{i}\zeta)^{k+2}}\Big(J_\lambda(\zeta)S[P_k^\lambda] + J_{\lambda-1}(\zeta)S[Q_k^\lambda]\Big)\\
&= \frac{A}{(\mathrm{i}\zeta)^{k+2}}\Big(P_{k+2}^{\lambda}(\zeta)J_\lambda(\zeta) + Q_{k+2}^{\lambda}(\zeta)J_{\lambda-1}(\zeta)\Big). \qquad \blacksquare
\end{aligned}$$

Some obvious properties of the polynomials P_n^λ and Q_n^λ, $n \in \mathbb{N}_0$, are stated in the following lemma.

Lemma 2.1. *For each $k \in \mathbb{N}_0$, we have $\deg(P_{2k}^\lambda) = \deg(P_{2k+1}^\lambda) = 2k$. Also, for each $k \in \mathbb{N}$, $\deg(Q_{2k}^\lambda) = \deg(Q_{2k-1}^\lambda) = 2k$.*

The free terms in the polynomials P_n^λ and Q_n^λ are given by $P_n^\lambda(0) = (-1)^n(2\lambda)_n$ and $Q_n^\lambda(0) = 0$ respectively. Their leading coefficients are

$$\begin{cases} (-1)^k, & n = 2k,\\ (-1)^k(2k\lambda+k-1), & n = 2k-1, \end{cases} \qquad \text{and} \qquad \begin{cases} (-1)^k k(2\lambda+1), & n = 2k,\\ (-1)^k, & n = 2k+1, \end{cases}$$

respectively.

As we can see immediately, for each $\lambda > -1/2$, if $\zeta > 0$ is an arbitrary zero of the Bessel function J_λ, the polynomials π_n orthogonal with respect to (2.1) do not exist, because $\Delta_0 = \mu_0 = AJ_\lambda(\zeta) = 0$ (see (2.4)).

In the next theorem we prove that for some pairs of λ and ζ these polynomials though exist.

Theorem 2.2. *Let λ be a positive rational number and ζ be a positive zero of the Bessel function $J_{\lambda-1}$. Then, the polynomials π_n orthogonal with respect to (2.1) exist.*

Proof. In order to prove the existence of polynomials π_n orthogonal with respect to the functional (2.1) we need to prove that the corresponding Hankel determinants are different from zero. Supposing that ζ be a nontrivial zero of the Bessel function $J_{\lambda-1}$, the moments (2.2) reduce to

$$\mu_k = \frac{A}{(\mathrm{i}\zeta)^k}P_k^\lambda(\zeta)J_\lambda(\zeta), \quad k \in \mathbb{N}_0.$$

Obviously, from the Hankel determinant (1.2) we can extract the factor $(AJ_\lambda(\zeta))^{n+1}/(\mathrm{i}\zeta)^{n(n+1)}$, so that

$$\Delta_n = \frac{(AJ_\lambda(\zeta))^n}{(\mathrm{i}\zeta)^{n(n-1)}}H_n, \qquad H_n = \begin{vmatrix} P_0^\lambda(\zeta) & P_1^\lambda(\zeta) & \dots & P_{n-1}^\lambda(\zeta)\\ P_1^\lambda(\zeta) & P_2^\lambda(\zeta) & \dots & P_n^\lambda(\zeta)\\ \vdots & \vdots & & \vdots\\ P_{n-1}^\lambda(\zeta) & P_n^\lambda(\zeta) & \dots & P_{2n-2}^\lambda(\zeta) \end{vmatrix}. \tag{2.6}$$

Note that all determinants H_n, $n \in \mathbb{N}$, are polynomials in ζ, i.e., $H_n = H_n(\zeta) = \sum_{\nu=0}^{n(n-1)} B_\nu(\lambda)\zeta^\nu$, with rational coefficients $B_\nu(\lambda)$ since $\lambda \in \mathbb{Q}$.

