# Summation identities for the Kummer confluent hypergeometric function $_1F_1(a; b; z)$

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**Abstract**

Motivated by the role which hypergeometric functions have in the numerical and symbolic calculation, especially in the fields of applied mathematics and mathematical physics, in this paper we derive a general summation identity for the special type of hypergeometric function, i.e., for the so-called Kummer confluent hypergeometric function $_1F_1(a; b; z)$. Special cases are also considered.
**Keywords:** Summation identities, Kummer confluent hypergeometric function, Gegenbauer polynomial, Legendre duplication formula, orthogonal polynomials

## 1. Introduction

The generalized hypergeometric function $_pF_q$ is defined by

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{\nu=0}^{\infty} \frac{(a_1)_\nu \cdots (a_p)_\nu}{(b_1)_\nu \cdots (b_q)_\nu} \frac{z^\nu}{\nu!}, $$

where the Pochhammer symbol $(\lambda)_\nu$ is given by

$$ (\lambda)_\nu = \lambda(\lambda+1)\cdots(\lambda+\nu-1) = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}, $$

and $\Gamma(\lambda)$ is familiar Euler's gamma function

$$ \Gamma(\lambda) = \int_0^\infty t^{\lambda-1} \mathrm{e}^{-t}\, \mathrm{d}t \quad \text{for} \quad \mathrm{Re}(\lambda) > 0. $$

In Wolfram's MATHEMATICA, the function $_pF_q$ is implemented as `HypergeometricPFQ` and it is suitable for both symbolic and numerical calculation (Wolfram, 2003). For $p = q + 1$, it has a branch cut discontinuity in the complex $z$ plane running from $1$ to $\infty$. When $p \le q$

this series converges for each $z \in \mathbf{C}$. For some recent results on this subject, especially on transformations, summations and other applications see (Milovanović *et al.*, 2018; Milovanović & Rathie, 2019).

In the case $p = q = 1$ the function ${}_1F_1(a; b; z)$ is known as the Kummer confluent hypergeometric function and it has the following integral representation (Olver *et al.*, 2010, p. 326)

$$
{}_1F_1(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1}\mathrm{e}^{zt}\,\mathrm{d}t, \quad \mathrm{Re}(b) > \mathrm{Re}(a) > 0. \tag{1}
$$

In this note we obtain a general summation identitity for the Kummer confluent hypergeometric function ${}_1F_1(a; b; z)$. The idea for this investigation comes from the theory of generalized Gauss-Rys quadrature formulas developed recently in (Milovanović, 2018; Milovanović & Vasović, 2022). In the next section we prove this general summation identity and consider several special cases, giving a corollary as a simplest case of the general result. Some conclusions and possible applications of the obtained results are mentioned at the end of this short note.

## 2. The main results: Summation identities

For a general sum of the form

$$
\sum_{\nu=0}^{k}(-1)^\nu \binom{k}{\nu}(\nu+\lambda+s+\varepsilon)_{k-s}\Bigl(\nu+\frac{1}{2}+\varepsilon\Bigr)_{s-1}{}_1F_1\left(\nu+s-\frac{1}{2}+\varepsilon;\nu+\lambda+s+\varepsilon;-x\right), \tag{2}
$$

where $\varepsilon$ is 0 or 1, $\lambda > -1/2$ and $s, k \in \mathbb{N}$, in this note we prove explicit expression in terms of the hypergeometric function ${}_2F_2$.

**Theorem 1** *Let $\lambda > -1/2$, $k, s \in \mathbf{N}$ and $1 \le s \le k$. Then we have*

$$
\sum_{\nu=0}^{k}(-1)^\nu \binom{k}{\nu}(\nu+\lambda+s+\varepsilon)_{k-s}\Bigl(\nu+\frac{1}{2}+\varepsilon\Bigr)_{s-1}{}_1F_1\left(\nu+s-\frac{1}{2}+\varepsilon;\nu+\lambda+s+\varepsilon;-x\right)
$$
$$
= \frac{(-1)^{s-1}k!\left(\lambda+\frac{1}{2}\right)_k x^{k-s+1}}{(k-s+1)!(\lambda+k+\varepsilon)_{k+1}}\,{}_2F_2\left(k+\frac{1+\varepsilon}{2}, k+\frac{\varepsilon}{2}+1; k-s+2, 2k+\varepsilon+\lambda+1; -x\right),
$$

