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Interpolation splines minimizing a semi-norm

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Dedicated to Claude Brezinski and Sebastiano Seatzu on their 70th birthday.

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Abstract Using S.L. Sobolev's method, we construct the interpolation splines minimizing the semi-norm in $K_2(P_2)$, where $K_2(P_2)$ is the space of functions ϕ such that ϕ' is absolutely continuous, ϕ'' belongs to $L_2(0,1)$ and $\int_0^1(\varphi''(x)+\varphi(x))^2dx<\infty$. Explicit formulas for coefficients of the interpolation splines are obtained. The resulting interpolation spline is exact for the trigonometric functions $\sin x$ and $\cos x$. Finally, in a few numerical examples the qualities of the defined splines and D^2-splines are compared. Furthermore the relationship of the defined splines with an optimal quadrature formula is shown.

Keywords Interpolation splines · Hilbert space · a seminorm minimizing property · S.L. Sobolev's method · discrete argument function · discrete analogue of a differential operator · coefficients of interpolation splines

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1 Introduction and Statement of the Problem

In order to find an approximate representation of a function φ by elements of a certain finite dimensional space, it is possible to use values of this function at some points x_β, $\beta=0,1,\dots,N$. The corresponding problem is called *the interpolation problem*, and the points x_β are *interpolation nodes*.

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Polynomial and spline interpolations are very wide subjects in approximation theory (cf. DeVore and Lorentz [8], Mastroianni and Milovanović [22]), The theory of splines as a relatively new area has undergone a rapid progress. Many books are devoted to the theory of splines, for example, Ahlberg *et al* [1], Arcangeli *et al* [2], Attea [3], Berlinet and Thomas-Agnan [4], Bojanov *et al* [5], de Boor [7], Eubank [10], Green and Silverman [13], Ignatov and Pevniy [19], Korneichuk *et al* [20], Laurent [21], Nürnberger [23], Schumaker [25], Stechkin and Subbotin [29], Vasilenko [30], Wahba [32] and others.

If the exact values $\varphi(x_\beta)$ of an unknown function $\varphi(x)$ are known, it is usual to approximate φ by minimizing

$$\int_a^b (g^{(m)}(x))^2 dx \tag{1.1}$$

on the set of interpolating functions (i.e., $g(x_\beta) = \varphi(x_\beta)$, $\beta = 0, 1, \ldots, N$) of the Sobolev space $L_2^{(m)}(a,b)$ of functions with a square integrable m-th generalized derivative. It turns out that the solution is the natural polynomial spline of degree $2m-1$ with knots $x_0, x_1, \ldots, x_N$. It is called the interpolating D^m-spline for the points $(x_\beta, \varphi(x_\beta))$. In the non-periodic case this problem was first investigated by Holladay [18] for $m = 2$, and the result of Holladay was generalized by de Boor [6] for any m. In the Sobolev space $\widetilde{L_2^{(m)}}$ of periodic functions the minimization problem of integrals of type (1.1) was investigated by I.J. Schoenberg [24], M. Golomb [14], W. Freeden [11,12] and others.

We consider the Hilbert space

$$K_2(P_2) := \left\{ \varphi : [0,1] \to \mathbb{R} \;\middle|\; \varphi' \text{ is absolutely continuous and } \varphi'' \in L_2(0,1) \right\},$$

with the semi-norm

$$\|\varphi\| := \left\{ \int_0^1 \left(P_2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) \varphi(x) \right)^2 \mathrm{d}x \right\}^{1/2}, \tag{1.2}$$

where

$$P_2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1 \qquad \text{and} \qquad \int_0^1 \left(P_2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) \varphi(x) \right)^2 \mathrm{d}x < \infty.$$

The equality (1.2) gives the semi-norm and $\|\varphi\| = 0$ if and only if $\varphi(x) = c_1 \sin x + c_2 \cos x$.

It should be noted that for a linear differential operator of order n, $L := P_n(\mathrm{d}/\mathrm{d}x)$, Ahlberg, Nilson, and Walsh in the book [1, Chapter 6] investigated the Hilbert spaces in the context of generalized splines. Namely, with the inner product

$$\langle \varphi, \psi \rangle := \int_0^1 L\varphi(x) \cdot L\psi(x)\, \mathrm{d}x,$$

$K_2(P_n)$ is a Hilbert space if we identify functions that differ by a solution of $L\varphi = 0$.

We consider the following interpolation problem:

Problem 1. *Find the function $S(x) \in K_2(P_2)$ which gives minimum to the semi-norm* (1.2) *and satisfies the interpolation condition*

$$S(x_\beta) = \varphi(x_\beta), \quad \beta = 0, 1, \ldots, N, \tag{1.3}$$

for any $\varphi \in K_2(P_2)$*, where* $x_\beta \in [0,1]$ *are the nodes of interpolation.*

It is known (cf. [21, Chapter 4]) that the solution $S(x)$ of Problem 1 is an interpolation spline function in the space $K_2(P_2)$ and the spline $S(x)$ uniquely exists when $N \geq 1$.

