

# Efficient computation of highly oscillatory integrals with Hankel kernel <sup>☆</sup>



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## ABSTRACT

In this paper, we consider the evaluation of two kinds of oscillatory integrals with a Hankel function as kernel. We first rewrite these integrals as the integrals of Fourier-type. By analytic continuation, these Fourier-type integrals can be transformed into the integrals on  $[0, +\infty)$ , the integrands of which are not oscillatory, and decay exponentially fast. Consequently, the transformed integrals can be efficiently computed by using the generalized Gauss–Laguerre quadrature rule. Moreover, the error analysis for the presented methods is given. The efficiency and accuracy of the methods have been demonstrated by both numerical experiments and theoretical results.

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## 1. Introduction

In this paper, we are concerned with the numerical approximation of the integrals with a highly oscillatory Hankel kernel of the form

$$I_1[f] = \int_a^b f(x)H_\nu^{(1)}(\omega x)dx \quad \text{and} \quad I_2[f] = \int_a^{+\infty} f(x)H_\nu^{(1)}(\omega x)dx, \quad (1.1)$$

where  $H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$  is Hankel function of the first kind of order  $\nu$ ,  $\omega \gg 1$  and  $b > a > 0$ . The integrals (1.1) play an important role in many areas of science and engineering, such as astronomy, optics, quantum mechanics, seismology image processing, electromagnetic scattering, such as [3,4,11,17].

For large values of  $\omega$ , the integrands become highly oscillatory, and the efficient and reliable numerical evaluation of the integrals is problematic. Moreover, a prohibitively number of quadrature nodes are needed to obtain satisfied accuracy if one uses classical numerical methods like Simpson rule, Gaussian quadrature, etc. In the last few years, many efficient methods have been devised for the integral  $\int_a^b f(x)H_\nu^{(1)}(\omega x)dx$ , such as Levin method [20,21], Levin-type method [25], modified Clenshaw–Curtis method [26], generalized quadrature rule [12,13], Filon-type method [30], Clenshaw–Curtis–Filon-type method [31], Gauss–Laguerre quadrature [5,6]. However, only a few methods to evaluate the integral  $\int_a^\infty f(x)H_\nu^{(1)}(\omega x)dx$  have been proposed. For the latest references, we refer the readers to [7,8] for a more general review.

All above-mentioned methods share the property that the larger the  $\omega$ , the higher the accuracy. The goal of this paper is to explore efficient and high order methods for the integrals (1.1) based on the idea of complex integration method (see

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Milovanović [23], Huybrechs and Vandewalle [16] and Chen [5,6]), which can also be applied to the computation of highly oscillatory Cauchy principal value integrals and highly oscillatory integrals with algebraic singularities [18,19,27,28,32].

An outline of this paper is as follows. In Section 2, we present the details of the proposed methods for computing the integrals (1.1). Meanwhile, error analysis for the presented methods is discussed. In Section 3, several numerical examples are given to show the efficiency and accuracy of presented methods.

**2. Numerical schemes for the integrals**

Thanks to the identity [15, p. 915]

$$H_v^{(1)}(\omega x) = \sqrt{\frac{2}{\pi \omega x}} \frac{e^{i(\omega x - \frac{\pi}{2} \nu - \frac{\pi}{4})}}{\Gamma(\nu + \frac{1}{2})} \int_0^{+\infty} \left(1 + \frac{it}{2\omega x}\right)^{\nu - \frac{1}{2}} t^{\nu - \frac{1}{2}} e^{-t} dt, \quad x > 0, \tag{2.1}$$

we can rewrite the integrals (1.1) in the following form

$$\begin{aligned} I_1[f] &= \sqrt{\frac{2}{\pi \omega}} \frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu + \frac{1}{2})} \int_a^b f(x)x^{-\frac{1}{2}} e^{i\omega x} \left[ \int_0^{+\infty} \left(1 + \frac{it}{2\omega x}\right)^{\nu - \frac{1}{2}} t^{\nu - \frac{1}{2}} e^{-t} dt \right] dx \\ &= \sqrt{\frac{2}{\pi \omega}} \frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu + \frac{1}{2})} \int_a^b f(x)x^{-\frac{1}{2}} g(x) e^{i\omega x} dx, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} I_2[f] &= \sqrt{\frac{2}{\pi \omega}} \frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu + \frac{1}{2})} \int_a^{+\infty} f(x)x^{-\frac{1}{2}} e^{i\omega x} \left[ \int_0^{+\infty} \left(1 + \frac{it}{2\omega x}\right)^{\nu - \frac{1}{2}} t^{\nu - \frac{1}{2}} e^{-t} dt \right] dx \\ &= \sqrt{\frac{2}{\pi \omega}} \frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu + \frac{1}{2})} \int_a^{+\infty} f(x)x^{-\frac{1}{2}} g(x) e^{i\omega x} dx, \end{aligned} \tag{2.3}$$

where

$$g(x) = \int_0^{+\infty} \left(1 + \frac{it}{2\omega x}\right)^{\nu - \frac{1}{2}} t^{\nu - \frac{1}{2}} e^{-t} dt. \tag{2.4}$$

From Eqs. (2.2) and (2.3), we can see that the integrals (1.1) are transformed into the integrals of Fourier form, which can be evaluated by complex integration method and quadrature rules of Gaussian type. In what follows, we will focus on the fast computation of the integrals (1.1) and error analysis for the presented methods.

**2.1. The evaluation of the integral  $I_1[f]$**

For the calculation of the integral  $I_1[f]$ , we first assume that  $f$  is an analytic function in the half-strip of the complex plane,  $a \leq \text{Re}(z) \leq b$ ,  $\text{Im}(z) \geq 0$ , and there exists two constants  $C$  and  $\omega_0$ , such that  $|f(x + iR)| \leq Ce^{\omega_0 R}$ ,  $a \leq x \leq b$ , with  $0 < \omega_0 < \omega$ .

Following the ideas of [5,16,23], consider the contour as shown in Fig. 1 for the integral (2.2), and let  $D$  denote the region

$$D = \{z \in \mathbb{C} \mid a \leq \text{Re}(z) \leq b, 0 \leq \text{Im}(z) \leq R\}.$$

Since the integrand

$$F(z) = f(z)z^{-\frac{1}{2}} g(z) \tag{2.5}$$

is analytic in the region  $D$ , by the Cauchy Residue Theorem [1], we have

$$\int_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4} F(z) e^{i\omega z} dz = 0, \tag{2.6}$$

with all the contours taken in the counterclockwise direction.