On the other side, non-trivial zeros ($\zeta > 0$) of the Bessel functions J_λ, with a rational index λ, are transcendental numbers (cf. [8, p. 220]) and they cannot be zeros of polynomials with rational coefficients unless polynomials are identically equal to zero.

Thus, we have to prove only that determinants H_n are not identically equal to zero. To prove this we emphasize that, according to Lemma 2.1, the free coefficient in the polynomial P_k^λ equals $P_k^\lambda(0) = (-1)^k(2\lambda)_k$, $k \in \mathbb{N}_0$, and that therefore the free coefficient in the polynomial $H_n(\zeta)$ equals to $H_n(0) = B_0(\lambda)$. Using equality $(2\lambda)_k = \Gamma(2\lambda+k)/\Gamma(2\lambda)$, we get

$$H_n(0) = \frac{1}{\Gamma(2\lambda)^n}\begin{vmatrix} \Gamma(2\lambda) & \Gamma(2\lambda+1) & \dots & \Gamma(2\lambda+n-1) \\ \Gamma(2\lambda+1) & \Gamma(2\lambda+2) & \dots & \Gamma(2\lambda+n) \\ \vdots & \vdots & & \vdots \\ \Gamma(2\lambda+n-1) & \Gamma(2\lambda+n) & \dots & \Gamma(2\lambda+2n-2) \end{vmatrix}.$$

According to $\Gamma(2\lambda+k) = \int_0^{+\infty} x^{k+2\lambda-1}\,\mathrm{e}^{-x}\,\mathrm{d}x$ we recognize the last determinant as a Hankel determinant for the generalized Laguerre measure $x^{2\lambda-1}\,\mathrm{e}^{-x}\,\chi_{[0,+\infty)}(x)\,\mathrm{d}x$ $(\lambda > 0)$. Because of that, $H_n(0)$ is evidently different from zero.

Accordingly, when λ is a positive rational number and ζ is a positive zero of the Beseel function $J_{\lambda-1}$, the sequence of orthogonal polynomials with respect to (2.1) exists. ∎

This result enables an application of computational methods for the construction of these polynomials, as well as some applications in numerical quadratures for the mentioned specific values of the parameters λ and ζ.

3. THREE-TERM RECURRENCE RELATION

In this section we suppose such parameters λ and ζ, which provide the existence of orthogonal polynomials π_n with respect to (2.1). As we mentioned in Section 1, the (quasi) inner-product $(p,q) := \mathcal{L}[w; p\cdot q]$, in our case (2.1), has the property $(zp,q) = (p,zq)$, and because of that the corresponding (monic) polynomials $\{\pi_n\}_{n\in\mathbb{N}_0}$ satisfy the fundamental three-term recurrence relation

$$\pi_{n+1}(x) = (x - \mathrm{i}\alpha_n)\pi_n(x) - \beta_n\pi_{n-1}(x), \quad n \in \mathbb{N}, \tag{3.1}$$

with $\pi_0(x) = 1$, $\pi_{-1}(x) = 0$. The recursion coefficients α_n and β_n can be expressed in terms of Hankel determinants as (cf. [5])

$$\mathrm{i}\alpha_n = \frac{\Delta'_{n+1}}{\Delta_{n+1}} - \frac{\Delta'_n}{\Delta_n} = \frac{1}{\mathrm{i}\zeta}\left(\frac{H'_{n+1}}{H_{n+1}} - \frac{H'_n}{H_n}\right), \quad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2} = \frac{1}{(\mathrm{i}\zeta)^2}\frac{H_{n+1}H_{n-1}}{H_n^2}, \tag{3.2}$$

where H_n is defined in (2.6), and Δ'_n is the Hankel determinant Δ_{n+1} with the penultimate column and the last row removed. The corresponding determinant H'_n is given by

