G. V. Milovanovic[1]

INTEGRAL EQUATIONS OF LOVE'S TYPE AND APPLICATIONS

In this paper we consider some methods for solving Fredholm integral equations of the second kind

$$f(x)+\frac{s}{\pi}\int_{-1}^{1}\frac{d}{d^2+(x-y)^2}f(y)\mathrm{d}y=g(x),\quad -1\le x\le 1,$$

where $f(x)$ is the unknown function, $k(x,y)=k(x-y)=d/(d^2+(x-y)^2)$ is the so-called difference kernel, with $d>0$, $s=\pm 1$, and $g(x)$ is a given function. This quasi-singular kernel has two complex conjugate poles $x\pm \mathrm{i}d$, which approach to the real axis when $d\to 0+$. There are many methods in the literature for this kind of equations, which are known as integral equations of Love's type. The simplest case with $g(x)=1$ is appeared in an electrostatic problem analysed first time by Eric Russell Love [Quart. J. Mech. Appl. Math. **2** *(1949), 428-451]. Beside numerical solutions, we propose also fast approximate analytic solutions of this type of equations and give applications in an electrostatic problem with a coaxial symmetry.*

INTRODUCTION

In 1949 Eric Russell Love (1912-2001) described the electrostatic potential in space, generated by a condenser consisting of two parallel equal circular plates of the radius R separated by a distance h (see Fig. 1.1). Taking a normalization so that $h=Rd$, it can be considered with dimensionless variables as two unit disks, where d is a distance between them.

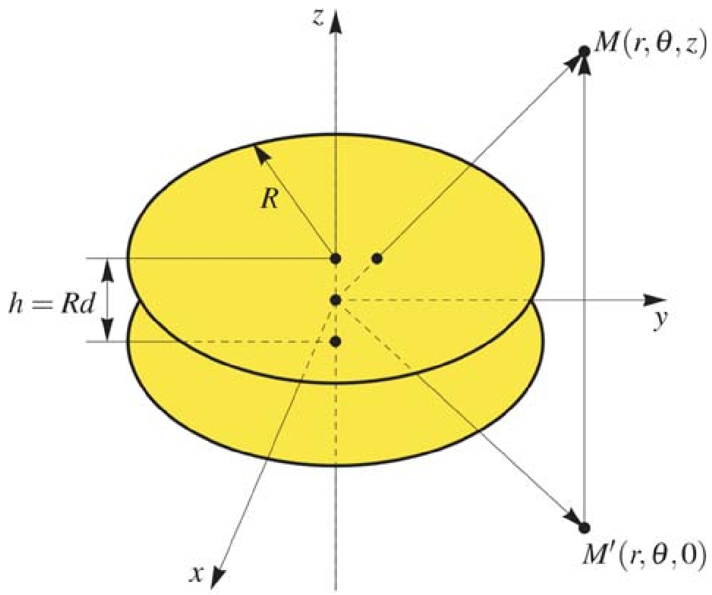

Fig. 1.1. Electrostatical system od two parallel equal circular plates

Supposing the equal and opposite potentials at these disks, e.g., the upper at $V=+1$ and the lower one at $V=-1$, and the potential at infinity being taken as zero, E. Love in [1] (Theorem 1) used a coaxial symmetry of this electrostatical system and proved that the potential in an arbitrary point $M(r,\theta,z)$ in the space $\mathbb{R}^3$, outside the circular plates, can be expressed in the form

$$V(r,z)=\frac{1}{\pi}\int_{-1}^{1}\left\{\frac{1}{\sqrt{r^2+\left(z-\frac{1}{2}d+\mathrm{i}x\right)^2}}-\frac{1}{\sqrt{r^2+\left(z+\frac{1}{2}d+\mathrm{i}x\right)^2}}\right\}f(x)\mathrm{d}x,$$

1 Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia

where both square roots in the previous integral have positive real part and $f(x)$ is the unique solution of the following Fredholm integral equation of the second kind

$$f(x)-\frac{1}{\pi}\int_{-1}^{1}\frac{d}{d^2+(x-y)^2}f(y)\mathrm{d}y=1, \quad -1\le x\le 1. \tag{1.1}$$

He also proved that there exists a unique, continuous, real and even solution $f(x)$ of this integral equation on the closed interval $[-1,1]$. We call this equation as *Love's first integral equation*.