For the integral over the contour  $\Gamma_2$ , we have

$$\begin{aligned} \int_{\Gamma_2} F(z) e^{i\omega z} dz &= i \int_0^R F(b + ip) e^{i\omega(b+ip)} dp \\ &= i e^{i\omega b} \int_0^R F(b + ip) e^{-\omega p} dp \\ &= \frac{i e^{i\omega b}}{\omega} \int_0^{\omega R} F\left(b + \frac{ip}{\omega}\right) e^{-p} dp. \end{aligned} \tag{2.7}$$

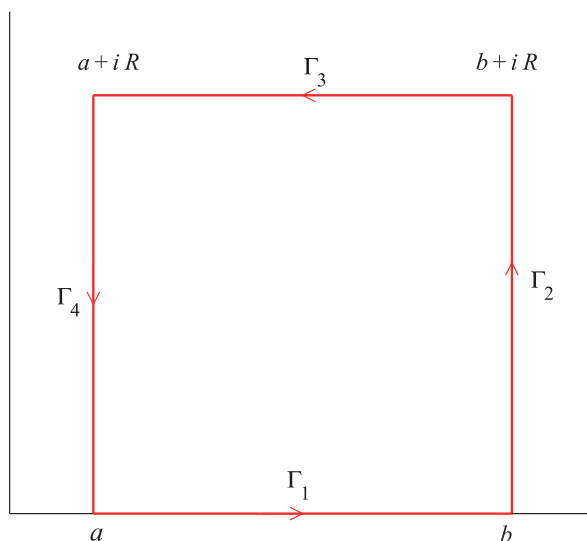


Fig. 1. The integration paths for the integral (2.2).

Similarly, for the integral over the contour  $\Gamma_4$ , there holds

$$\begin{aligned} \int_{\Gamma_4} F(z)e^{i\omega z} dz &= i \int_R^0 F(a + ip)e^{i\omega(a+ip)} dp \\ &= ie^{i\omega a} \int_R^0 F(a + ip)e^{-\omega p} dp \\ &= -\frac{ie^{i\omega a}}{\omega} \int_0^{\omega R} F\left(a + \frac{ip}{\omega}\right) e^{-p} dp. \end{aligned} \tag{2.8}$$

For the integral over the contour  $\Gamma_3$ , we evaluate  $\int_{\Gamma_3} F(z)e^{i\omega z} dz$  by letting  $z = x + iR, x \in [a, b]$ , then it yields that

$$\begin{aligned} \int_{\Gamma_3} F(z)e^{i\omega z} dz &= \int_b^a F(x + iR)e^{i\omega(x+iR)} dx \\ &= e^{-\omega R} \int_b^a F(x + iR)e^{i\omega x} dx. \end{aligned} \tag{2.9}$$

It follows from the analytic continuation of (2.4) to  $D$ , there exists some constant  $M$ , such that  $|z^{-\frac{1}{2}}g(z)| \leq M$ , so that for  $z \in \Gamma_3$ ,

$$|F(z)e^{i\omega z}| = |f(z)| \cdot |z^{-1/2}g(z)| \cdot |e^{i\omega z}| \leq |f(x + iR)| \cdot M \cdot e^{-\omega R} \leq CM e^{-(\omega - \omega_0)R}.$$

Thus, from (2.9) it follows

$$\left| \int_{\Gamma_3} F(z)e^{i\omega z} dz \right| \leq \int_{\Gamma_3} |F(z)e^{i\omega z}| \cdot |dz| \leq CM e^{-(\omega - \omega_0)R} (b - a).$$

Since  $\omega_0 < \omega$ , we conclude that

$$\int_{\Gamma_3} F(z)e^{i\omega z} dz \rightarrow 0 \quad \text{as } R \rightarrow +\infty. \tag{2.10}$$

Now, according to (2.6), we have

$$\begin{aligned} I_1[f] &= \int_a^b f(x)H_\nu^{(1)}(\omega x) dx = \sqrt{\frac{2}{\pi\omega}} \frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu + \frac{1}{2})} \int_{\Gamma_1} F(z)e^{i\omega z} dz \\ &= -\sqrt{\frac{2}{\pi\omega}} \frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu + \frac{1}{2})} \int_{\Gamma_2 \cup \Gamma_3 \cup \Gamma_4} F(z)e^{i\omega z} dz. \end{aligned}$$

Taking  $R \rightarrow +\infty$  and using (2.7), (2.8), and (2.9), the previous equality reduces to

$$I_1[f] = \frac{i}{\omega} \sqrt{\frac{2}{\pi\omega}} \frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu + \frac{1}{2})} (G(a) - G(b)), \tag{2.11}$$

where

$$G(c) = e^{i\omega c} \int_0^{+\infty} F\left(c + \frac{i}{\omega}t\right)e^{-t} dt.$$

According to (2.4) and using (2.5), we get

$$G(c) = e^{i\omega c} \int_0^{+\infty} \int_0^{+\infty} \frac{f\left(c + \frac{i}{\omega}t\right)}{\left(c + \frac{i}{\omega}t\right)^{1/2}} \left(1 + \frac{is}{2\omega\left(c + \frac{i}{\omega}t\right)}\right)^{\nu-1/2} s^{\nu-1/2} e^{-t-s} ds dt,$$

i.e.,

$$G(c) = e^{i\omega c} \int_0^{+\infty} \int_0^{+\infty} \frac{f\left(c + \frac{i}{\omega}t\right)}{\left(c + \frac{i}{\omega}t\right)^\nu} \left(c + \frac{i}{\omega}t + \frac{i}{2\omega}s\right)^{\nu-1/2} s^{\nu-1/2} e^{-t-s} ds dt. \tag{2.12}$$

In order to calculate  $G(c)$  (for  $c = a$  and  $c = b$ ), we can apply two Gaussian quadratures

$$\int_{a_\ell}^{b_\ell} h(x)w_\ell(x)dx = \sum_{k=1}^n A_{n,k}^{(\ell)} h(x_{n,k}^{(\ell)}) + R_n^{(\ell)}[h], \quad \ell = 1, 2, \tag{2.13}$$

where  $x_{n,k}^{(\ell)}$  and  $A_{n,k}^{(\ell)}$ ,  $k = 1, \dots, n$ , are the nodes and the Christoffel numbers, respectively.

It is known that the nodes  $x_{n,k}^{(\ell)}$  are eigenvalues of the following symmetric tridiagonal Jacobi matrix

$$J_n(w_\ell) = \begin{bmatrix} \alpha_0^{(\ell)} & \sqrt{\beta_1^{(\ell)}} & & & \mathbf{0} \\ \sqrt{\beta_1^{(\ell)}} & \alpha_1^{(\ell)} & \sqrt{\beta_2^{(\ell)}} & & \\ & \sqrt{\beta_2^{(\ell)}} & \alpha_2^{(\ell)} & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}^{(\ell)}} \\ \mathbf{0} & & & \sqrt{\beta_{n-1}^{(\ell)}} & \alpha_{n-1}^{(\ell)} \end{bmatrix}, \tag{2.14}$$

and the weight coefficients  $A_{n,k}^{(\ell)}$  are given by

$$A_{n,k}^{(\ell)} = \beta_0^{(\ell)} v_{k,1}^2, \quad k = 1, \dots, n,$$

where  $v_{k,1}$  is the first component of the eigenvector  $\mathbf{v}_k (= [v_{k,1} \dots v_{k,n}]^T)$  corresponding to the eigenvalue  $x_{n,k}^{(\ell)}$  and normalized such that  $\mathbf{v}_k^T \mathbf{v}_k = 1$  (cf. [22, p. 100]). Here,  $\alpha_k^{(\ell)}$  and  $\beta_k^{(\ell)}$  are recursive coefficients in the three-term recurrence relation for monic orthogonal polynomials with respect to the weight function  $w_\ell$  on  $(a_\ell, b_\ell)$ ,