We give a definition of the interpolation spline function in the space $K_2(P_2)$ following [21, Chapter 4, pp. 217–219].

Let $\Delta : 0 = x_0 < x_1 < \cdots < x_N = 1$ be a mesh on the interval $[0,1]$, then the interpolation spline function with respect to Δ is a function $S(x) \in K_2(P_2)$ and satisfies the following conditions:

(i) $S(x)$ is a linear combination of functions $\sin x$, $\cos x$, $x \sin x$ and $x \cos x$ on each open mesh interval $(x_\beta, x_{\beta+1})$, $\beta = 0, 1, \ldots, N-1$;

(ii) $S(x)$ is a linear combination of functions $\sin x$ and $\cos x$ on intervals $(-\infty, 0)$ and $(1, \infty)$;

(iii) $S^{(\alpha)}(x_\beta^-) = S^{(\alpha)}(x_\beta^+)$, $\alpha = 0, 1, 2$, $\beta = 0, 1, \ldots, N$;

(iv) $S(x_\beta) = \varphi(x_\beta)$, $\beta = 0, 1, \ldots, N$ for any $\varphi \in K_2(P_2)$.

We consider the fundamental solution

$$G(x) = \frac{\mathrm{sign}(x)}{4} (\sin x - x \cos x) \tag{1.4}$$

of the differential operator $\dfrac{\mathrm{d}^4}{\mathrm{d}x^4} + 2\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + 1$, i.e., the solution of the equation

$$G^{(4)}(x) + 2G^{(2)}(x) + G(x) = \delta(x),$$

where $\delta(x)$ is Dirac's delta function.

Remark 2. The following rule for finding a fundamental solution of a linear differential operator

$$P\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) := \frac{\mathrm{d}^m}{\mathrm{d}x^m} + a_1 \frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}} + \cdots + a_m,$$

where a_j are real numbers, is given in [31, pp. 88–89]: Replacing $\frac{\mathrm{d}}{\mathrm{d}x}$ by p we get a polynomial $P(p)$. Then we expand the expression $1/P(p)$ to partial fractions

$$\frac{1}{P(p)} = \prod_j (p-\lambda)^{-k_j} = \sum_j \left[c_{j,k_j}(p-\lambda_j)^{-k_j} + \cdots + c_{j,1}(p-\lambda_j)^{-1} \right]$$

and to every partial fraction $(p-\lambda)^{-k}$ we correspond $\dfrac{x^{k-1}\mathrm{sign}\, x}{(k-1)!} \cdot e^{\lambda x}$.

Using this rule, it is found the function $G(x)$ which is the fundamental solution of the operator $\dfrac{\mathrm{d}^4}{\mathrm{d}x^4} + 2\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + 1$ and has the form (1.4).

It is clear that the third derivative of the function

$$G(x - x_\gamma) = \frac{\mathrm{sign}\,(x - x_\gamma)}{4} \big(\sin(x - x_\gamma) - (x - x_\gamma) \cos(x - x_\gamma) \big)$$

has a discontinuity equal to 1 at the point x_γ, and the first and the second derivatives of $G(x - x_\gamma)$ are continuous. Suppose a function $p_\gamma(x)$ coincides with the spline $S(x)$

on the interval $(x_\gamma, x_{\gamma+1})$, i.e., $p_\gamma(x) := p_{\gamma-1}(x) + C_\gamma G(x - x_\gamma)$, $x \in (x_\gamma, x_{\gamma+1})$, where C_γ is the jump of the function $S'''(x)$ at x_γ:

$$C_\gamma = S'''(x_\gamma^+) - S'''(x_\gamma^-).$$

Then the spline $S(x)$ can be written in the following form

$$S(x) = \sum_{\gamma=0}^{N} C_\gamma G(x - x_\gamma) + p_{-1}(x), \tag{1.5}$$

where $p_{-1}(x) = d_1 \sin x + d_2 \cos x$, d_1, d_2 are real numbers.

Furthermore, the function $S(x)$ satisfies the condition (ii) and therefore the function

$$\frac{1}{4} \sum_{\gamma=0}^{N} C_\gamma (\sin(x - x_\gamma) - (x - x_\gamma)\cos(x - x_\gamma))$$

is a linear combination of the functions $\sin x$ and $\cos x$. It leads to the following conditions for C_γ,

$$\sum_{\gamma=0}^{N} C_\gamma \sin x_\gamma = 0, \quad \sum_{\gamma=0}^{N} C_\gamma \cos x_\gamma = 0.$$

Taking into account the last two equations and the interpolation condition (iv) for the coefficients C_γ, $\gamma = 0, 1, 2, \ldots, N$, d_1, d_2 of the spline (1.5) we obtain the following system of $N + 3$ linear equations,

$$\sum_{\gamma=0}^{N} C_\gamma G(x_\beta - x_\gamma) + d_1 \sin x_\beta + d_2 \cos x_\beta = \varphi(x_\beta), \quad \beta = 0, 1, \ldots, N, \tag{1.6}$$

$$\sum_{\gamma=0}^{N} C_\gamma \sin x_\gamma = 0, \tag{1.7}$$

$$\sum_{\gamma=0}^{N} C_\gamma \cos x_\gamma = 0, \tag{1.8}$$

where $\varphi \in K_2(P_2)$.