For the corresponding equation with the sign + , i.e. when $s=1$ in

$$f(x)+\frac{s}{\pi}\int_{-1}^{1}\frac{d}{d^2+(x-y)^2}f(y)\mathrm{d}y=1, \quad -1\le x\le 1, \tag{1.2}$$

we say that it is *Love's second integral equation*. In the case when the potentials of the plates in Fig. 1.1 are equal in magnitude and sign (suppose both are positive), then for the potential $V(r,z)$ in an arbitrary point $M(r,\theta,z)$ in $\mathbb{R}^3$, outside the circular plates, a similar formula holds, where only the sign - between two terms on the right-hand side should be replaced by the + sign.

An approximative analytic solution of (1.1), in the case $d=1$, was given by Love [2],

$$f(x)\approx f_L(x)=1.919200-0.311717x^2+0.015676x^4+0.019682x^6-0.000373x^8.$$

Recently, Norgren & Jonsson [3] have calculated the capacitance of the circular parallel plate capacitor by expanding the solution of Love's integral equation (1.1) into a Fourier cosine series. For some other approaches see [4]-[7]. In 2010, A.S. Kumar [8] presented a method for finding an analytical solution of Love's integral equation, based on the so-called Boubaker polynomials expansion scheme (BPES) [9]. However, in his approach a sirious mistake has appeared.

In this paper we give an account on some very efficient methods of Nyström type for numerical solving the general Fredholm integral equations of the second kind (FK2), based on recent progress in the weighted polynomial interpolation (see Mastroianni & Milovanović [10] and [11]), as well as a method for getting approximations of Love's equations in an analytic form (Milovanović & Joksimović [12]). The paper is organized as follows. In Section 2 we give some preliminaries and basic facts on this class of integral equations. Sections 3 and 4 are devoted to the previous mentioned methods, and finally in Section 5 some numerical examples are presented.

PRELIMINARIES AND BASIC FACTS

Integral equations appear in many fields including continuum and quantum mechanics, kinetic theory of gases, optimization and optimal control systems, communication theory, potential theory, geophysics, electricity and magnetism, biology and population genetics, mathematical economics, queueing theory, etc. Most of the boundary value problems involving differential equations can be converted to integral equations. There are also some problems that can be expressed only in terms of integral equations. Here, we are interested only in Fredholm integral equations of the second kind on a finite interval $[-1,1]$,

$$f(x)+\mu\int_{-1}^{1}k(x,y)f(y)w(y)\mathrm{d}y=g(x), \quad -1\le x\le 1, \tag{2.1}$$

where $k(x,y)$ is the kernel, $w(y)$ is a given weight function, $g(x)$ is a known function, μ is a real parameter, and $f(x)$ is a unknown function. In the case of Love's equations (1.2), $k(x,y)$ is the so-called *difference* kernel

$$k(x,y)=k(x-y)=\frac{1}{\pi}\cdot\frac{d}{d^2+(x-y)^2}, \quad d>0,$$

which has two complex conjugate poles $x\pm \mathrm{i}d$. As we can see these poles approach the real axis when $d\to 0^+$, and therefore the kernel is quasi-singular. Letting

$$(Kf)(x)=\frac{1}{\pi}\int_{-1}^{1}\frac{d}{d^2+(x-y)^2}f(y)\mathrm{d}y,$$

the operator form of Love's integral equations (1.2) is $f\mp Kf=(I\mp K)f=g$ where I denotes the identity operator, and K is compact with

$$\|K\|_\infty=\frac{2}{\pi}\tan^{-1}\frac{1}{d}<1.$$

It is clear that $\|K\|_\infty$ tends to one when $d\to 0^+$.