$$\pi_{k+1}^{(\ell)}(x) = (x - \alpha_k^{(\ell)})\pi_k^{(\ell)}(x) - \beta_k^{(\ell)}\pi_{k-1}^{(\ell)}(x), \quad k = 0, 1, \dots, \tag{2.15}$$

with  $\pi_0^{(\ell)}(x) = 1$ ,  $\pi_{-1}^{(\ell)}(x) = 0$ , and the coefficient  $\beta_0^{(\ell)}$  is taken so that  $\beta_0^{(\ell)} \equiv \int_{a_\ell}^{b_\ell} w_\ell(x)dx$ . The most popular method for solving the eigenvalue problem for (2.14) is the Golub–Welsch procedure, obtained by a simplification of QR algorithm [14]. Thus, when recursive coefficients in (2.15) are known, then the construction of a Gaussian formula is a very easy task.

The quadrature formulas (2.13), with respect to the weight functions  $w_\ell$ ,  $\ell = 1, 2$ , provide an approximation  $Q_{n,m}[u]$  for double integrals of the form

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} u(x,y)w_1(x)w_2(y)dx dy \approx Q_{n_1,n_2}[u] = \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} A_{n_1,k}^{(1)} A_{n_2,j}^{(2)} u(x_{n_1,k}^{(1)}, x_{n_2,j}^{(2)}). \tag{2.16}$$

For calculating (2.12), we can take  $a_1 = a_2 = 0$ ,  $b_1 = b_2 = +\infty$ ,  $w_1(t) = e^{-t}$  (Laguerre weight), and  $w_2(s) = s^{\nu-1/2} e^{-s}$  (generalized Laguerre weights), with the recursion coefficients

$$\begin{aligned} \alpha_k^{(1)} &= 2k + 1, & \beta_0^{(1)} &= 1, & \beta_k^{(1)} &= k^2; \\ \alpha_k^{(2)} &= 2k + \nu + \frac{1}{2}, & \beta_0^{(2)} &= \Gamma\left(\nu + \frac{1}{2}\right), & \beta_k^{(2)} &= k\left(k + \nu - \frac{1}{2}\right). \end{aligned}$$

According to (2.12), we can introduce the function  $\varphi(\cdot, \cdot; c)$  by

$$\varphi(t, s; c) = e^{i\omega c} \frac{f\left(c + \frac{i}{\omega}t\right)}{\left(c + \frac{i}{\omega}t\right)^\nu} \left(c + \frac{i}{\omega}t + \frac{i}{2\omega}s\right)^{\nu-1/2}, \tag{2.17}$$

then the numerical method for the evaluation of the integral (2.2) can be denoted by

$$Q_{n_1,n_2}[f] = \frac{i}{\omega} \sqrt{\frac{2}{\pi\omega}} \frac{e^{-i\pi(2\nu+1)/4}}{\Gamma\left(\nu + \frac{1}{2}\right)} \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} A_{n_1,k}^{(1)} A_{n_2,j}^{(2)} (\varphi(x_{n_1,k}^{(1)}, x_{n_2,j}^{(2)}; a) - \varphi(x_{n_1,k}^{(1)}, x_{n_2,j}^{(2)}; b)), \tag{2.18}$$

where  $x_{n_1,k}^{(1)}$  and  $A_{n_1,k}^{(1)}$  denote the nodes and weights of the  $n_1$ -point Gauss–Laguerre quadrature rule, and  $x_{n_2,j}^{(2)}$  and  $A_{n_2,j}^{(2)}$  denote the nodes and weights of the  $n_2$ -point generalized Gauss–Laguerre quadrature rule with the weight function  $w_2(s) = s^{\nu-1/2}e^{-s}$ .

**Theorem 2.1.** Suppose that  $f$  is an analytic function in the half-strip of the complex plane,  $a \leq \operatorname{Re}(z) \leq b$ ,  $\operatorname{Im}(z) \geq 0$ , and there exist two constants  $C$  and  $\omega_0$ , such that  $|f(x + iR)| \leq Ce^{\omega_0 R}$ ,  $a \leq x \leq b$ , with  $0 < \omega_0 < \omega$ . Then the error bound of the method (2.18) for the integral (2.2) is given by

$$I_1[f] - Q_{n_1,n_2}[f] = O(\omega^{-\frac{3}{2}-2\tau}), \quad \omega \gg 1, \tag{2.19}$$

where  $\tau = \min\{n_1, n_2\}$ .

**Proof.** The error of the  $n$ -point generalized Gauss–Laguerre quadrature rule applied to the integral  $\int_0^{+\infty} h(x)x^\alpha e^{-x} dx$ ,  $\alpha > -1$  is given by the following formula (cf. [10, p. 223])

$$E_n[h] = \frac{n! \Gamma(n + \alpha + 1)}{(2n)!} h^{(2n)}(\xi), \quad 0 < \xi < +\infty.$$

According to this formula, we have

$$\begin{aligned} G(a) &= \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} A_{n_1,k}^{(1)} A_{n_2,j}^{(2)} \varphi(x_{n_1,k}^{(1)}, x_{n_2,j}^{(2)}; a) \\ &= \int_0^{+\infty} \int_0^{+\infty} \varphi(t, s; a) s^{\nu-1/2} e^{-t-s} ds dt - \sum_{k=1}^{n_1} A_{n_1,k}^{(1)} \int_0^{+\infty} \varphi(x_{n_1,k}^{(1)}, s; a) s^{\nu-1/2} e^{-s} ds \\ &\quad + \left( \sum_{k=1}^{n_1} A_{n_1,k}^{(1)} \int_0^{+\infty} \varphi(x_{n_1,k}^{(1)}, s; a) s^{\nu-1/2} e^{-s} ds - \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} A_{n_1,k}^{(1)} A_{n_2,j}^{(2)} \varphi(x_{n_1,k}^{(1)}, x_{n_2,j}^{(2)}; a) \right) \\ &= \frac{(n_1!)^2}{(2n_1)!} \frac{d^{2n_1}}{dt^{2n_1}} \left\{ \int_0^{+\infty} \varphi(t, s; a) s^{\nu-1/2} e^{-s} ds \right\} \Big|_{t=\xi_1} + \sum_{k=1}^{n_1} A_{n_1,k}^{(1)} \frac{(n_2!) \Gamma(n_2 + \nu + \frac{1}{2})}{(2n_2)!} \frac{d^{2n_2}}{ds^{2n_2}} \{ \varphi(x_{n_1,k}^{(1)}, s; a) \} \Big|_{s=\xi_2} \\ &= \frac{(n_1!)^2}{(2n_1)!} O(\omega^{-2n_1}) + \frac{n_2! \Gamma(n_2 + \nu + \frac{1}{2})}{(2n_2)!} \sum_{k=1}^{n_1} A_{n_1,k}^{(1)} O(\omega^{-2n_2}) \\ &= O(\omega^{-2\tau}), \end{aligned} \tag{2.20}$$

where  $\tau = \min\{n_1, n_2\}$  and  $\xi_1, \xi_2 \in (0, +\infty)$ .

