Mathematica
Balkanica

New Series Vol. 18, 2004, Fasc. 3-4

On Generalized Stirling Numbers and Polynomials

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Presented at Internat. Congress "MASSEE' 2003", 4th Symposium "TMSF"

In this paper we prove that some results concerned the generalized Stirling numbers, are consequence of the results of Toscano and Chak. New explicit expressions for generalized Stirling numbers are also given.

AMS Subj. Classification: 05A15, 05A19, 05A10

Key Words: Stirling numbers, Stirling polynomials

1. Introduction

The Stirling numbers of the second kind and the corresponding polynomials, the so called single variable Bell polynomials (see [7],[12]), are defined by

$$S(n,k)=\frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n=\frac{(-1)^k}{k!}\Delta^k 0^n \tag{1}$$

and

$$A_n(x)=\sum_{k=0}^{n}S(n,k)x^k, \tag{2}$$

respectively.

Singh ([14]), Sinha and Dhawan ([17]), and Shrivastava ([13]), studied the generalized Stirling numbers and polynomials defined, respectively, by

$$S^{(\alpha)}(n,k,r)=\frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\alpha+rj)^n \tag{3}$$

and

$$T_n^{(\alpha)}(x,r,-p)=\sum_{k=0}^{n} S^{(\alpha)}(n,k,r)p^k x^{rk}. \tag{4}$$

It is easily verified from (4) that

$$T_n^{(\alpha)}(x,r,p)=x^{-\alpha}e^{px^r}(xD)^n x^{\alpha}e^{-px^r},$$

i.e. the generalized Truesdell polynomials ([19]).

Singh's generalization was motivated by the generalization of Hermite polynomials of Gould-Hopper ([8]) given by

$$H_n^{(r)}(x,\alpha,p)=(-1)^n x^{-\alpha}e^{px^r}D^n(x^{\alpha}e^{-px^r}).$$

Singh Chandel introduced the following generalizations of the Stirling numbers and polynomials (see [15]):

$$S^{(\alpha,\lambda)}(n,k,r)=\frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\alpha+rj)^{(\lambda-1,n)}, \tag{5}$$

and

$$T_n^{(\alpha,\lambda)}(x,r,-p)=x^{n(\lambda-1)}\sum_{k=o}^{n} S^{(\alpha,\lambda)}(n,k,r)p^k x^{rk}, \tag{6}$$

where

$$a^{(\lambda-1,n)}=\left(\frac{a}{\lambda-1}\right)_n(\lambda-1)^n.$$

Evidently, when $\lambda\to 1$, equations (5) and (6) reduce to (3) and (4) respectively, which in turns, yields (1) and (2), respectively for $r-1=\alpha=0$.

In this paper we prove that the generalizations (3) and (4) and related properties are an old result, published in 1887 by d'Ocagne [11]. Also, we prove that Singh Chandel's result is the consequence of the fundamental results of Toscano ([19]) and Chak ([6]).

New explicit expressions for numbers (3) and (5) are also given.

2. On a result of R. P. Singh

One of the first generalization of the Stirling numbers of the second kind (1) is given by d'Ocagne (see [11]):

$$S^{(\alpha)}(n,k)=\frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\alpha+j)^n \tag{7}$$

with the recurrence relation

$$S^{(\alpha)}(n,k) = S^{(\alpha)}(n-1,k-1) + (k+\alpha)S^{(\alpha)}(n-1,k). \tag{8}$$

Toscano introduced similar numbers and related polynomials in many papers (see for example [18] and [19] for references).

For the numbers defined by (7), we have the generating function, recurrence relations (see [19]):

$$\sum_{n=0}^{\infty} S^{(\alpha)}(n,k)\frac{t^n}{n!} = \frac{(-1)^k}{k!}e^{\alpha t}(1-e^t)^k, \tag{9}$$

$$S^{(\alpha)}(n+1,k) - \alpha S^{(\alpha)}(n,k) = \sum_{i=0}^{n}\binom{n}{i}S^{(\alpha)}(i,k-1), \tag{10}$$

$$(\alpha+j)^n = \sum_{i=0}^{n}\binom{j}{i}i!S^{(a)}(n,k), \tag{11}$$

and related polynomials

$$A_n(x,\alpha) = \sum_{k=0}^{n} S^{(\alpha)}(n,k)x^k$$

with generating function (see [6]):

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}A_n(x,\alpha) = e^{\alpha t - x(1-e^t)}. \tag{12}$$

Starting with the recurrence relation for numbers $S^{(\alpha)}(n,k)$ (8), and using substitution $\alpha \to \frac{\alpha}{r}$, we get

$$r^n S^{(\frac{\alpha}{r})}(n,k) = rr^{n-1}S^{(\frac{\alpha}{r})}(n-1,k-1) + (\alpha+rk)r^{n-1}S^{(\frac{\alpha}{r})}(n-1,k).$$

We see that

$$S^{(\alpha)}(n,k,r) = r^n S^{(\frac{\alpha}{r})}(n,k), \tag{13}$$

because the recurrence relation for numbers $S^{(\alpha)}(n,k,r)$ is

$$S^{(\alpha)}(n,k,r) = rS^{(\alpha)}(n-1,k-1,r) + (\alpha+rk)S^{(\alpha)}(n-1,k,r).$$

In the same way from (10) and (11), we have other recurrences which are proved by Sinha and Dhawan [17] and Shrivastava [13].