NYSTRÖM INTERPOLANTS FOR FK2

There are several computational approaches to the solution of general Fredholm equations of the second kind, e.g., classical methods, projection-variation and Nyström methods, iteration methods, etc. (cf. Atkinson [13], Kythe & Puri [14]). Sometimes, these methods are developed for specific type of kernels, so that for singular equations there are some special methods (cf. Prössdorf & Silbermann [15]). The most of methods lead to a system of equations, but very often the condition number of the corresponding matrix is very large! The solution can be done in a polynomial form, as a picewise polynomial, spline, etc. In this section we give a type of Nyström interpolants for solving Fredholm integral equations on the finite interval $[-1,1]$ with respect to the Jacobi weight function $v^{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta$, with parameters $\alpha,\beta>-1$. This is based on a new approach in the weighted polynomial interpolation (see Mastroianni & Milovanović [10] and [11]).

The basic idea is to take another Jacobi weight $v^{\gamma,\delta}(x)=(1-x)^\gamma(1+x)^\delta$, with parameters γ and δ, such that $0\le\gamma<1-\alpha$ and $0\le\delta<1-\beta$, and to consider the solution of the Fredholm integral equation (2.1) in the following space

$$C_{v^{\gamma,\delta}}=\left\{f\in C^0((-1,1)):\lim_{x\to\pm1}(fv^{\gamma,\delta})(x)=0\right\},\quad \|f\|_{C_{v^{\gamma,\delta}}}=\|fv^{\gamma,\delta}\|_\infty .$$

The convergence of Nyström interpolants is connected with *best error weighted approximation* of the function f in the space $C_{v^{\gamma,\delta}}$ by polynomials of degree at most n,

$$E_n(f)_{v^{\gamma,\delta}}=\inf_{P_n\in\mathcal{P}_n}\|(f-P_n)v^{\gamma,\delta}\|_\infty.$$

We approximate the integral operator

$$(Kf)(x)=\mu\int_{-1}^{1}k(x,y)f(y)w(y)\,\mathrm{d}\,y$$

by a discrete operaror K_n, using the *Gauss-Jacobi* quadrature formula, i.e.,

$$(K_nf)(x)=\mu\sum_{k=1}^{n}\lambda_k(v^{\alpha,\beta})k(x,x_k)f(x_k),$$

where x_k are zeros of the (orthonormal) Jacobi polynomal $p_n(v^{\alpha,\beta})$ and $\lambda_k(v^{\alpha,\beta})$, $k=1,\dots,n$, are the corresponding Christoffel numbers. Then, FK2 reduces to $(I+K_n)f_n=g_n$, $n=1,2,\dots$, wherefrom, by multiplication with $v^{\gamma,\delta}$, and taking the collocation points x_i, $i=1,\dots,n$, we get the following system of n linear equations

$$\sum_{k=1}^{n}\left[\delta_{i,k}+\mu\frac{v^{\gamma,\delta}(x_i)}{v^{\gamma,\delta}(x_k)}k(x_i,x_k)\lambda_k(v^{\alpha,\beta})\right]a_k=g(x_i)v^{\gamma,\delta}(x_i),\quad i=1,\dots,n,$$

where $a_k=f_n(x_k)v^{\gamma,\delta}(x_k)$, $k=1,\dots,n$, are unknowns. If for a sufficiently large n, this system admits the unique solution $(a_1^*,\dots,a_n^*)$, then we construct the Nyström interpolant

$$f_n^*(x)=g(x)-\mu\sum_{k=1}^{n}k(x,x_k)\frac{\lambda_k(v^{\alpha,\beta})}{v^{\gamma,\delta}(x_k)}a_k^*,\quad n=1,2,\dots.$$