By the same argument, we can easily get

$$G(b) - \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} A_{n_1,k}^{(1)} A_{n_2,j}^{(2)} \varphi(x_{n_1,k}^{(1)}, x_{n_2,j}^{(2)}; a) = O(\omega^{-2\tau}). \tag{2.21}$$

A combination of Eqs. (2.11), (2.18), (2.20), (2.21) leads to desired result.  $\square$

**Remark 1.** An alternative expression for  $G(c)$  can be done by a transformation of the first quadrant of the  $(t, s)$ -plane to the half-strip

$$S = \{(\tau, \sigma) : 0 \leq \tau \leq 1, 0 \leq \sigma < +\infty\}, \tag{2.22}$$

by taking  $t = \sigma\tau, s = \sigma(1 - \tau)$ , i.e.,

$$\tau = \frac{t}{t+s}, \quad \sigma = t+s.$$

The corresponding Jacobian determinant is  $D(t, s)/D(\tau, \sigma) = \sigma$ , so that

$$G(c) = e^{i\omega c} \int_0^{+\infty} \int_0^1 \frac{f(c + \frac{i}{\omega}\sigma\tau)}{(c + \frac{i}{\omega}\sigma\tau)^\nu} \left( c + \frac{i}{2\omega}\sigma(1 + \tau) \right)^{\nu-1/2} \sigma^{\nu+1/2} e^{-\sigma} (1 - \tau)^{\nu-1/2} d\tau d\sigma. \tag{2.23}$$

Similarly, by introduce the function

$$\varphi(\sigma, \tau; c) = e^{i\omega c} \frac{f(c + \frac{i}{\omega}\sigma\tau)}{(c + \frac{i}{\omega}\sigma\tau)^\nu} \left( c + \frac{i}{2\omega}\sigma(1 + \tau) \right)^{\nu-1/2}, \tag{2.24}$$

we present another method for the integral (2.2) by the following formula

$$\widehat{Q}_{n_1,n_2}[f] = \frac{i}{\omega} \sqrt{\frac{2}{\pi\omega}} \frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu + \frac{1}{2})} \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} A_{n_1,k}^{(3)} A_{n_2,j}^{(4)} (\varphi(x_{n_1,k}^{(3)}, x_{n_2,j}^{(4)}; a) - \varphi(x_{n_1,k}^{(3)}, x_{n_2,j}^{(4)}; b)), \tag{2.25}$$

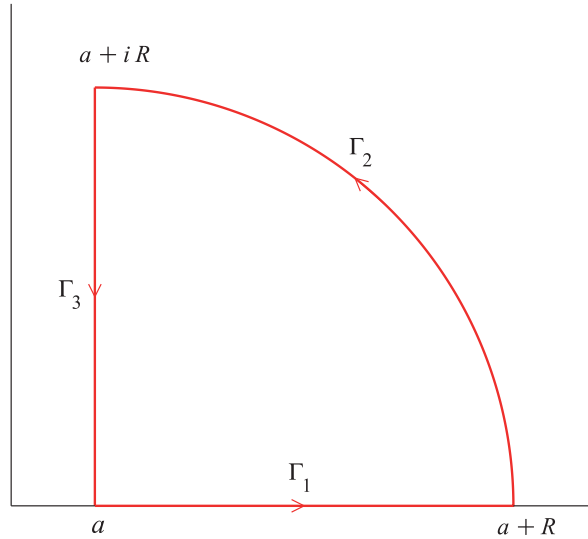


Fig. 2. The integration paths for the integral (2.3).

where  $x_{n_1,k}^{(3)}$  and  $A_{n_1,k}^{(3)}$  denote the nodes and weights of the  $n_1$ -point generalized Gauss–Laguerre quadrature rule with the weight function  $\sigma^{\nu+1/2}e^{-\sigma}$  on  $(0, +\infty)$ , and  $x_{n_2,j}^{(4)}$  and  $A_{n_2,j}^{(4)}$  denote the nodes and weights of the  $n_2$ -point Gaussian quadrature rule with shifted Jacobi weight,  $(1 - \tau)^{\nu-1/2}$  on  $(0, 1)$ . The corresponding recursion coefficients are

$$\alpha_k^{(3)} = 2k + \nu + \frac{3}{2}, \quad \beta_0^{(3)} = \Gamma\left(\nu + \frac{3}{2}\right), \quad \beta_k^{(3)} = k\left(k + \nu + \frac{1}{2}\right);$$

$$\alpha_k^{(4)} = \frac{2[4k^2 + 2(2\nu + 1)k + 2\nu - 1]}{(4k + 2\nu - 1)(4k + 2\nu + 3)}, \quad \beta_0^{(4)} = \frac{2}{2\nu + 1}, \quad \beta_k^{(4)} = \frac{4k^2(2k + 2\nu - 1)^2}{(4k + 2\nu - 3)(4k + 2\nu - 1)^2(4k + 2\nu + 1)}.$$

2.2. The evaluation of the integral  $I_2[f]$ .

In this section, we suppose that  $f$  is an analytic function in the complex plane  $\{0 \leq \arg(z) \leq \frac{\pi}{2}\}$ , and there exists some constant  $C_1$ , such that  $|f(z)| \leq C_1$  as  $|z| \rightarrow +\infty$ .

According to the idea of [23], we adopt the integration paths as shown in Fig. 2 to derive the desired result. Noting that  $F(z) = f(z)z^{-\frac{1}{2}}g(z)$  is analytic in the complex plane  $\{0 \leq \arg(z) \leq \frac{\pi}{2}\}$ , from Cauchy Residue Theorem [1], we have

$$\int_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3} F(z)e^{i\omega z} dz = 0, \tag{2.26}$$

For the integral over the contour  $\Gamma_3$ , it is easy to show that

$$\int_{\Gamma_3} f(z)z^{-\frac{1}{2}}g(z)e^{i\omega z} dz = -\frac{ie^{i\omega a}}{\omega} \int_0^{\omega R} F\left(a + \frac{ip}{\omega}\right) e^{-p} dp. \tag{2.27}$$

For the integral over the contour  $\Gamma_2$ , setting

$$z - a = Re^{i\theta}, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

derives

$$\int_{\Gamma_2} F(z)e^{i\omega z} dz = iR \int_0^{\pi/2} e^{i\theta} F(a + Re^{i\theta}) e^{i\omega(a + Re^{i\theta})} d\theta. \tag{2.28}$$