Note that the analytic representation (1.5) of the interpolation spline $S(x)$ and the system (1.6)–(1.8) for the coefficients can be also obtained from [30, pp. 45–47, Theorem 2.2].

It should be noted that systems for coefficients of D^m-splines similar to the system (1.6)–(1.8) were investigated, for example, in [2,9,19,21,30].

The main aim of this paper is to solve Problem 1, i.e., to solve the system of equations (1.6)–(1.8) for equal spaced nodes $x_\beta = h\beta$, $\beta = 0, 1, \ldots, N$, $h = 1/N$, $N = 1, 2, \ldots$ and to find analytic formulas for coefficients C_γ, $\gamma = 0, 1, \ldots, N$, d_1 and d_2 of $S(x)$.

The rest of the paper is organized as follows. In Section 2 we give an algorithm for solving the system of equations (1.6)–(1.8) for equally-spaced nodes x_β. Using this algorithm the coefficients of the interpolation spline $S(x)$ are computed in Section 3. Some numerical examples for comparing defined splines $S(x)$ and D^2-splines are given in Section 4, as well as a relationship of such splines with the optimal quadrature formulas derived recently in [17]. is shown.

2 An algorithm for computing the coefficients of interpolation splines

In this section we give an algorithm for solving the system of equations (1.6)–(1.8), when the nodes x_β are equally spaced. Here we use a similar method proposed by S.L. Sobolev [28, 26] for finding the coefficients of optimal quadrature formulas in the space $L_2^{(m)}$.

We use mainly the concept of discrete argument functions and operations on them. The theory of discrete argument functions is given in [27, 28]. For completeness we give some definitions.

Assume that the nodes x_β are equally spaced, i.e., $x_\beta = h\beta$, $h = 1/N$, $N = 1, 2, \ldots$.

Definition 3. The function $\varphi(h\beta)$ is a *function of discrete argument* if it is given on some set of integer values of β.

Definition 4. *The inner product* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by

$$[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),$$

if the series on the right hand side of the last equality converges absolutely.

Definition 5. *The convolution* of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the inner product

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

Now we turn to our problem.

Suppose that $C_\beta = 0$ when $\beta < 0$ and $\beta > N$. Using convolution, we write equalities (1.6)–(1.8) as follows:

$$G(h\beta) * C_\beta + d_1 \sin(h\beta) + d_2 \cos(h\beta) = \varphi(h\beta), \quad \beta = 0, 1, \ldots, N, \tag{2.1}$$

$$\sum_{\beta=0}^{N} C_\beta \sin(h\beta) = 0, \tag{2.2}$$

$$\sum_{\beta=0}^{N} C_\beta \cos(h\beta) = 0, \tag{2.3}$$

where $G(h\beta)$ is a function of discrete argument corresponding to the function G given in (1.4).

Thus, we have the following problem.

Problem 6. *Find the coefficients C_β, $\beta = 0, 1, \ldots, N$, and the constants d_1, d_2 which satisfy the system* (2.1)–(2.3).

Further we investigate Problem 6 which is equivalent to Problem 1. Namely, instead of C_β we introduce the following functions

$$v(h\beta) = G(h\beta) * C_\beta, \tag{2.4}$$

$$u(h\beta) = v(h\beta) + d_1 \sin(h\beta) + d_2 \cos(h\beta). \tag{2.5}$$

In such a statement it is necessary to express the coefficients C_β by the function $u(h\beta)$. For this we have to construct such an operator $D(h\beta)$ which satisfies the equality

$$D(h\beta) * G(h\beta) = \delta(h\beta),$$

where $\delta(h\beta)$ is equal to 0 when $\beta \neq 0$ and is equal to 1 when $\beta = 0$, i.e., $\delta(h\beta)$ is the discrete delta-function.

The construction of the discrete analogue $D(h\beta)$ of the differential operator $\frac{\mathrm{d}^4}{\mathrm{d}x^4} + 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1$is given in [16].