This system of equations is well-conditioned. Namely, if its matrix we denote by V_n, then for each n

we can prove that $\operatorname{cond}(V_n) \le \operatorname{cond}(I + K_n) < \text{const}$ (see [11]). Moreover, if the kernel $k(x,y)$ and the function $g(x)$ are in the space $C_{v^{\gamma,\delta}}$, under some additional conditions (see [11]), we can prove the convergence of the sequence $\{f_n^*(x)\}_n$ to the solution $f^*(x)$ in terms of *best error approximation*, i.e.,

$$\left\| f^* - f_n^* \right\|_{C_{v^{\gamma,\delta}}} \le C \left\{ \|f\|_{C_{v^{\gamma,\delta}}} \sup_{|x|\le 1} v^{\gamma,\delta}(x) E_{n-1}(k_x) + \sup_{|x|\le 1} v^{\gamma,\delta}(x) \|k_x\|_\infty E_{n-1}(f)_{v^{\gamma,\delta}} \right\}.$$

We mention also that the Nyström method can be used for weekly singular kernels (for details see [11]).

APPROXIMATE ANALYTICAL SOLUTIONS FOR LOVE'S INTEGRAL EQUATIONS

As we mentioned in Section 1, A.S. Kumar [8] tried to apply the Boubaker polynomials expansion scheme for finding an analytical solution of Love's integral equation, but his method has a severely flawed. Otherwise, solutions to several applied physics problems are based on the BPES (cf. [16]), using only the subsequence $\{B_{4m}(x)\}$ of these polynomials, which satisfy the relation

$$B_{4(m+1)}(x) = (x^4 - 4x^2 + 2)B_{4m}(x) - \beta_m B_{4(m-1)}(x), \quad m \ge 1,$$

with $B_0(x) = 1$ and $B_4(x) = x^4 - 2$, where $\beta_0 = 0,\ \beta_1 = -2,$ and $\beta_m = 1$ for $m \ge 2$. Otherwise, these polynomials are very similar to Chebyshev polynomials; their three-term recurrence relation is $B_{n+1}(x) = xB_n(x) - B_{n-1}(x),\ n = 2,3,\ldots$, where $B_0(x) = 1,\ B_1(x) = 1,\ B_2(x) = x^2 + 2$. In [12] we gave several new properties of Boubaker polynomials, including their zero dstribution, and presented an application to Love's integral equations (for some other procedures see [17] and [18]).

Since the solution of Love's equation (1.2) is an even function on [−1, 1], we try to find it in the set of all algebraic polynomials of degree $2n$ as a linear combination of Boubaker polynomials $B_0, B_2, \ldots, B_{2n}$, i.e.,

$$f_{2n}(x) = \sum_{m=0}^{n} c_m B_{2m}(x). \tag{4.1}$$

Then, putting it in (1.2) we obtain

$$\sum_{m=0}^{n} c_m B_{2m}(x) + \frac{s}{\pi} \int_{-1}^{1} \frac{d}{d^2 + (x-y)^2} \sum_{m=0}^{n} c_m B_{2m}(y)\,\mathrm{d}y = 1,$$

i.e.,

$$\sum_{m=0}^{n} \left(B_{2m}(x) + \frac{s}{\pi} \int_{-1}^{1} \frac{d\, B_{2m}(y)}{d^2 + (x-y)^2}\,\mathrm{d}y \right) c_m = 1.$$

Since the solution of Love's equation is an even function on [-1,1], we can take $n+1$ mutually different nonnegative points in [0,1] as collocation points $\tau_k,\ k = 1,\ldots,n$. Thus, for $x = \tau_k,\ k = 0,1,\ldots,n$, we get a system of linear equations for determining the coefficients $c_m,\ m = 0,1,\ldots,n$,

$$\begin{aligned} a_{0,0}c_0 + a_{0,1}c_1 + \cdots + a_{0,n}c_n &= 1, \\ a_{1,0}c_0 + a_{1,1}c_1 + \cdots + a_{1,n}c_n &= 1, \\ &\vdots \\ a_{n,0}c_0 + a_{n,1}c_1 + \cdots + a_{n,n}c_n &= 1, \end{aligned}$$

with the matrix $A_n = [a_{k,m}]_{k=0,m=0}^{n,n}$, where

$$a_{k,m} = B_{2m}(\tau_k) + \frac{s}{\pi} \int_{-1}^{1} \frac{d\, B_{2m}(y)}{d^2 + (\tau_k - y)^2}\,\mathrm{d}y, \quad k,m = 0,1,\ldots,n,$$

and s takes -1 and $+1$ for first and second Love's equation, respectively.