Using the well-known inequality [1]

$$\frac{2\theta}{\pi} \leq \sin(\theta), \quad 0 \leq \theta \leq \frac{\pi}{2},$$

and the fact that  $|f(z)| \leq C_1$ ,  $|g(z)| \leq C_2$  for some constant  $C_2$ , yields

$$\left| \int_{\Gamma_2} F(z)e^{i\omega z} dz \right| \leq \frac{RC_1C_2}{|R-a|^{1/2}} \int_0^{\pi/2} e^{-(2/\pi)\omega R\theta} d\theta = \frac{\pi C_1 C_2}{2\omega |R-a|^{1/2}} (1 - e^{-\omega R}) \rightarrow 0, \tag{2.29}$$

when  $R \rightarrow +\infty$ . By means of Eqs. (2.26)–(2.27) and (2.29), when  $R \rightarrow +\infty$ , we obtain

$$\begin{aligned} \int_a^{+\infty} f(x)H_\nu^{(1)}(\omega x)dx &= \sqrt{\frac{2}{\pi\omega}} \frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu + \frac{1}{2})} \lim_{R \rightarrow +\infty} \int_{\Gamma_1} F(z)e^{i\omega z} dz \\ &= -\sqrt{\frac{2}{\pi\omega}} \frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu + \frac{1}{2})} \lim_{R \rightarrow +\infty} \int_{\Gamma_2 \cup \Gamma_3} F(z)e^{i\omega z} dz \\ &= \frac{i}{\omega} \sqrt{\frac{2}{\pi\omega}} \frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu + \frac{1}{2})} G(a), \end{aligned} \tag{2.30}$$

where  $G(a)$  is denoted by (2.12).

Then, the numerical method for the evaluation of the integral (2.3) can be denoted by

$$\bar{Q}_{n_1, n_2}[f] = \frac{i}{\omega} \sqrt{\frac{2}{\pi\omega}} \frac{e^{-i\pi(2\nu+1)/4}}{\Gamma(\nu + \frac{1}{2})} \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} A_{n_1, k}^{(1)} A_{n_2, j}^{(2)} \varphi(x_{n_1, k}^{(1)}, x_{n_2, j}^{(2)}; a). \tag{2.31}$$

where  $x_{n_1, k}^{(1)}$  and  $A_{n_1, k}^{(1)}$  denote the nodes and weights of the  $n_1$ -point Gauss–Laguerre quadrature rule, and  $x_{n_2, j}^{(2)}, A_{n_2, j}^{(2)}$  denote the nodes and weights of the  $n_2$ -point generalized Gauss–Laguerre quadrature rule with the weight function  $s^{\nu-1/2}e^{-s}$ .

**Theorem 2.2.** Suppose that  $f$  is an analytic function in the complex plane  $\{0 \leq \arg(z) \leq \frac{\pi}{2}\}$ , and there exists some constant  $C_1$ , such that  $|f(z)| \leq C_1$  as  $|z| \rightarrow +\infty$ . Then the error bound of the method (2.31) for the integral (2.3) is given by

$$I_2[f] - \bar{Q}_{n_1, n_2}[f] = O(\omega^{-\frac{3}{2}-2\tau}), \quad \omega \gg 1, \tag{2.32}$$

where  $\tau = \min\{n_1, n_2\}$ .

**Proof.** The proof of this result is quite similar to Theorem 2.1 and so is omitted.  $\square$

**Remark 2.** We can also obtain (2.30) by letting  $b \rightarrow +\infty$  in  $I_1[f]$ . In order to do it, we use the expression for  $G(b)$  given by (2.23) and a simple crude estimate of (2.24) on the half-strip  $S$ , defined by (2.22). Under conditions for a function  $f$  given in Theorem 2.1, we note that  $|f(b + i(\sigma\tau/\omega))| \leq Ce^{(\omega_0/\omega)\sigma}$ , with  $0 < \omega_0 < \omega$ . Then

$$|\varphi(\sigma, \tau; b)| = \frac{|f(b + i\frac{\sigma\tau}{\omega})|}{\sqrt{b}} \cdot \frac{(1 + (\frac{\sigma(1+\tau)}{2\omega b})^2)^{(2\nu-1)/4}}{(1 + (\frac{\sigma\tau}{\omega b})^2)^{\nu/2}} < \frac{Ce^{(\omega_0/\omega)\sigma}}{\sqrt{b}} \left(1 + \left(\frac{\sigma}{\omega b}\right)^2\right)^m,$$

where  $m$  is a nonnegative number which depends on  $\nu$ . For example, for  $\nu > 1/2$  we can take  $m = [(2\nu - 1)/4] + 1$  ( $[x]$  denotes the integer part of  $x$ ), and  $m = 0$  for  $0 < \nu \leq 1/2$ .

Since

$$\begin{aligned} |G(b)| &\leq \int_0^{+\infty} \int_0^1 |\varphi(\sigma, \tau; b)| \sigma^{\nu+1/2} e^{-\sigma} (1-\tau)^{\nu-1/2} d\tau d\sigma \\ &< \frac{2C}{(2\nu+1)\sqrt{b}} \int_0^{+\infty} \left(1 + \left(\frac{\sigma}{\omega b}\right)^2\right)^m \sigma^{\nu+1/2} e^{-(\omega-\omega_0)\sigma/\omega} d\sigma \\ &= \frac{2C}{(2\nu+1)\sqrt{b}} \left(\frac{\omega}{\omega-\omega_0}\right)^{\nu+3/2} \sum_{k=0}^m \binom{m}{k} \frac{\Gamma(2k + \nu + \frac{3}{2})}{(\omega-\omega_0)^{2k} b^{2k}} \rightarrow 0, \end{aligned}$$

when  $b \rightarrow +\infty$ , we conclude that (2.30) holds.

**Remark 3.** In [29], Xiang considered the convergence rate of Gauss–Laguerre-type quadrature for the integral  $I[f] = \int_0^{+\infty} x^\alpha e^{-x} f(x) dx$ ,  $\alpha > -1$ . Suppose that  $f, f', \dots, f^{(k-1)}$  are absolutely continuous in  $[0, +\infty)$  and satisfy for  $j = 0, 1, \dots, k$  for some  $k \geq 1$  that

$$\lim_{x \rightarrow +\infty} e^{x/2} x^{1+j+\alpha} f^{(j)}(x) = 0, \quad \text{and} \quad V = \left(\int_0^{+\infty} x^{1+k+\alpha} e^{-x} [f^{(k+1)}(x)]^2 dx\right)^{1/2} < +\infty,$$

then the convergence rate of Gauss–Laguerre-type quadrature satisfies

$$I[f] - Q_N^{GL}[f] = O(N^{-(k-1-|\alpha|)/2}).$$

From this formula, we can see that the presented methods in this paper are uniformly convergent in  $n_1$  and  $n_2$  for fixed  $\omega$ .

**Remark 4.** Based on the asymptotic series of Whittaker functions [2], Chen [6] presented two methods for the evaluation of the integrals (1.1) by using the complex integration method. For convenience, let  $n_1$  denote the number of the nodes of Gauss–Laguerre quadrature rule, and  $n_2$  denote the number of terms truncated of the series in Whittaker functions. Then the method for the first integral in [6] can be denoted by  $Q_{n_1, n_2, H_\nu^{(1)}}^S[f]$ , and method for the second integral in (1.1) can be denoted by  $\bar{Q}_{n_1, n_2, H_\nu^{(1)}}^S[f]$ .

**Table 1**

Relative errors for the integral (3.3) by the presented method  $\bar{Q}_{n_1, n_2}[f]$  and the method  $\bar{Q}_{n_1, n_2, H_v^{(1)}}^S[f]$  in [6] with  $n_1 = n_2 = 1, 2, 3, 4$  for different  $\omega$ .