*where $\varepsilon$ is equal to 0 or 1.*

*Proof.* We start from the formula for Gegenbauer polynomials (Prudnikov *et al.*, 1986, p. 529, Eq. 10)

$$
\int_0^a z^{\alpha-1}(a^2-z^2)^{\lambda-1/2}\mathrm{e}^{-xz^2}C^\lambda_{2k+\varepsilon}\Bigl(\frac{z}{a}\Bigr)\,\mathrm{d}z = \frac{(-1)^k a^{\alpha+2\lambda-1}}{2(2k+\varepsilon)!}(2\lambda)_{2k+\varepsilon}\Bigl(\frac{1+\varepsilon-\alpha}{2}\Bigr)_k
$$
$$
\times \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(\frac{\alpha+\varepsilon}{2})}{\Gamma(\frac{1+\alpha+\varepsilon}{2}+\lambda+k)}\,{}_2F_2\Bigl(\frac{\alpha}{2},\frac{\alpha+1}{2};\frac{1+\alpha-\varepsilon}{2}-k,\frac{1+\alpha+\varepsilon}{2}+\lambda+k;-a^2x\Bigr), \tag{3}
$$

which holds for $\varepsilon = 0$ or $1$, $a > 0$, $\lambda > -1/2$ and $\mathrm{Re}\,(\alpha) > -\varepsilon$.

Let $s \in \mathbb{N}$ and $1 \le s \le k$. First, we consider the right-hand side in (3) for $a = 1$. For these values of $s$ we put

$$
\alpha = \begin{cases} 2(s+\gamma)-1, & \varepsilon = 0, \\ 2(s+\gamma), & \varepsilon = 1, \end{cases} \tag{4}
$$

where $0<|\gamma|\ll 1$. Since

$$\frac{\alpha}{2}=s-\frac{1}{2}(1-\varepsilon)+\gamma,\quad \frac{\alpha+1}{2}=s+\frac{1}{2}\varepsilon+\gamma,\quad \frac{\alpha+\varepsilon}{2}=s-\frac{1}{2}+\varepsilon+\gamma,$$
$$\frac{1+\alpha-\varepsilon}{2}=s+\gamma,\quad \frac{1+\alpha+\varepsilon}{2}=s+\varepsilon+\gamma$$

and

$$\Big(\frac{1+\varepsilon-\alpha}{2}\Big)_k=(-1)^k\Big(\frac{1+\alpha-\varepsilon}{2}-k\Big)_k=(-1)^k(s-k+\gamma)_k,$$

we split now the right-hand side of (3) into two parts, replacing there the previous values for $\alpha$,

$$\begin{aligned}R_k^{(1)}(\alpha,\lambda,\varepsilon)&=\frac{(-1)^k(2\lambda)_{2k+\varepsilon}}{2(2k+\varepsilon)!}\frac{\Gamma(\lambda+\frac{1}{2})\Gamma(\frac{\alpha+\varepsilon}{2})}{\Gamma(\frac{1+\alpha+\varepsilon}{2}+\lambda+k)}\\&=\frac{(-1)^k(2\lambda)_{2k+\varepsilon}}{2(2k+\varepsilon)!}\frac{\Gamma(\lambda+\frac{1}{2})\Gamma(s-\frac{1}{2}+\varepsilon+\gamma)}{\Gamma(s+\varepsilon+\lambda+k+\gamma)}\end{aligned}$$

and

$$\begin{aligned}R_k^{(2)}(\alpha,\lambda,\varepsilon,x)&=\Big(\frac{1+\varepsilon-\alpha}{2}\Big)_k\,{}_2F_2\Big(\frac{\alpha}{2},\frac{\alpha+1}{2};\frac{1+\alpha-\varepsilon}{2}-k,\frac{1+\alpha+\varepsilon}{2}+\lambda+k;-x\Big)\\&=(-1)^k(s-k+\gamma)_k\\&\quad\times{}_2F_2\Big(s-\frac{1}{2}(1-\varepsilon)+\gamma,s+\frac{1}{2}\varepsilon+\gamma;s-k+\gamma,s+\varepsilon+\lambda+k+\gamma;-x\Big).\end{aligned}$$