In this approach we need the integrals

$$J_{2m}(x,d)=(KB_{2m})(x)=\frac{1}{\pi}\int_{-1}^{1}\frac{d}{d^2+(x-y)^2}B_{2m}(y)\mathrm{d}y,\ m\geq 0.$$

Using the recurrence relation

$$B_{2m+2}(x)=(x^2-a_m)B_{2m}(x)-b_mB_{2m-2}(x),\quad m\geq 0,$$

where $a_0=b_1=-2$ and $a_m=2,\ b_{m+1}=1,\ m\geq 1$, we can obtain the corresponding recurence relation for the integrals $J_{2m}=J_{2m}(x,d)$ in the form

$$J_{2m+2}+(d^2+a_m-x^2)J_{2m}+b_mJ_{2m-2}=\frac{2d}{\pi}I_{2m}+\frac{xd}{\pi}\left\{B_{2m}(1)\log\frac{d^2+(1-x)^2}{d^2+(1+x)^2}+K_{2m}\right\},$$

where

$$J_0(x,d)=\frac{1}{\pi}\left[\tan^{-1}\left(\frac{1-x}{d}\right)+\tan^{-1}\left(\frac{x+1}{d}\right)\right],$$

$$J_2(x,d)=\frac{1}{\pi}\left[2d+\left(2-d^2+x^2\right)\tan^{-1}\left(\frac{x+1}{d}\right)-2xd\tanh^{-1}\left(\frac{2x}{d^2+x^2+1}\right)+\left(d^2-2\right)\tan^{-1}\left(\frac{x-1}{d}\right)+x^2\tan^{-1}\left(\frac{1-x}{d}\right)\right]$$

and

$$I_{2m}=\int_0^1 B_{2m}(y)\ \mathrm{d}y,\quad K_{2m}=K_{2m}(x,d)=\int_0^1\log\frac{d^2+(x+y)^2}{d^2+(x-y)^2}B'_{2m}(y)\ \mathrm{d}y.$$

It is easy to prove that

$$I_{2m}=\frac{6}{2m-1}\sin\left(\frac{(4m+1)\pi}{6}\right)+\frac{2}{2m+1}\cos\left(\frac{(2m+1)\pi}{3}\right).$$

On the other side, the integral K_{2m} can be expressed as a linear combination of the integrals (see [12])

$$S_k=S_k(x,d)=\int_0^1\log\frac{d^2+(x+y)^2}{d^2+(x-y)^2}y^{2k-1}\ \mathrm{d}y,\ \ k\in\mathbb{N}.$$

NUMERICAL EXAMPLES

In this section we present some results for first Love's equation (1.1), obtained using the previous method (Section 4), with the positive zeros of the Chebyshev polynomial $T_{2n+2}(x)$, i.e.,

$$\tau_k=\cos\frac{(2k+1)\pi}{4(n+1)},\ k=0,1,\ldots,n,$$

as collocation points. All computations were performed in Mathematica, Ver. 9.0.1.0, on MacBook Pro Retina, OS X 10.9.2.

The obtained coefficients $c_m=c_m^{(n)}$ in approximation $f_{2n}(x)$ given by (4.1), for $n=1,2,3,4,$ are presented in Table 5.1. The corresponding expansions (in power form) are