	$\omega = 25$	$\omega = 50$	$\omega = 100$	$\omega = 200$	$\omega = 400$
$\bar{Q}_{1,1}[f]$	3.22 (-3)	8.06 (-4)	2.02 (-4)	5.04 (-5)	1.26 (-5)
$\bar{Q}_{1,1, H_v^{(1)}}^S[f]$	1.23 (-2)	3.90 (-3)	7.75 (-4)	1.94 (-4)	4.85 (-5)
$\bar{Q}_{2,2}[f]$	6.35 (-6)	3.26 (-7)	1.90 (-8)	1.16 (-8)	7.24 (-11)
$\bar{Q}_{2,2, H_v^{(1)}}^S[f]$	2.08 (-4)	2.68 (-5)	3.37 (-6)	4.23 (-7)	5.29 (-8)
$\bar{Q}_{3,3}[f]$	9.29 (-8)	1.05 (-9)	1.27 (-11)	2.05 (-13)	3.88 (-15)
$\bar{Q}_{3,3, H_v^{(1)}}^S[f]$	3.48 (-6)	2.19 (-7)	1.37 (-8)	8.59 (-10)	5.37 (-11)
$\bar{Q}_{4,4}[f]$	2.32 (-9)	6.31 (-12)	1.46 (-12)	4.10 (-15)	2.77 (-16)
$\bar{Q}_{4,4, H_v^{(1)}}^S[f]$	1.52 (-7)	4.89 (-9)	1.54 (-10)	4.83 (-12)	1.51 (-13)

**Table 2**

Relative errors for the integral (3.4) by the presented method  $Q_{n_1, n_2}[f]$  and the method  $Q_{n_1, n_2, H_v^{(1)}}^S[f]$  in [6] with  $n_1 = n_2 = 1, 2, 3, 4$  for different  $\omega$ .

	$\omega = 25$	$\omega = 50$	$\omega = 100$	$\omega = 200$	$\omega = 400$
$Q_{1,1}[f]$	9.29 (-4)	4.13 (-4)	4.41 (-5)	1.45 (-5)	4.85 (-6)
$Q_{1,1, H_v^{(1)}}^S[f]$	2.78 (-4)	2.64 (-4)	2.10 (-5)	9.21 (-6)	3.29 (-6)
$Q_{2,2}[f]$	2.63 (-6)	1.87 (-7)	6.11 (-9)	3.54 (-10)	2.58 (-11)
$Q_{2,2, H_v^{(1)}}^S[f]$	1.47 (-5)	4.10 (-6)	1.85 (-7)	3.48 (-8)	6.11 (-9)
$Q_{3,3}[f]$	5.03 (-9)	8.99 (-11)	7.24 (-13)	3.19 (-13)	6.60 (-16)
$Q_{3,3, H_v^{(1)}}^S[f]$	9.94 (-8)	1.59 (-8)	3.79 (-10)	3.59 (-11)	3.11 (-12)
$Q_{4,4}[f]$	7.67 (-12)	4.13 (-14)	8.84 (-14)	3.19 (-13)	5.49 (-16)
$Q_{4,4, H_v^{(1)}}^S[f]$	2.31 (-9)	1.79 (-10)	2.15 (-12)	3.19 (-13)	4.29 (-15)

Both methods have the same error bound  $O(\omega^{-\frac{3}{2}-\min\{2n_1, n_2+1\}})$ . For more details, one can refer to [6]. From Theorems 2.1–2.2, we can see that the presented methods in this paper are of higher order error bounds than the methods in [6]. In Section 3, we also present two examples to show that the methods in this paper are more efficient than the methods in [6] (see Tables 1 and 2).

**3. Numerical examples**

In what follows, we will test several examples to illustrate the qualities of the approximation provided by (2.18) and (2.31). All computations were performed in MATHEMATICA, Ver. 10.0.1, on MacBook Pro Retina, OS X 10.9.5, and alternatively using the R2012a version of the MATLAB system on a 2.50 GHz PC with 4 GB of RAM. In construction of Gaussian quadrature rules with high precision, we use the MATHEMATICA package `OrthogonalPolynomials` (cf. [9] and [24]), which is freely downloadable from the web site <http://www.mi.sanu.ac.rs/~gvm/>.

In order to conduct the experiments, we require knowledge on the exact values of all the considered integrals  $I[f]$  (precisely,  $I_1[f]$  and  $I_2[f]$ ). The values that we assumed to be exact are computed in the MATHEMATICA using the high precision arithmetic. In order to show the asymptotic order on  $\omega$  for the presented method, in some experiments we consider the scaled absolute errors

$$\omega^{\frac{3}{2}+2\min\{n_1, n_2\}} |I[f] - Q_{n_1, n_2}[f]|.$$

**Example 3.1.** Let us consider the integrals

$$\int_1^{10} \cos(x)H_1^{(1)}(\omega x)dx \quad \text{and} \quad \int_1^2 \frac{1}{1+x^2}H_2^{(1)}(\omega x)dx \tag{3.1}$$

by the method  $Q_{n_1, n_2}[f]$  with  $\omega$  from 5 to 200.

The scaled absolute errors for the first integral in (3.1) are displayed in Fig. 3 (left) for certain selected number of nodes  $(n_1, n_2) = (1, 2)$  (solid line),  $(2, 3)$  (dotted line), and  $(3, 4)$  (dashed line). Scaled absolute errors for inverted number of nodes  $(n_1, n_2) = (2, 1), (3, 2)$ , and  $(4, 3)$  are given in the same figure (right).

The corresponding graphics for the second integral in (3.1) are displayed in Fig. 4.

**Example 3.2.** Let us consider the integrals over  $(1, +\infty)$ ,

$$\int_1^{+\infty} \frac{1}{1+x^2}H_3^{(1)}(\omega x)dx \quad \text{and} \quad \int_1^{+\infty} \frac{\cos x}{x^4}H_2^{(1)}(\omega x)dx \tag{3.2}$$

by the method  $\bar{Q}_{n_1, n_2}[f]$  with  $\omega$  from 5 to 200, Figs. 5–6 show the asymptotic order on  $\omega$  for the presented method.