Now, we consider the generating function (9). Using substitution $\alpha \to \frac{\alpha}{r}$ and relation (13), we get

$$\sum_{n=0}^{\infty} S^{(\alpha)}(n,k,r)\frac{t^n}{n!} = \sum_{n=0}^{\infty} S^{(\frac{\alpha}{r})}(n,k)\frac{(rt)^n}{n!} = \frac{(-1)^k}{k!}e^{\alpha t}(1-e^{rt})^k,$$

i.e the result from [17] and [13].

For the polynomials $T_n^{(\alpha)}(x,r,-p)$, using substitution $x \to px^r, \alpha \to \frac{\alpha}{r}$ and $t \to rt$, we have from (12)

$$\sum_{n=0}^{\infty} \frac{(rt)^n}{n!} A_n(px^r, \frac{\alpha}{r}) = e^{\alpha t - px^r(1-e^{rt})}$$

which is the result for the generalized polynomials $T_n^{(\alpha)}(x,r,-p)$ from [17] and [13].

Remark 1. From (3) we get directly relation (13), because $(\alpha+rj)^n = r^n(\frac{\alpha}{r}+j)^n$.

Remark 2. The fact $T_n^{(\alpha)}(x,r,-p) = r^n A_n(px^r, \frac{\alpha}{r})$ is also found by Toscano in [19], in a different way, using differential operators.

3. On a result of R. C. Singh Chandel

The main results from the paper [15] are consequence of the results from A.M. Chak [6].

Starting with the following notation by Singh Chandel

$$\alpha^{(k-1.n)} = \left(\frac{\alpha}{k-1}\right)_n (k-1)^n = \alpha(\alpha+k-1)\cdots(\alpha+(k-1)(n-1)),$$

we see that

$$(\alpha+rj)^{(k-1,n)} = r^n\left(\frac{\alpha}{r}+j\right)^{(\frac{k-1}{r},n)},$$

and so, from (5) we have

$$S^{(\alpha,\lambda)}(n,k,r) = \frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}r^n(\frac{\alpha}{r}+j)^{(\frac{\lambda-1}{r},n)} \quad = \quad r^n A_{n,k}^{(\frac{\alpha}{r},\frac{\lambda}{r})},$$

where $A_{n.k}^{(\alpha,\lambda)}$ are numbers defined by A.M. Chak [6] in the following way

$$(x^\lambda D)^n = x^{(n-1)\lambda}\sum_{i=0}^{n} A_{n,i}^{(\alpha,\lambda)} x^{i+\alpha} D^i x^{-\alpha}.$$

Thus, using the equations proved and the recurrence from [6]

$$A_{n+1,k}^{(\alpha,\lambda)} = A_{n,k-1}^{(\alpha,\lambda)} + (n\lambda - n + \alpha + k) A_{n,k}^{(\alpha,\lambda)},$$

we get a recurrence from [15]

$$S^{(\alpha,\lambda)}(n+1,k,r) = rS^{(\alpha,\lambda)}(n,k-1,r) + (n\lambda - n + \alpha + rk) S^{(\alpha,\lambda)}(n,k,r).$$

For polynomials treated in [15] we get from (6) easily

$$T_n^{(\alpha,\lambda)}(x,r,-p) = r^n x^{n(\lambda-1)} G_{n,\frac{\lambda}{r}}^{(\frac{\alpha}{r})}(-px^r),$$

where $G_{n,\lambda}^{(\alpha)}(x)$ are a class of polynomials from [6].

4. Explicit expressions for $S^{(\alpha)}(n,k,r)$ and $S^{(\alpha,\lambda)}(n,k,r)$

In this section we use the results from Cakić [1] (see Theorem 4, and comment by L. Carlitz) and Milovanović and Cakić [10] (see Theorem 3).

Applying Carlitz's [2] extension of a result of Wang [20], Singh Chandel and Dwivedi [16] proved the following explicit formula for $S^{(0,\lambda)}(n,k,r)$:

$$S^{(0,\lambda)}(n,k,r) = \frac{n!}{k!} \sum_{i_1+\cdots+i_k=n} \prod_{l=1}^{k} \frac{r^{(\lambda-1,i_l)}}{i_l!},$$

where $i_s > 0$.