Following [16] we have:

Theorem 2.1 *The discrete analogue of the differential operator* $\frac{\mathrm{d}^4}{\mathrm{d}x^4} + 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1$ *has the form*

$$D(h\beta) = p \begin{cases} A_1\, \lambda_1^{|\beta|-1}, & |\beta| \geq 2, \\ 1 + A_1, & |\beta| = 1, \\ C + \dfrac{A_1}{\lambda_1}, & \beta = 0, \end{cases} \tag{2.6}$$

where

$$p = \frac{2}{\sin h - h\, \cos h},\ A_1 = \frac{4h^2 \sin^4 h\, \lambda_1^2}{(\lambda_1^2 - 1)(\sin h - h\cos h)^2},\ C = \frac{2h\cos 2h - \sin 2h}{\sin h - h\cos h} \tag{2.7}$$

and

$$\lambda_1 = \frac{2h - \sin 2h - 2\sin h\sqrt{h^2 - \sin^2 h}}{2(h\cos h - \sin h)} \tag{2.8}$$

is a zero of the polynomial

$$Q_2(\lambda) = \lambda^2 + \frac{2h - \sin(2h)}{\sin h - h\cos h}\lambda + 1,$$

and $|\lambda_1| < 1$ *and* h *is a small parameter.*

Theorem 2.2 *The discrete analogue* $D(h\beta)$ *of the differential operator* $\frac{\mathrm{d}^4}{\mathrm{d}x^4}+2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+1$ *satisfies the following equalities:*

1) $D(h\beta) * \sin(h\beta) = 0$,
2) $D(h\beta) * \cos(h\beta) = 0$,
3) $D(h\beta) * (h\beta)\sin(h\beta) = 0$,
4) $D(h\beta) * (h\beta)\cos(h\beta) = 0$,
5) $D(h\beta) * G(h\beta) = \delta(h\beta)$.

Then, taking into account (2.5) and Theorems 2.1, 2.2, for optimal coefficients we have

$$C_\beta = D(h\beta) * u(h\beta). \tag{2.9}$$

Thus, if we find the function $u(h\beta)$ then the coefficients C_β can be obtained from equality (2.9). In order to calculate the convolution (2.9) we need a representation of the function $u(h\beta)$ for all integer values of β. From equality (2.1) we get that $u(h\beta) = \varphi(h\beta)$ when $h\beta \in [0,1]$. Now we need to find a representation of the function $u(h\beta)$ when $\beta < 0$ and $\beta > N$.

Since $C_\beta = 0$ when $h\beta \notin [0,1]$ then $C_\beta = D_m(h\beta) * u(h\beta) = 0$, $h\beta \notin [0,1]$. We calculate now the convolution $v(h\beta) = G(h\beta) * C_\beta$ when $\beta \le 0$ and $\beta \ge N$.

Supposing $\beta \le 0$ and taking into account equalities (1.4), (2.2), (2.3), we have

$$\begin{aligned}
v(h\beta) &= \sum_{\gamma=-\infty}^{\infty} C_\gamma\, G(h\beta - h\gamma)\\
&= \sum_{\gamma=0}^{N} C_\gamma \frac{\operatorname{sign}(h\beta - h\gamma)}{4}\Big(\sin(h\beta - h\gamma) - (h\beta - h\gamma)\cos(h\beta - h\gamma)\Big)\\
&= -\frac{1}{4}\sum_{\gamma=0}^{N} C_\gamma \Big\{\sin(h\beta)\cos(h\gamma) - \cos(h\beta)\sin(h\gamma) - (h\beta)\Big[\cos(h\beta)\cos(h\gamma)\\
&\quad + \sin(h\beta)\sin(h\gamma)\Big] + (h\gamma)\Big[\cos(h\beta)\cos(h\gamma) + \sin(h\beta)\sin(h\gamma)\Big]\Big\}\\
&= -\frac{1}{4}\cos(h\beta)\sum_{\gamma=0}^{N} C_\gamma (h\gamma)\cos(h\gamma) - \frac{1}{4}\sin(h\beta)\sum_{\gamma=0}^{N} C_\gamma (h\gamma)\sin(h\gamma).
\end{aligned}$$

Denoting $b_1 = \frac{1}{4}\sum_{\gamma=0}^{N} C_\gamma (h\gamma)\sin(h\gamma)$ and $b_2 = \frac{1}{4}\sum_{\gamma=0}^{N} C_\gamma (h\gamma)\cos(h\gamma)$ we get for $\beta \le 0$

$$v(h\beta) = -b_1 \sin(h\beta) - b_2 \cos(h\beta).$$

and for $\beta \ge N$

$$v(h\beta) = b_1 \sin(h\beta) + b_2 \cos(h\beta).$$

Now, setting

$$d_1^- = d_1 - b_1,\ d_2^- = d_2 - b_2,\ d_1^+ = d_1 + b_1,\ d_2^+ = d_2 + b_2$$

we formulate the following problem:

Problem 7. *Find the solution of the equation*

$$D(h\beta) * u(h\beta) = 0, \quad h\beta \notin [0,1], \tag{2.10}$$

in the form:

$$u(h\beta) = \begin{cases} d_1^- \sin(h\beta) + d_2^- \cos(h\beta), & \beta \le 0,\\ \varphi(h\beta), & 0 \le \beta \le N,\\ d_1^+ \sin(h\beta) + d_2^+ \cos(h\beta), & \beta \ge N. \end{cases} \tag{2.11}$$

where d_1^-, d_2^-, d_1^+, d_2^+ are unknown coefficients.