$$H'_n = \begin{vmatrix} P_0^\lambda(\zeta) & P_1^\lambda(\zeta) & \dots & P_{n-2}^\lambda(\zeta) & P_n^\lambda(\zeta) \\ P_1^\lambda(\zeta) & P_2^\lambda(\zeta) & \dots & P_{n-1}^\lambda(\zeta) & P_{n+1}^\lambda(\zeta) \\ \vdots & \vdots & & \vdots & \vdots \\ P_{n-1}^\lambda(\zeta) & P_n^\lambda(\zeta) & \dots & P_{2n-3}^\lambda(\zeta) & P_{2n-1}^\lambda(\zeta) \end{vmatrix}. \tag{3.3}$$

Although β_0 can be arbitrary, as usual it is convenient to take $\beta_0 = \mu_0 = AJ_\lambda(\zeta)$.

In this case, however, the values of Hankel determinants cannot be found easily, but, it is clear that the recursion coefficients are rational functions in ζ. Using our software package [9] we can generate coefficients even in symbolic form for some reasonable small values of n (for $n \le 2$ see Table 3.1).

Increasing n, the complexity of expressions for α_n and β_n increases quite rapidly. On the other side, using the Chebyshev algorithm, similarly as in [2], a numerical construction of recursion coefficients can be done. In Table 3.2 we give numerical values of α_n and β_n, $n \le 14$, when $\lambda = 1$ and $\zeta \approx 8.653727912911012$ (a zero of $J_0(z)$). Numbers in parentheses indicate decimal exponents.

According to a very extensive numerical calculations we can state the following conjecture.

Table 3.1. Recursion coefficients α_n and β_n for $n \le 2$

n	α_n	β_n
0	$\frac{2\lambda}{\zeta}$	$\left(\frac{2}{\zeta}\right)^{\lambda}\sqrt{\pi}\,\Gamma\left(\lambda+\frac{1}{2}\right)J_\lambda(\zeta)$
1	$\frac{4\lambda(1+\lambda)-\zeta^2}{\zeta(2\lambda-\zeta^2)}$	$\frac{\zeta^2-2\lambda}{\zeta^2}$
2	$\frac{(7+6\lambda)\zeta^6-4\lambda(4+\lambda)(3+2\lambda)\zeta^4+32\lambda^2(3+2\lambda)\zeta^2-32\lambda^3(2+\lambda)}{2\zeta(\zeta^2-2\lambda)[\zeta^4-\lambda(5+2\lambda)\zeta^2+4\lambda^2]}$	$-\frac{2(1+2\lambda)[\zeta^4-\lambda(5+2\lambda)\zeta^2+4\lambda^2]}{\zeta^2(-2\lambda+\zeta^2)^2}$

Table 3.2. Recursion coefficients α_n and β_n for $0 \le n \le 14$, $\lambda = 1$, $\zeta \approx 8.653727912911012$

n	α_n	β_n
0	2.311142689171081(−1)	9.854626351063405(−2)
1	1.060445637373780(−1)	9.732930973514553(−1)
2	7.651132178528472(−1)	−7.673248270228832(−2)
3	−6.295009390649605(−2)	9.653084647774708(−1)
4	1.932207097354234	−1.506275573638140(−1)
5	−8.605888996725327(−1)	2.187766221402496
6	5.664110881321379(−2)	1.588977014752849(−1)
7	−4.402172898193667(−3)	2.582628598423771(−1)
8	2.654519062269625(−4)	2.494455025986602(−1)
9	−1.308066174957451(−5)	2.500301915274877(−1)
10	5.346570353604267(−7)	2.499986477921526(−1)
11	−1.839775190533208(−8)	2.500000506256831(−1)
12	5.397928683729530(−10)	2.499999983937877(−1)
13	−1.365470127894177(−11)	2.500000000437017(−1)
14	3.007166162157992(−13)	2.499999999989697(−1)

Conjecture 3.1. *For the recursion coefficients the following asymptotic relations are true*

$$\alpha_n \to 0,\ \beta_n \to \frac{1}{4}, \quad \text{as } n \to +\infty.$$

Notice that for $\lambda = 0$, from the result given in [4], we have $\alpha_n \to 0$ and $\beta_n \to 1/4$, as $n \to +\infty$.