We note that

$$\widehat{R}_k^{(1)}(2s-1+\varepsilon,\lambda,\varepsilon)=\lim_{\gamma\to 0}R_k^{(1)}(\alpha,\lambda,\varepsilon)=\frac{(-1)^k(2\lambda)_{2k+\varepsilon}}{2(2k+\varepsilon)!}\frac{\Gamma(\lambda+\frac{1}{2})\Gamma(s-\frac{1}{2}+\varepsilon)}{\Gamma(s+\varepsilon+\lambda+k)}.\tag{5}$$

To find the corresponding value $\widehat{R}_k^{(2)}(2s-1+\varepsilon,\lambda,\varepsilon,x)=\lim_{\gamma\to 0}R_k^{(2)}(\alpha,\lambda,\varepsilon,x)$, we consider the series

$$\begin{aligned}&(-1)^k(s-k+\gamma)_k\,{}_2F_2\Big(s-\frac{1}{2}(1-\varepsilon)+\gamma,s+\frac{1}{2}\varepsilon+\gamma;s-k+\gamma,s+\varepsilon+\lambda+k+\gamma;-x\Big)\\&\qquad=(-1)^k(s-k+\gamma)_k\sum_{\nu=0}^{\infty}\frac{\big(s-\frac{1}{2}(1-\varepsilon)+\gamma\big)_\nu\big(s+\frac{1}{2}\varepsilon+\gamma\big)_\nu}{(s-k+\gamma)_\nu(s+\varepsilon+\lambda+k+\gamma)_\nu}\cdot\frac{(-x)^\nu}{\nu!}.\end{aligned}$$

Because of

$$\lim_{\gamma\to 0}\frac{(s-k+\gamma)_k}{(s-k+\gamma)_\nu}=\begin{cases}0, & \text{for } 0\le\nu\le k-s,\\[4pt] \dfrac{(s-1)!}{(s-k+\nu-1)!}, & \text{for } k-s+1\le\nu\le k,\end{cases}$$

we conclude that the first $k-s+1$ terms (for $\nu=0,1,\dots,k-s$) of this series, in the limit case, vanish, so that we consider the corresponding series with terms starting from the index $\nu=m=k-s+1$, i.e.,

$$\begin{aligned}&(-1)^k(s-k+\gamma)_k\sum_{\nu=k-s+1}^{\infty}\frac{\big(s-\frac{1}{2}(1-\varepsilon)+\gamma\big)_\nu\big(s+\frac{1}{2}\varepsilon+\gamma\big)_\nu}{(s-k+\gamma)_\nu(s+\varepsilon+\lambda+k+\gamma)_\nu}\cdot\frac{(-x)^\nu}{\nu!}\\&\quad=(-1)^k(s-k+\gamma)_k\sum_{\nu=0}^{\infty}\frac{\big(s-\frac{1}{2}(1-\varepsilon)+\gamma\big)_{\nu+m}\big(s+\frac{1}{2}\varepsilon+\gamma\big)_{\nu+m}}{(s-k+\gamma)_{\nu+m}(s+\varepsilon+\lambda+k+\gamma)_{\nu+m}}\cdot\frac{(-x)^{\nu+m}}{(\nu+m)!}.\end{aligned}$$

Using the elementary identity $(p)_{\nu+m} = (p+m)_\nu (p)_m$ and letting $\gamma \to 0$, we get

$$\begin{aligned}
&\widehat{R}_k^{(2)}(2s-1+\varepsilon, \lambda, \varepsilon, x) \\
&\quad = (-1)^{s-1} \frac{(s-1)!\left(s-\frac{1-\varepsilon}{2}\right)_m \left(s+\frac{\varepsilon}{2}\right)_m}{(s+\varepsilon+\lambda+k)_m} x^m \sum_{\nu=0}^{\infty} \frac{\left(k+\frac{1+\varepsilon}{2}\right)_\nu \left(k+1+\frac{\varepsilon}{2}\right)_\nu}{(2k+\lambda+1+\varepsilon)_\nu (\nu+m)!} \cdot \frac{(-x)^\nu}{\nu!} \\
&\quad = (-1)^{s-1} \frac{(s-1)!\left(s-\frac{1-\varepsilon}{2}\right)_{k-s+1} \left(s+\frac{\varepsilon}{2}\right)_{k-s+1}}{(s+\varepsilon+\lambda+k)_{k-s+1} (k-s+1)!} x^{k-s+1} \\
&\qquad\qquad \times {}_2F_2\left(k+\frac{\varepsilon}{2}+\frac{1}{2}, k+\frac{\varepsilon}{2}+1; k-s+2, 2k+\varepsilon+\lambda+1; -x\right).
\end{aligned}$$