Table 5.1. Coefficients in approximation $f_{2n}(x)$ for n=1,2,3,4

$c_m^{(n)}$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$n=1$	2.47659519	-0.2807209050			
$n=2$	2.63988705	-0.3201403668	0.04005435		
$n=3$	2.53975466	-0.2766963025	0.04573868	0.012093154	
$n=4$	2.46662496	-0.2641585212	0.01602553	0.000730762	-0.00565549

$$f_2(x) \approx 1.915152 - 0.280721x^2,$$
$$f_4(x) \approx 1.919498 - 0.320140x^2 + 0.0400543x^4,$$
$$f_6(x) \approx 1.919071 - 0.312976x^2 + 0.02155237x^4 + 0.01209315x^6,$$
$$f_8(x) \approx 1.919029 - 0.311595x^2 + 0.014564x^4 + 0.0233527x^6 - 0.00565549x^8.$$

Maximal relative errors of the previous approximate solutions, including ones for $n=5$ and $n=6,$ as well as for Love's solution $f_L(x)$, are presented in Table 5.2, where we used as the exact solution the one obtained by an efficient method for solving Fredholm integral equations of the second kind [1]. Alternatively, we can also use $f_{16}(x)$ as $f(x)$. Numbers in parentheses indicate decimal exponents.

Table 5.2. Maximal relative errors of the approximate solutions

Approximation	f_L	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
Relative errors	1.69(-3)	3.21(-3)	2.43(-4)	2.78(-5)	1.37(-6)	2.91(-7)	9.65(-9)

Also, graphs of the relative errors $|(f_{2n}(x)-f(x))/f(x)|$ and $|(f_L(x)-f(x))/f(x)|$ are displayed in Figure 5.1. Notice that the both approximate solutions $f_8(x)$ (for $n=4$) and $f_L(x)$ are polynomials of the same degree eight.

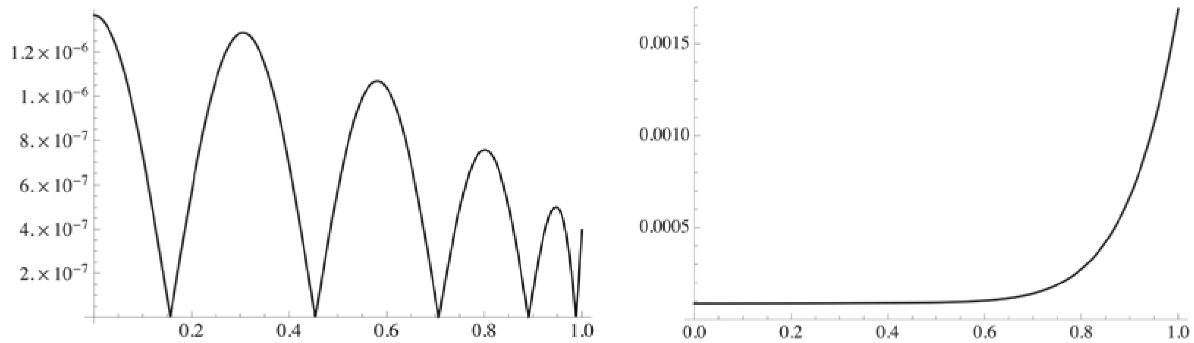

Fig. 5.1. Relative errors of the approximate solutions of degree eight $f_8(x)$ (left) and $f_L(x)$ (right)

The solutions $f_8(x)$ for different values of the distance d ($d=10,$ $d=1,$ $d=1/10,$ and $d=1/100$) are presented in Figure 5.2.