It is clear that

$$d_1 = \frac{1}{2}\left(d_1^+ + d_1^-\right), \quad d_2 = \frac{1}{2}\left(d_2^+ + d_2^-\right), \tag{2.12}$$

$$b_1 = \frac{1}{2}\left(d_1^+ - d_1^-\right), \quad b_2 = \frac{1}{2}\left(d_2^+ - d_2^-\right).$$

These unknowns d_1^-, d_2^-, d_1^+, d_2^+ can be found from equation (2.10), using the function $D(h\beta)$. Then the explicit form of the function $u(h\beta)$ and coefficients C_β, d_1, d_2 can be found. Thus, Problem 7 and respectively Problems 3 and 1 can be solved.

In the next section we realize this algorithm for computing the coefficients C_β, $\beta = 0, 1, \dots, N$, d_1 and d_2 of the interpolation spline (1.5) for any $N = 1, 2, \dots$.

3 Computation of coefficients of the interpolation spline

In this section using the algorithm from the previous section we obtain explicit formulae for coefficients of interpolation spline (1.5) which is the solution of Problem 1.

It should be noted that the interpolation spline (1.5), which is the solution of Problem 1, is exact for the functions $\sin(x)$ and $\cos(x)$.

Theorem 3.1 *Coefficients of interpolation spline* (1.5) *which minimizes the semi-norm* (1.2) *with equally spaced nodes in the space* $K_2(P_2)$ *have the following form:*

$$
\begin{aligned}
C_0 &= Cp\varphi(0) + p[\varphi(h) - d_1^- \sin h + d_2^- \cos h] + \frac{A_1 p}{\lambda_1}\left[\sum_{\gamma=0}^{N} \lambda_1^{\gamma}\varphi(h\gamma) + M_1 + \lambda_1^N N_1\right],\\
C_\beta &= Cp\varphi(h\beta) + p[\varphi(h(\beta-1)) + \varphi(h(\beta+1))]\\
&\quad + \frac{A_1 p}{\lambda_1}\left[\sum_{\gamma=0}^{N} \lambda_1^{|\beta-\gamma|}\varphi(h\gamma) + \lambda_1^{\beta} M_1 + \lambda_1^{N-\beta} N_1\right], \quad \beta = 1, 2, \dots, N-1,\\
C_N &= Cp\varphi(1) + p[\varphi(h(N-1)) + d_1^+ \sin(1+h) + d_2^+ \cos(1+h)]\\
&\quad + \frac{A_1 p}{\lambda_1}\left[\sum_{\gamma=0}^{N} \lambda_1^{N-\gamma}\varphi(h\gamma) + \lambda_1^N M_1 + N_1\right],\\
d_1 &= \frac{1}{2}(d_1^+ + d_1^-), \qquad d_2 = \frac{1}{2}(d_2^+ + d_2^-),
\end{aligned}
$$

where p, A_1, C *and* λ_1 *are defined by* (2.7), (2.8),

$$
M_1 = \frac{\lambda_1[d_2^-(\cos h - \lambda_1) - d_1^- \sin h]}{\lambda_1^2 - 2\lambda_1 \cos h + 1}, \tag{3.1}
$$

$$
N_1 = \frac{\lambda_1[d_2^+(\cos(1+h) - \lambda_1 \cos 1) + d_1^+(\sin(1+h) - \lambda_1 \sin 1)]}{\lambda_1^2 - 2\lambda_1 \cos h + 1}, \tag{3.2}
$$

and d_1^+, d_1^-, d_2^+, d_2^- *are defined by* (3.3) *and* (3.9).

Proof First we find the expressions for d_2^- and d_2^+. When $\beta = 0$ and $\beta = N$, from (2.11) we get

$$
d_2^- = \varphi(0), \quad d_2^+ = \frac{\varphi(1)}{\cos 1} - d_1^+ \tan 1. \tag{3.3}
$$

Now we have two unknowns d_1^- and d_1^+, which can be found from (2.10) when $\beta = -1$ and $\beta = N + 1$.

Taking into account (2.11) and Definition 2.3 from (2.10) we have

$$\sum_{\gamma=-\infty}^{-1} D(h\beta-h\gamma)[d_1^- \sin(h\gamma)+d_2^- \cos(h\gamma)]+\sum_{\gamma=0}^{N} D(h\beta-h\gamma)\varphi(h\gamma)$$
$$+\sum_{\gamma=N+1}^{\infty} D(h\beta-h\gamma)[d_1^+ \sin(h\gamma)+d_2^+ \cos(h\gamma)]=0,$$

where $\beta<0$ and $\beta>N$.