Now, we obtain the right-hand side of (3), under the assumed conditions, as a product of $\widehat{R}_k^{(1)}(2s-1+\varepsilon, \lambda, \varepsilon)$ and $\widehat{R}_k^{(2)}(2s-1+\varepsilon, \lambda, \varepsilon, x)$:

$$\begin{aligned}
RHS = &\frac{(-1)^{k+s-1}(2\lambda)_{2k+\varepsilon}}{2(2k+\varepsilon)!} \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(s-\frac{1}{2}+\varepsilon)(s-1)!\left(s-\frac{1-\varepsilon}{2}\right)_{k-s+1}\left(s+\frac{\varepsilon}{2}\right)_{k-s+1}}{\Gamma(2k+\varepsilon+\lambda+1)(k-s+1)!} \\
&\times x^{k-s+1} {}_2F_2\left(k+\frac{\varepsilon}{2}+\frac{1}{2}, k+\frac{\varepsilon}{2}+1; k-s+2, 2k+\varepsilon+\lambda+1; -x\right). \qquad (6)
\end{aligned}$$

On the other side, for $a = 1$ and $z = \sqrt{t}$, the integral in (3) reduces to

$$\frac{1}{2}\int_0^1 t^{\alpha/2-1}(1-t)^{\lambda-1/2} \mathrm{e}^{-xt} C_{2k+\varepsilon}^{\lambda}(\sqrt{t})\, \mathrm{d}t,$$

which, after using the polynomial representation of the Gegenbauer polynomials

$$C_n^\lambda(x) = \sum_{\nu=0}^{[n/2]} \frac{(-1)^\nu (\lambda)_{n-\nu}}{\nu!(n-2\nu)!}(2x)^{n-2\nu},$$

and taking $\alpha = 2s-1+\varepsilon$ (as before in (4), with $\gamma = 0$), we get the following expression for the left-hand side of (3)

$$\begin{aligned}
LHS = &\frac{1}{2}\sum_{\nu=0}^{k} \frac{(-1)^\nu (\lambda)_{2k-\nu+\varepsilon} 2^{2(k-\nu)+\varepsilon}}{\nu!(2k-2\nu+\varepsilon)!} \int_0^1 t^{s-\frac{3}{2}+\varepsilon+k-\nu}(1-t)^{\lambda-1/2}\mathrm{e}^{-xt}\, \mathrm{d}t \\
= &\frac{(-1)^k}{2}\sum_{\nu=0}^{k} \frac{(-1)^\nu (\lambda)_{k+\nu+\varepsilon} 2^{2\nu+\varepsilon}}{(k-\nu)!(2\nu+\varepsilon)!} \int_0^1 t^{s-\frac{3}{2}+\varepsilon+\nu}(1-t)^{\lambda-1/2}\mathrm{e}^{-xt}\, \mathrm{d}t.
\end{aligned}$$

However, it can be expressed in terms of the Kummer confluent hypergeometric function (1) by taking $a = s-\frac{1}{2}+\varepsilon+\nu$, $b = s+\nu+\lambda+\varepsilon$, $z = -x$, so that

$$\begin{aligned}
LHS = &\frac{(-1)^k}{2}\sum_{\nu=0}^{k} \frac{(-1)^\nu (\lambda)_{k+\nu+\varepsilon} 2^{2\nu+\varepsilon}}{(k-\nu)!(2\nu+\varepsilon)!} \frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\nu+s-\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\nu+\lambda+s+\varepsilon\right)} \\
&\times {}_1F_1\left(\nu+s-\frac{1}{2}+\varepsilon; \nu+\lambda+s+\varepsilon; -x\right),
\end{aligned}$$

i.e.,

$$LHS = \frac{(-1)^k}{2}\frac{\sqrt{\pi}\,\Gamma\left(\lambda+\frac{1}{2}\right)}{k!\Gamma(\lambda)}\sum_{\nu=0}^{k}(-1)^\nu\binom{k}{\nu}(\nu+\lambda+s+\varepsilon)_{k-s}\Big(\nu+\frac{1}{2}+\varepsilon\Big)_{s-1}$$
$$\times\,{}_1F_1\left(\nu+s-\frac{1}{2}+\varepsilon;\nu+\lambda+s+\varepsilon;-x\right), \tag{7}$$