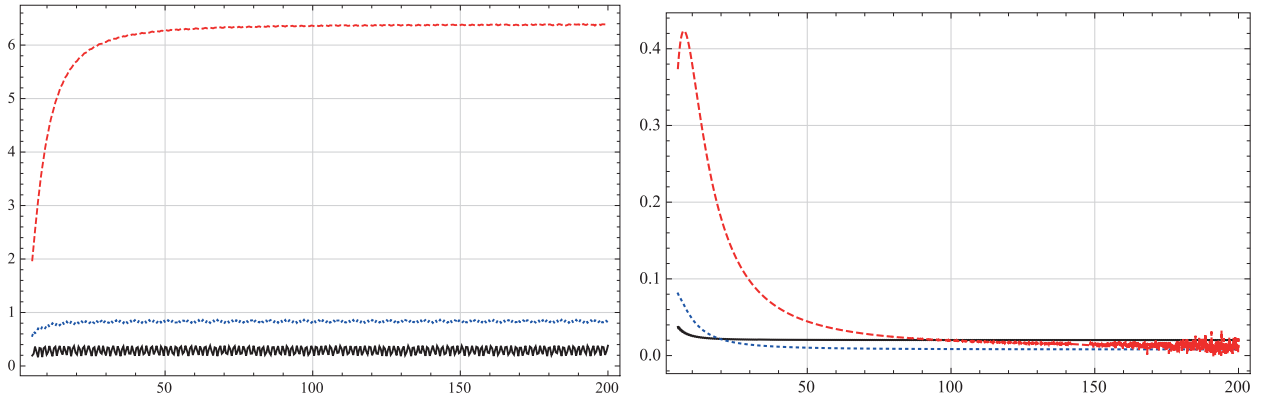


Fig. 3. Scaled absolute errors for the first integral in (3.1), obtained with  $(n_1, n_2) = (1, 2), (2, 3), (3, 4)$  (left) and with  $(n_1, n_2) = (2, 1), (3, 2), (4, 3)$  (right), when  $\omega$  runs over  $[5, 200]$ .

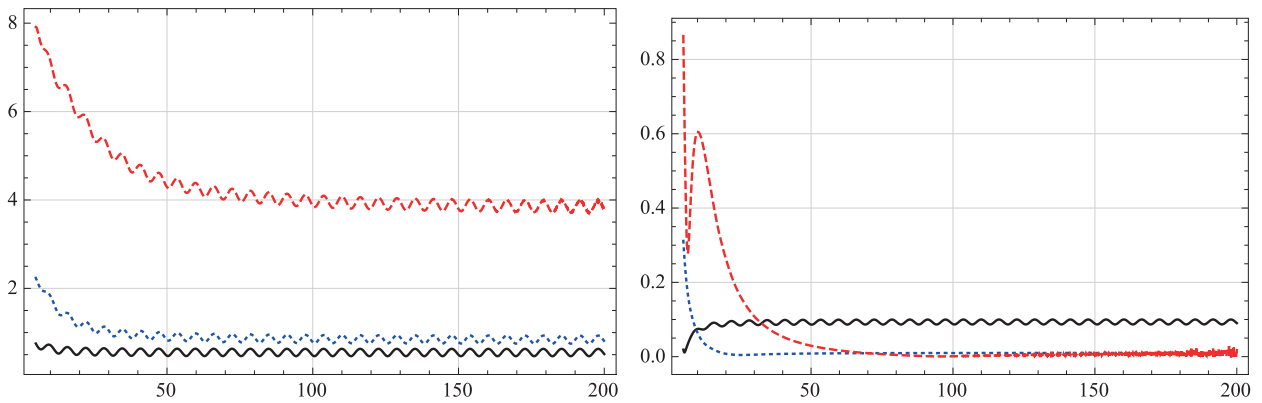


Fig. 4. Scaled absolute errors for the second integral in (3.1), obtained with  $(n_1, n_2) = (1, 2), (2, 3), (3, 4)$  (left) and with  $(n_1, n_2) = (2, 1), (3, 2), (4, 3)$  (right), when  $\omega$  runs over  $[5, 200]$ .

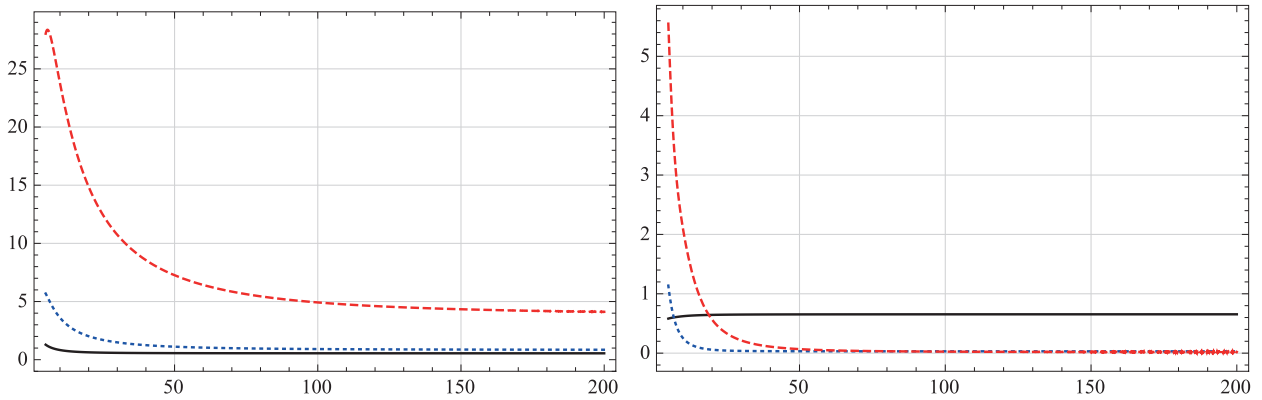


Fig. 5. Scaled absolute errors for the first integral in (3.2), obtained with  $(n_1, n_2) = (1, 3), (2, 3), (3, 4)$  (left) and with  $(n_1, n_2) = (3, 1), (3, 2), (4, 3)$  (right), when  $\omega$  runs over  $[5, 200]$ .

**Example 3.3.** Let us consider the integral

$$\int_1^{+\infty} e^{-x(x^2 + 1)} H_3^{(1)}(\omega x) dx \tag{3.3}$$

by the presented method  $\bar{Q}_{n_1, n_2}[f]$  and the method  $\bar{Q}_{n_1, n_2, H_V^{(1)}}^S[f]$  in [6] with  $n_1 = n_2 = 1, 2, 3, 4$  (Table 1, numbers in parentheses indicate decimal exponents).

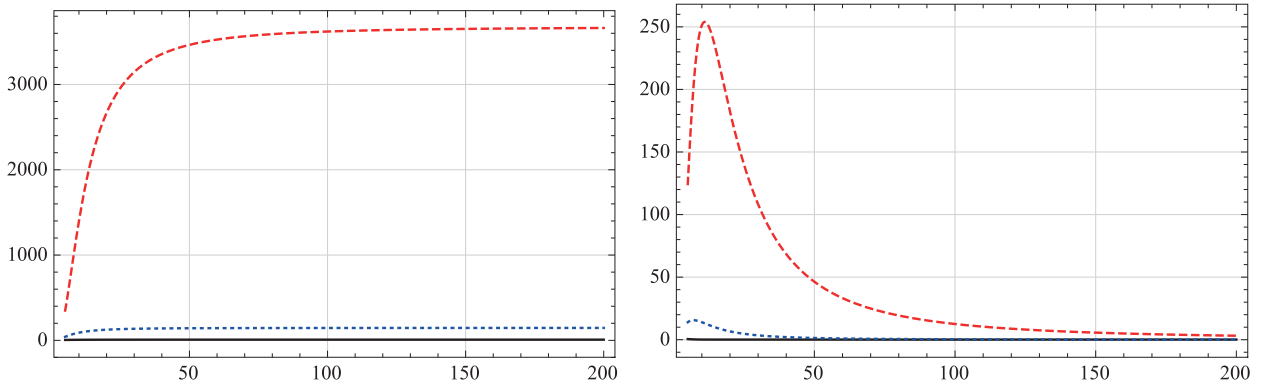


Fig. 6. Scaled absolute errors for the second integral in (3.2), obtained with  $(n_1, n_2) = (1, 2), (2, 3), (3, 4)$  (left) and with  $(n_1, n_2) = (2, 1), (3, 2), (4, 3)$  (right), when  $\omega$  runs over  $[5, 200]$ .