Hence for $\beta=-1$ and $\beta=N+1$ we get the following system of equations for $d_1^-,\ d_1^+,\ d_2^-,\ d_2^+$:

$$\sum_{\gamma=-\infty}^{-1} D(h\beta-h\gamma)[d_1^- \sin(h\gamma)+d_2^- \cos(h\gamma)]+\sum_{\gamma=0}^{N} D(h\beta-h\gamma)\varphi(h\gamma)$$
$$+\sum_{\gamma=N+1}^{\infty} D(h\beta-h\gamma)[d_1^+ \sin(h\gamma)+d_2^+ \cos(h\gamma)]=0,$$

where $\beta<0$ and $\beta>N$.
Hence for $\beta=-1$ and $\beta=N+1$ we get the following system of linear equations for $d_1^-,\ d_1^+,\ d_2^-,\ d_2^+$:

$$\begin{aligned}-d_1^-\sum_{\gamma=1}^{\infty} D(h\gamma-h)\sin(h\gamma)+d_2^-\sum_{\gamma=1}^{\infty} D(h\gamma-h)\cos(h\gamma)\\
+d_1^+\sum_{\gamma=1}^{\infty} D(h(N+\gamma)+h)\sin(1+h\gamma)\\
+d_2^+\sum_{\gamma=1}^{\infty} D(h(N+\gamma)+h)\cos(1+h\gamma)\\
=-\sum_{\gamma=0}^{N} D(h\gamma+h)\varphi(h\gamma),\end{aligned}\tag{3.4}$$

$$\begin{aligned}-d_1^-\sum_{\gamma=1}^{\infty} D(h(N+\gamma)+h)\sin(h\gamma)+d_2^-\sum_{\gamma=1}^{\infty} D(h(N+\gamma)+h)\cos(h\gamma)\\
+d_1^+\sum_{\gamma=1}^{\infty} D(h\gamma-h)\sin(1+h\gamma)\\
+d_2^+\sum_{\gamma=1}^{\infty} D(h\gamma-h)\cos(1+h\gamma)\\
=-\sum_{\gamma=0}^{N} D(h(N+1)-h\gamma)\varphi(h\gamma).\end{aligned}\tag{3.5}$$

Since $|\lambda_1|<1$, the series in the previous systemof equations are convergent.

Using (3.3) and taking into account (2.6), after some calculations and simplifications, from (3.4) and (3.5) we obtain

$$B_{11}d_1^-+B_{12}d_1^+=T_1,\qquad B_{21}d_1^-+B_{22}d_1^+=T_2,$$

where

$$B_{11}=\lambda_1\sin h,\ B_{12}=-\frac{\lambda_1^{N+1}\sin h}{\cos 1},\ B_{21}=\lambda_1^{N+1}\sin h,\ B_{22}=-\frac{\lambda_1\sin h}{\cos 1}, \tag{3.6}$$

$$T_1=\frac{2h\lambda_1\sin^2 h}{h\cos h-\sin h}\sum_{\gamma=0}^{N}\lambda_1^{\gamma}\varphi(h\gamma)$$
$$+(\lambda_1\cos h-1)\varphi(0)+\lambda_1^{N+1}(\cos h-\lambda_1-\tan 1\sin h)\varphi(1), \tag{3.7}$$

$$T_2=\frac{2h\lambda_1\sin^2 h}{h\cos h-\sin h}\sum_{\gamma=0}^{N}\lambda_1^{N-\gamma}\varphi(h\gamma)$$
$$+\lambda_1^{N+1}(\cos h-\lambda_1)\varphi(0)+(\lambda_1\cos h-1-\lambda_1\tan 1\sin h)\varphi(1). \tag{3.8}$$

Hence, we get

$$d_1^-=\frac{T_1B_{22}-T_2B_{12}}{B_{11}B_{22}-B_{12}B_{21}},\quad d_1^+=\frac{T_2B_{11}-T_1B_{21}}{B_{11}B_{22}-B_{12}B_{21}}. \tag{3.9}$$

Combaining (2.12), (3.3) and (3.9) we obtain d_1 and d_2 wich are given in the statemant of Theorem 3.1.

Now, we calculate the coefficients C_β, $\beta=0,1,\dots,N$. Taking into account (2.11) from (2.9) for C_β we get

$$\begin{aligned}
C_\beta &= D(h\beta)*u(h\beta)\\
&=\sum_{\gamma=-\infty}^{\infty}D(h\beta-h\gamma)u(h\gamma)\\
&=\sum_{\gamma=1}^{\infty}D(h\beta+h\gamma)[-d_1^-\sin(h\gamma)+d_2^-\cos(h\gamma)]+\sum_{\gamma=0}^{N}D(h\beta-h\gamma)\varphi(h\gamma)\\
&\quad+\sum_{\gamma=1}^{\infty}D(h(N+\gamma)-h\beta)[d_1^+\sin(1+h\gamma)+d_2^+\cos(1+h\gamma)],
\end{aligned}$$

from which, using (2.6) and taking into account notations (3.1), (3.2), when $\beta=0,1,\dots,N$, for C_β we obtain the expressions given in Theorem 3.1.