Finally, comparing (7) and (6) we obtain

$$\sum_{\nu=0}^{k}(-1)^\nu\binom{k}{\nu}(\nu+\lambda+s+\varepsilon)_{k-s}\Big(\nu+\frac{1}{2}+\varepsilon\Big)_{s-1}{}_1F_1\left(\nu+s-\frac{1}{2}+\varepsilon;\nu+\lambda+s+\varepsilon;-x\right)$$
$$=\frac{(-1)^k 2k!\Gamma(\lambda)}{\sqrt{\pi}\,\Gamma\left(\lambda+\frac{1}{2}\right)}\frac{(-1)^{k+s-1}(2\lambda)_{2k+\varepsilon}\Gamma(\lambda+\frac{1}{2})\Gamma(s-\frac{1}{2}+\varepsilon)\Gamma(s)\big(s-\frac{1-\varepsilon}{2}\big)_{k-s+1}\big(s+\frac{\varepsilon}{2}\big)_{k-s+1}}{2\Gamma(2k+\varepsilon+1)\Gamma(2k+\varepsilon+\lambda+1)(k-s+1)!}$$
$$\times\; x^{k-s+1}{}_2F_2\left(k+\frac{\varepsilon}{2}+\frac{1}{2},k+\frac{\varepsilon}{2}+1;k-s+2,2k+\varepsilon+\lambda+1;-x\right)$$
$$=\frac{(-1)^{s-1}k!\left(\lambda+\frac{1}{2}\right)_k x^{k-s+1}}{(k-s+1)!(\lambda+k+\varepsilon)_{k+1}}\,{}_2F_2\left(k+\frac{\varepsilon}{2}+\frac{1}{2},k+\frac{\varepsilon}{2}+1;k-s+2,2k+\varepsilon+\lambda+1;-x\right),$$

which had to be proved.

Theorem 1 for $\varepsilon=0$ and $\varepsilon=1$ gives the following sums:

**Corollary 1** *For $\lambda>-1/2$, $k,s\in\mathbf{N}$ and $1\le s\le k$, we have*

$$\sum_{\nu=0}^{k}(-1)^\nu\binom{k}{\nu}(\nu+\lambda+s)_{k-s}\Big(\nu+\frac{1}{2}\Big)_{s-1}{}_1F_1\left(\nu+s-\frac{1}{2};\nu+\lambda+s;-x\right)$$
$$=\frac{(-1)^{s-1}k!\left(\lambda+\frac{1}{2}\right)_k x^{k-s+1}}{(k-s+1)!(\lambda+k)_{k+1}}\,{}_2F_2\left(k+\frac{1}{2},k+1;k-s+2,2k+\lambda+1;-x\right) \tag{8}$$

*and*

$$\sum_{\nu=0}^{k}(-1)^\nu\binom{k}{\nu}(\nu+\lambda+s+1)_{k-s}\Big(\nu+\frac{3}{2}\Big)_{s-1}{}_1F_1\left(\nu+s+\frac{1}{2};\nu+\lambda+s+1;-x\right)$$
$$=\frac{(-1)^{s-1}k!\left(\lambda+\frac{1}{2}\right)_k x^{k-s+1}}{(k-s+1)!(\lambda+k+1)_{k+1}}\,{}_2F_2\left(k+1,k+\frac{3}{2};k-s+2,2k+\lambda+2;-x\right). \tag{9}$$

In the case $s=1$ the function ${}_2F_2$ reduces to ${}_1F_1$, because $k-s+2=k+1$, so that Theorem 1 gives the following result:

**Corollary 2** *Let $\lambda>-1/2$ and $k\in\mathbf{N}$. Then we have*

$$\sum_{\nu=0}^{k}(-1)^\nu\binom{k}{\nu}(\nu+\lambda+1)_{k-1}\,{}_1F_1\Big(\nu+\frac{1}{2};\nu+\lambda+1;-x\Big)$$
$$=\frac{(\lambda+\frac{1}{2})_k}{(k+\lambda)_{k+1}}x^k\,{}_1F_1\Big(k+\frac{1}{2};2k+\lambda+1;-x\Big)$$

*and*

$$\sum_{\nu=0}^{k}(-1)^{\nu}\binom{k}{\nu}(\nu+\lambda+2)_{k-1}\,{}_1F_1\Big(\nu+\frac{3}{2};\nu+\lambda+2;-x\Big)$$
$$=\frac{(\lambda+\frac{1}{2})_k}{(k+\lambda+1)_{k+1}}x^k\,{}_1F_1\Big(k+\frac{3}{2};2k+\lambda+2;-x\Big).$$

The sums from Corollary 1 are displayed in Figure 1 as function in $x$, in the case $k=4$ and $\lambda=1/2$ and $s=1,2,3,4$.

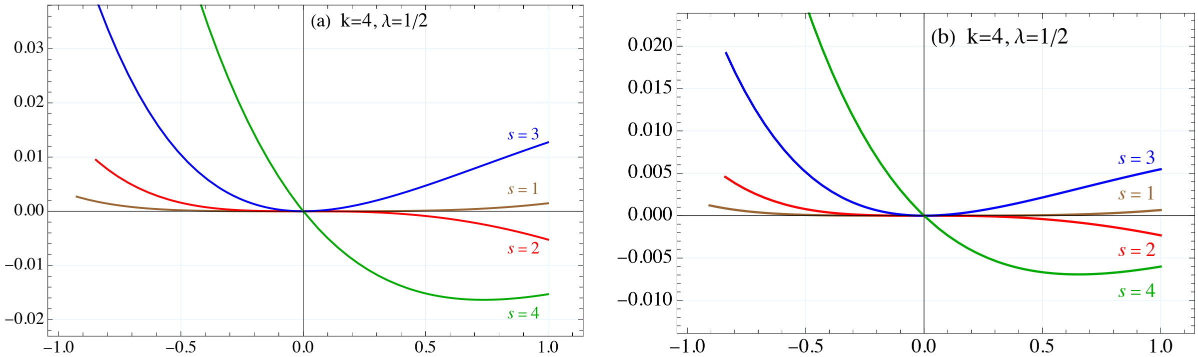


**Fig. 1.** The sums (8) (left) and (9) (right) for $x\in[-1,1]$, when $\lambda=1/2$, $k=4$ and $s=1,2,3,4$.

**Remark 1** The case $s>k$ is obvious. The general sum (2) should be written in the form

$$\sum_{\nu=0}^{k}(-1)^{\nu}\binom{k}{\nu}\frac{\left(\nu+\frac{1}{2}+\varepsilon\right)_{s-1}}{(\nu+\lambda+k+\varepsilon)_{s-k}}\,{}_1F_1\left(\nu+s-\frac{1}{2}+\varepsilon;\nu+\lambda+s+\varepsilon;-x\right),$$

$\widehat{R}_k^{(1)}(2s-1+\varepsilon,\lambda,\varepsilon)$ is given by (5), and $\widehat{R}_k^{(2)}(2s-1+\varepsilon,\lambda,\varepsilon,x)$ is given directly by

$$(-1)^k(s-k)_k\,{}_2F_2\Big(s-\frac{1}{2}(1-\varepsilon),s+\frac{1}{2}\varepsilon;s-k,s+\varepsilon+\lambda+k;-x\Big).$$

## 3. Conclusion

In this paper, new summation identities for the Kummer confluent hypergeometric function have been obtained. As possible applications of given summation identities, we mention applications in representing results from the theory of orthogonal polynomials, theory of special functions, integral equations, etc. (Mastroianni & Milovanović, 2008; Asanov *et al.*, 2017; Mastroianni & Milovanović, 2009).

**ACKNOWLEDGEMENTS**. The authors are deeply grateful to the anonymous referees and the editor for their valuable comments and constructive suggestions for improvements of this paper. The work of the first author was supported in part by the Serbian Academy of Sciences and Arts (Φ-96).

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