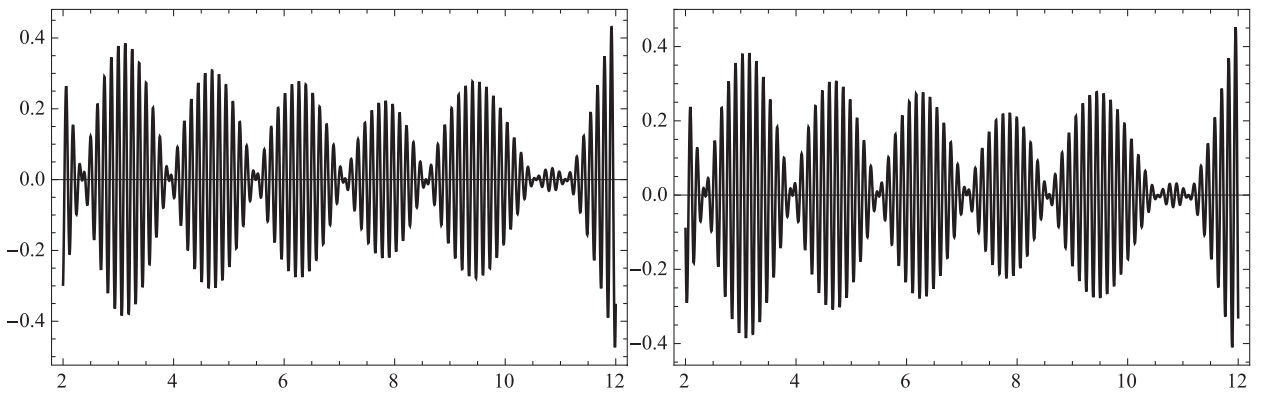


Fig. 7. Real part (left) and imaginary part (right) of the integrand in (3.4) for  $\omega = 50$ .

Table 3  
Relative errors in calculating  $I(\omega)$  for  $\omega = 50, 100, 1000$ .

n	$\omega = 50$		$\omega = 100$		$\omega = 1000$	
	(2.18)	(2.25)	(2.18)	(2.25)	(2.18)	(2.25)
1	4.13 (-4)	4.13 (-4)	4.41 (-5)	4.41 (-5)	1.16 (-6)	1.16 (-6)
2	1.87 (-7)	1.28 (-7)	6.11 (-9)	3.47 (-9)	4.78 (-13)	5.86 (-13)
3	8.99 (-11)	1.17 (-10)	6.82 (-13)	8.69 (-13)	5.39 (-19)	1.96 (-18)
4	7.18 (-14)	1.04 (-13)	3.88 (-17)	1.45 (-16)	3.09 (-24)	5.45 (-24)
5	1.46 (-16)	1.61 (-16)	5.91 (-20)	6.16 (-20)	1.72 (-29)	2.03 (-29)
6	8.19 (-19)	3.19 (-19)	8.35 (-23)	3.20 (-23)	2.89 (-34)	1.10 (-34)
7	6.87 (-21)	9.83 (-22)	1.82 (-25)	2.71 (-26)	6.33 (-39)	9.56 (-40)
8	7.59 (-23)	5.86 (-24)	5.24 (-28)	4.40 (-29)		
9	1.05 (-24)	6.63 (-26)	1.90 (-30)	1.30 (-31)		
10	1.75 (-26)	1.05 (-27)	8.42 (-33)	5.45 (-34)		
11	3.43 (-28)	2.04 (-29)	4.43 (-35)	2.85 (-36)		
12	7.76 (-30)	4.56 (-31)	2.72 (-37)	1.75 (-38)		
13	2.00 (-31)	1.16 (-32)	1.92 (-39)	1.24 (-40)		
14	5.79 (-33)	3.34 (-34)				

Example 3.4. Let us consider the integral

$$I(\omega) = \int_2^{12} \left( \frac{e^x}{1 + 100(x - 1/2)^2} + 6 \cos(2x) \right) H_3^{(1)}(\omega x) dx \tag{3.4}$$

by the presented method  $Q_{n_1, n_2}[f]$  and the method  $Q_{n_1, n_2, H_y^{(1)}}^S[f]$  in [6] with  $n_1 = n_2 = 1, 2, 3, 4$  (Table 2).

From Figs. 3–6 and Tables 1–2, we can see that the presented methods (2.18) and (2.31) are efficient for the evaluation of the integrals (1.1) and the error bounds for both methods are attainable. Particularly, Tables 1–2 also show that the methods (2.18)

and (2.31) are more efficient than the methods presented in [6]. In addition, more accurate approximations can be obtained as  $\omega$  increases for fixed  $n_1, n_2$ . For fixed  $\omega$ , adding additional quadrature nodes can also result in more accurate approximations.

In the following we also consider the efficiency of the method (2.25) in calculating the integral  $I(\omega)$  given by (3.4). For  $\omega = 50, 100, 1000$ , the exact value of this integral (rounded to 41 decimal digits) are

$$(-5.2606540619478132888162873836227788326071 + i0.8714309117651169287615634081029444029520) \times 10^{-3},$$

$$(1.6334254404141130052256106886480207660342 - i4.6946054645754247713765147952435215400185) \times 10^{-3},$$

$$(1.7173769379846547903397290523767095071656 - i4.4010850898305029428016414358209690900748) \times 10^{-5},$$

respectively. Real and imaginary parts of the integrand in (3.4) are presented in Fig. 7.

In numerical calculation of  $I(50)$  we apply the methods (2.18) and (2.25), taking  $n_1 = n_2 = n = 1, \dots, 14$ . The corresponding relative errors are given in Table 3. Numbers in parentheses indicate decimal exponents. The convergence in both methods is very fast. Precisely, it is slightly faster for the second method. In the same table we also present results for  $\omega = 100$  and  $\omega = 1000$ .

#### 4. Concluding remarks

In this paper, we present efficient methods for the integrals in (1.1). By transforming these integrals into the Fourier-type, we evaluate the integrals using complex integration method and Gauss–Laguerre quadrature rules. Error analysis for the presented methods is given, and the presented methods are uniformly convergent in  $n_1$  and  $n_2$  for fixed  $\omega$ . In addition, the method can be extended to the computation of integral  $\int_a^b f(x)(x-a)^\alpha(b-x)^\beta H_\nu^{(1)}(\omega x) dx$ ,  $\alpha, \beta > -1, b > a > 0$ . Theoretical results and numerical examples show that the methods are very efficient in obtaining very high precision approximations if  $\omega$  is sufficiently large.

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