4 Numerical results

In this section, in numerical examples, we compare the interpolation spline (1.5) with the natural cubic spline (D^2-spline) in the same points. For these splines we use the notations

$$S(x)\equiv S_N(f;x)\quad\text{and}\quad S^{\text{cubic}}(x)\equiv S_N^{\text{cubic}}(f;x),$$

respectively, and we calculate the the corresponding absolute errors

$$e_N(f;x):=|f(x)-S_N(f;x)|\quad\text{and}\quad e_N^{\text{cubic}}(f;x):=|f(x)-S_N^{\text{cubic}}(f;x)|.$$

It is known (see, for instance, [2,6,9,18,19,21,30]) that the natural cubic spline minimizes the integral $\int_0^1 (\varphi''(x))^2 dx$ in the Sobolev space $L_2^{(2)}(0,1)$ of functions with a square integrable 2^{nd} generalized derivative. In numerical examples we use the standard MAPLE function "`spline(X,Y,x,cubic)`" for calculating the natural cubic spline.

In our examples we apply the interpolation spline (1.5) and the natural cubic spline to approximate the functions

$$f_1(x) = \tan x \quad \text{and} \quad f_2(x) = \frac{313x^4 - 6900x^2 + 15120}{13x^4 + 660x^2 + 15120}$$

on $[0,1]$.

Thus, using Theorem 3.1 and the standard Maple function "`spline(X,Y,x,cubic)`", with $N = 5$ and $N = 10$, we get the corresponding interpolation splines $S_N(f_k;x)$, and $S_N^{\text{cubic}}(f_k;x)$, for $f(x) = f_k(x)$, $k = 1,2$.

The corresponding absolute errors $e_N(f_k;x)$ and $e_N^{\text{cubic}}(f_k;x)$ for these approximations are displayed in Figures 4.1 and 4.2 for functions $f = f_1$ and $f = f_2$, respectively.

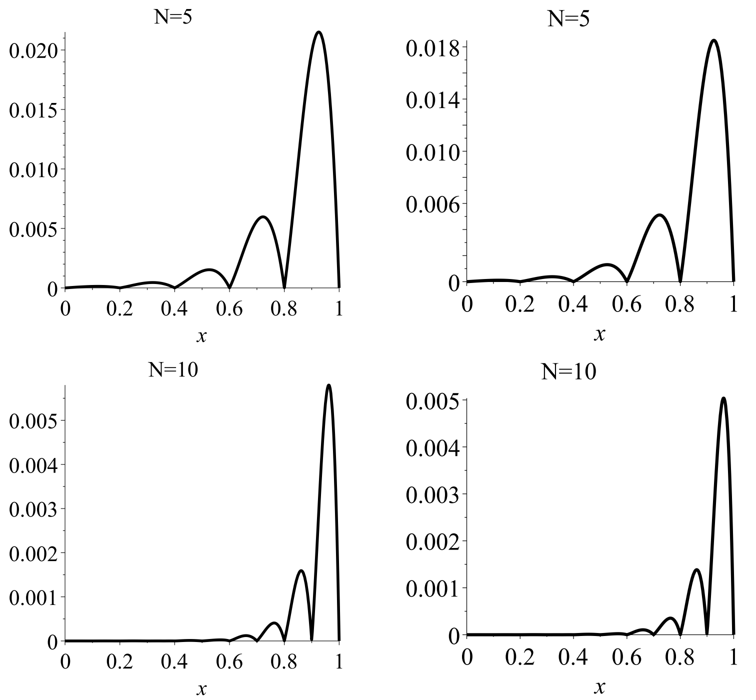

Fig. 4.1 Graphs of absolute errors $e_N(f_1;x)$ and $e_N^{\text{cubic}}(f_1;x)$ for $N = 5$ (top) and $N = 10$ (bottom)

In the first example ($f = f_1$) the results are quite similar, but in the second case the approximation by $S_N(f_2;x)$ is much better than the approximation by the corresponding cubic spline $S_N^{\text{cubic}}(f_2;x)$, because $f_2(x)$ is a rational approximation for the function $\cos x$ (cf. [15, p. 66]), and the interpolation spline $S_N(f_2;x)$ is exact for the trigonometric functions $\sin x$ and $\cos x$.

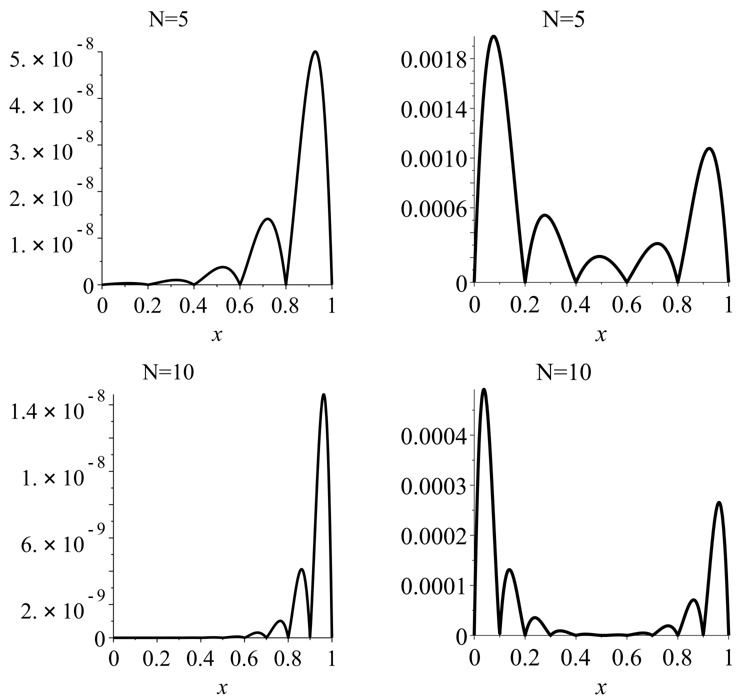

Fig. 4.2 Graphs of absolute errors $e_N(f_2;x)$ and $e_N^{\mathrm{cubic}}(f_2;x)$ for $N=5$ (top) and $N=10$ (bottom)