According to $\overline{\mu}_k = (-1)^k \mu_k$, $k \in \mathbb{N}_0$ (see (2.2)) we can prove:

Lemma 3.1. *If the sequence of monic orthogonal polynomials* $\{\pi_n\}_{n\in\mathbb{N}_0}$ *exists, then* $\pi_n(z) = (-1)^n\overline{\pi_n(-\overline{z})}$ *and the coefficients* α_n *and* β_n *are real.*

Using the Pearson equation for the "weight" $(1-x^2)^{\lambda-1/2}\,\mathrm{e}^{\mathrm{i}\zeta x}$, several interesting properties of π_n can be done. Because of the limited space, it will be given elsewhere. In the next section we mention only certain remarks on the corresponding Gaussian quadrature rule.

4. GAUSSIAN QUADRATURE RULE

Suppose that the polynomials π_n, orthogonal with respect to the functional (2.1), exist. Their zeros $x_k^{(n)}$, $k = 1, \ldots, n$, according to Lemma 3.1, are distributed symmetrically with respect to the imaginary axis. The corresponding quadrature of Gaussian type,

$$\int_{-1}^{1} f(x)(1-x^2)^{\lambda-1/2}\,\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x = \sum_{k=1}^{n} w_k^{(n)} f(x_k^{(n)}) + R_n(f), \tag{4.1}$$

where $R_n(f) = 0$ for each polynomial of degree at most $2n-1$, can be considered as a quadrature for highly oscillating integrals.

For the construction of the Gaussian quadrature rules once we computed three-term recurrence coefficients we used software package OrthogonalPolynomials (see [9]). Using functions implemented there in extended arithmetics we are able to construct Gaussian rules. Table 4.1 holds constructed Gaussian rule for $\lambda = 1/4$ and $\zeta \approx 57.72405855079898$ (a zero of $J_{-3/4}(z)$).

Table 4.1. Nodes and weights of the Gaussian quadrature rule with $n = 10$ nodes for $\lambda = 1/4$ and $\zeta \approx 57.72405855079898$

nodes	weights
$\mp 0.9997482190437303 + \mathrm{i}3.207821698581004(-3)$	$2.908955255104732(-2) \mp \mathrm{i}1.197225833309330(-3)$
$\mp 0.9983271285763714 + \mathrm{i}2.127094150299141(-2)$	$1.713966152501626(-2) \pm \mathrm{i}7.355386171553665(-4)$
$\mp 0.9954829078277354 + \mathrm{i}5.721633058272207(-2)$	$2.857288658737018(-3) \pm \mathrm{i}6.121828050618464(-4)$
$\mp 0.9908103300221749 + \mathrm{i}1.157235270974207(-1)$	$1.167827067290065(-4) \pm \mathrm{i}6.189820871051405(-5)$
$\mp 0.9831915502178804 + \mathrm{i}2.099164913530890(-1)$	$4.952507800753575(-7) \pm \mathrm{i}6.753546610410324(-7)$

We apply our Gaussian quadrature formula (4.1) to the integral

$$I(\zeta) = \mathrm{Im}\left\{\int_{-1}^{1} \frac{1}{x-\mathrm{i}}(1-x^2)^{-1/4}\,\mathrm{e}^{\mathrm{i}\zeta x}\,\mathrm{d}x\right\} \approx G_n(\zeta) = \mathrm{Im}\left\{\sum_{k=1}^{n} \frac{w_k^{(n)}}{x_k^{(n)}-\mathrm{i}}\right\},$$