In this section we prove the theorem about new explicit expressions for numbers $S^{(\alpha)}(n,k,r)$ and $S^{(\alpha,\lambda)}(n,k,r)$.

Theorem. *We have*

$$S^{(\alpha)}(n,k,r) = r^n \sum \binom{\alpha/r}{i_0} \prod_{s=1}^{n-1} \binom{\alpha/r + s - S_s}{i_s}, \tag{14}$$

and

$$S^{(\alpha,\lambda)}(n,k,r) = r^n \sum \binom{\alpha/r}{i_0} \prod_{s=1}^{n-1} \binom{\alpha/r + s(\frac{\lambda-1}{r}+1) - S_s}{i_s} \tag{15}$$

where $S_s = \sum_{t=0}^{s-1} i_t$. The summation on the right in the above sums is over all $i_0, \ldots, i_{n-1} \in \{0,1\}$ such that $i_0 + i_1 + \cdots + i_{n-1} = n - k$.

P r o o f. Introduce the notations

$$A(i_0,\ldots,i_{n-2}) = \binom{\alpha/r}{i_0} \prod_{s=1}^{n-2} \binom{\alpha/r + s - S_s}{i_s},$$

and

$$B(i_0,\ldots,i_{n-2})=\binom{\alpha/r}{i_0}\prod_{s=1}^{n-2}\binom{\alpha/r+s(\frac{\lambda-1}{r}+1)-S_s}{i_s}.$$

For $i_{n-1}\in\{0,1\}$ we get from (14) and (15), respectively:

$$S^{(\alpha)}(n,k,r)=r^n\sum_{S_{n-1}=(n-1)-(k-1)}A(i_0,\ldots,i_{n-2})+$$
$$r^n(\alpha/r+n-1-(n-k-1))\sum_{S_{n-1}=(n-1)-k}A(i_0,\ldots,i_{n-2}),$$

and

$$S^{(\alpha,\lambda)}(n,k,r)=r^n\sum_{S_{n-1}=(n-1)-(k-1)}B(i_0,\ldots,i_{n-2})+$$
$$r^n(\alpha/r+(n-1)((\lambda-1)/r+k))\sum_{S_{n-1}=(n-1)-k}B(i_0,\ldots,i_{n-2}),$$

i.e.

$$S^{(\alpha)}(n,k,r)=rS^{(\alpha)}(n-1,k-1,r)+(\alpha+rk)S^{(\alpha)}(n-1,k,r)$$

and

$$S^{(\alpha,\lambda)}(n,k,r)=$$
$$rS^{(\alpha,\lambda)}(n-1,k-1,r)+((n-1)(\lambda-1)+\alpha+rk)S^{(\alpha,\lambda)}(n-1,k,r).$$

For $k=0$ and $k=n$, we have

$$S^{(\alpha)}(n,0,r)=\alpha^n,\ S^{(\alpha)}(n,n,r)=r^n$$

and

$$S^{(\alpha,\lambda)}(n,0,r)=r^n\prod_{s=0}^{n-1}(\frac{\alpha}{r}+s\frac{\lambda-1}{r})=\alpha^{(\lambda-1,n)},$$

$$S^{(\alpha,\lambda)}(n,n,r)=r^n.$$

This completes the proof. ∎

Remark 3. In above theorem we use the convention: if $\alpha=0$, then $i_0=0$.

Notice that for bigger values of $k\,(k > [n/2])$ this formulas are simpler than classical expressions (3) and (5). For example, if we take $k = n-1$, then from (3) and (5) we find, respectively

$$S^{(\alpha)}(n, n-1, r) = \frac{(-1)^{n-1}}{(n-1)!}\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(\alpha + rj)^n,$$

$$S^{(\alpha,\lambda)}(n, n-1, r) = \frac{(-1)^{n-1}}{(n-1)!}\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(\alpha + rj)^{(\lambda-1,n)},$$

while representations (14) and (15) give, respectively

$$S^{(\alpha)}(n, n-1, r) = n\alpha r^{n-1} + \binom{n}{2}r^n,$$

$$S^{(\alpha,\lambda)}(n, n-1, r) = n\alpha r^{n-1} + \binom{n}{2}((\lambda-1)/r + 1)r^n.$$

5. Special cases

Some important special cases of numbers $S^{(\alpha)}(n, k, r)$ and $S^{(\alpha,\lambda)}(n, k, r)$ are: "Weighted Stirling numbers of the second kind" ([4], [5]): $R(n, k, \lambda) = S^{(\lambda)}(n, k, 1) = S^{(\lambda,1)}(n, k, 1)$, " Degenerate Stirling numbers of the second kind " ([3]): $S(n, k|\theta) = S^{(0,1-\theta)}(n, k, 1)$, " Degenerate weighted Stirling numbers of the second kind " ([9]): $S(n, k, \lambda|\theta) = S^{(\lambda,1-\theta)}(n, k, 1)$.

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Received: 30.09.2003

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