Finally, it should be noted that we used these functions $f_1(x)$ and $f_2(x)$ to test an optimal quadrature formula in the sense of Sard in the space $K_2(P_2)$,

$$I(\varphi) := \int_0^1 \varphi(x)\,\mathrm{d}x \cong \sum_{\nu=0}^{N} C_\nu \varphi(x_\nu) =: Q_N(\varphi), \tag{4.1}$$

which have been constructed recently in our paper [17]. The weight coefficients in (4.1) are

$$C_0 = C_N = \frac{2\sin h - (h+\sin h)\cos h}{(h+\sin h)\sin h} + \frac{h-\sin h}{(h+\sin h)\sin h(1+\lambda_1^N)}\big(\lambda_1 + \lambda_1^{N-1}\big)$$

and

$$C_\nu = \frac{4(1-\cos h)}{h+\sin h} + \frac{2h(h-\sin h)\sin h}{(h+\sin h)(h\cos h - \sin h)(1+\lambda_1^N)}\big(\lambda_1^\nu + \lambda_1^{N-\nu}\big),$$

for $\nu = 1,\ldots,N-1$, where λ_1 is given as in Theorem 2.1, with $|\lambda_1|<1$.

In [17] we have obtained quadrature sums $Q_N(f) = \sum\limits_{\nu=0}^{N} C_\nu f(x_\nu)$ as approximations of the integral $I(f)$, taking $N = 10, 100$, and 1000. These approximate values we can also obtain if we integrate our interpolation splines $S_N(f;x)$ over $(0,1)$, i.e., $Q_N(f) = I(S_N(f;x))$.

According to Theorem 3.1, we find the interpolation splines $S_N(f;x)$, given by (1.5). For example, for $N=1,2,3$, we have:

CASE $N=1$:

$$S_1(f;x) = d_1 \sin x + d_2 \cos x,$$

where

$$\begin{aligned} d_1 &= -0.6420926134\ f(0) + 1.188395105\ f(1) \\ d_2 &= f(0); \end{aligned}$$

CASE $N=2$:

$$S_2(f;x) = \sum_{\beta=0}^{2} C_\beta G(x-h\beta) + d_1 \sin x + d_2 \cos x,$$

where

$$\begin{aligned} C_0 &= 12.6159872\ f(0) - 22.14314029\ f(1/2) + 12.61598692\ f(1), \\ C_1 &= -22.14314054\ f(0) + 38.8648702\ f(1/2) - 22.14314053\ f(1), \\ C_1 &= 12.6159872\ f(0) - 22.14314029\ f(1/2) + 12.61598692\ f(1), \\ d_1 &= -1.038130553 f(0) + 0.6951119199\ f(1/2) + 0.7923572208\ f(1), \\ d_2 &= 0.2750574804\ f(0) + 1.272393834\ f(1/2) - 0.7249425490\ f(1); \end{aligned}$$

CASE $N=3$:

$$S_3(f;x) = \sum_{\beta=0}^{3} C_\beta G(x-h\beta) + d_1 \sin x + d_2 \cos x,$$

where

$$\begin{aligned} C_0 &= 44.237493\ f(0) - 94.7879290\ f(1/3) + 65.37216114\ f(2/3) - 11.18287316\ f(1), \\ C_1 &= -94.7879264\ f(0) + 244.513164\ f(1/3) - 218.3356748\ f(2/3) + 65.37216058\ f(1), \\ C_2 &= 65.37216057\ f(0) - 218.3356748\ f(1/3) + 244.513164\ f(2/3) - 94.7879264\ f(1), \\ C_4 &= -11.18287317\ f(0) + 65.37216115\ f(1/3) - 94.7879286\ f(2/3) + 44.237493\ f(1), \\ d_1 &= = -1.913065961\ f(0) + 2.741871157\ f(1/3) - 1.913243332\ f(2/3) + 1.528235894\ f(1), \\ d_2 &= 0.5877825293\ f(0) - 1.292209243\ f(1) - 0.5131537868 f(1/3) + 2.029946851\ f(2/3). \end{aligned}$$

Integrating these splines $S_1(f;x)$, $S_2(f;x)$, and $S_3(f;x)$ we get the optimal quadrature formulas of the form (4.1), with $N=1,2,3$ respectively.

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