for $\zeta \in \{\zeta_1, \zeta_2, \zeta_3\}$, where $\zeta_1 \approx 4.284053812724698$, $\zeta_2 \approx 57.72405855079898$ and $\zeta_3 \approx 10000.86749799776$ (zeros of $J_{-3/4}(z)$). Note here that $\lambda = 1/4$. Then, the exact values of $I(\zeta)$ are

$$\begin{aligned} I(\zeta_1) &= -0.22704556734867622316075985410884415910636254160 8\ldots, \\ I(\zeta_2) &= 0.04897630980933288349923692660715801514277536657 6\ldots, \\ I(\zeta_3) &= -0.00103035570621453967814156967741662901262658696 8\ldots. \end{aligned}$$

Their integrands are displayed in Fig. 4.1.

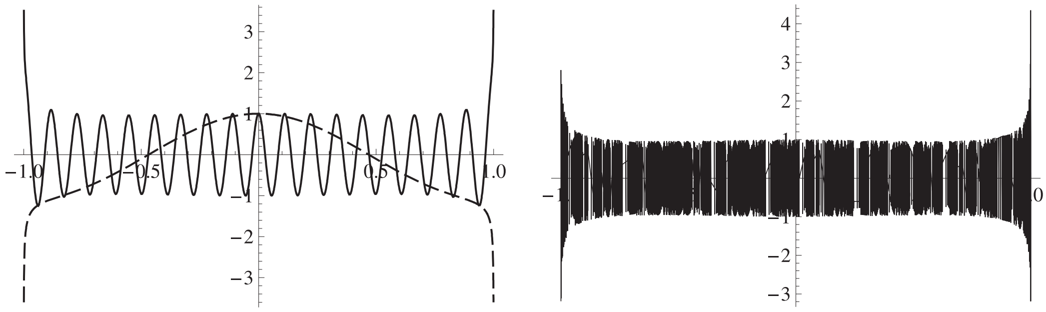

Figure 4.1. Integrand of $I(\zeta)$ for $\zeta = \zeta_1$ (dashed line) and $\zeta = \zeta_2$ (solid line) (left figure) and $\zeta = \zeta_3$ (right figure)

In Table 4.2 the relative errors in Gaussian approximations, $r_n = |(G_n(\zeta_\nu) - I(\zeta_\nu))/I(\zeta_\nu)|$, $\nu = 1, 2, 3$, for some selected number of nodes n are given. In numerical construction we use our software package [9].

Table 4.2. Relative errors r_n and r_n^G, for $n = 5,\ 7,\ 10(5)30$, when $\zeta = \zeta_1,\ \zeta_2,\ \zeta_3$

ζ	ζ_1		ζ_2		ζ_3	
n	r_n	r_n^G	r_n	r_n^G	r_n	r_n^G
5	1.35(−2)	2.15(−3)	3.55(−7)	1.93(1)	2.63(−16)	3.09(2)
7	3.82(−4)	2.50(−6)	1.32(−9)	1.86(1)	2.17(−23)	1.76(3)
10	1.93(−6)	4.51(−9)	6.47(−14)	9.11	7.77(−37)	3.87(2)
15	2.88(−10)	6.73(−13)	5.45(−17)	1.68	1.28(−49)	1.26(2)
20	4.29(−14)	1.00(−16)	1.71(−21)	3.02	6.16(−68)	8.81(2)
25	6.39(−18)	1.49(−20)	3.86(−23)	7.06	1.05(−79)	2.86(1)
30	9.50(−22)	2.22(−24)	9.40(−24)	4.60	7.52(−97)	2.40(2)

In order to compare these results we also apply the corresponding Gauss-Gegenbauer quadrature formula with respect to the weight function $x \mapsto (1-x^2)^{-1/4}$ and give its relative errors r_n^G. As we can see the Gauss-Gegenbauer quadrature is faster for small ζ, but when ζ increases our formula is much faster. Because of a highly oscillatory integrand the Gauss-Gegenbauer quadrature becomes unusable.

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