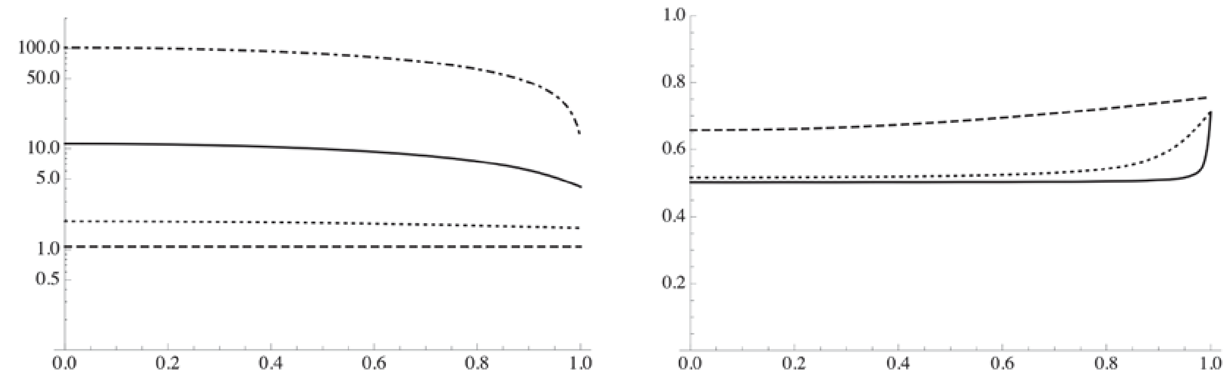

Fig. 5.2. (left) The solutions of first Love's equation as log-plots for *d* =10 (dashed line), *d* =1 (dotted line), *d* =1/10 (solid line), and *d* =1/100 (dot-dashed line); (right) The solutions of second Love's equation for *d* =1 (dashed line), *d* =1/10 (dotted line) and *d* =1/100 (solid line)

In the case when $d \to \infty$, the solution of first Love's equation (1.1) tends to the constant $f(x)=1$. For example, in the case $d=10$, the first two solutions are

$$f_2(x) \approx 1.067734116 - 0.00065980x^2,$$
$$f_4(x) \approx 1.067734911236 - 0.00066617763x^2 + 6.3737080810\cdot 10^{-6}x^4$$

with maximal relative errors on [-1,1], 7.40(-7) and 1.79(-9), respectively.

Finaly, using the expresion for $V(r,z)$ (see Section 1) we can calculate and plot the equipotential lines (see Figure 5.3 for two cases *d* =1 and *d* =1/10).

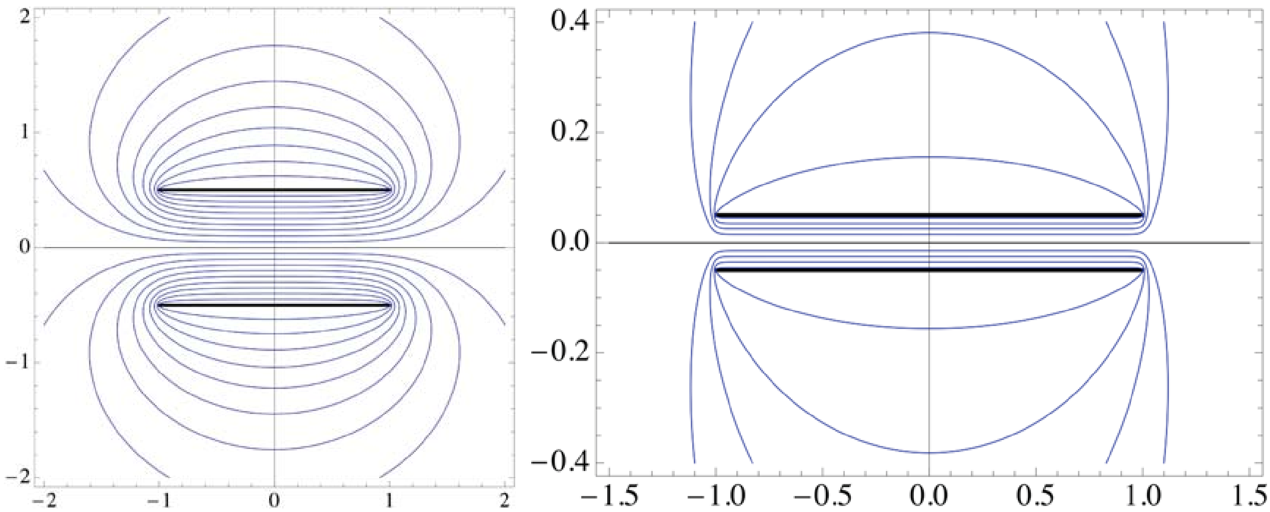

Fig. 5.3. Equipotential lines for $\pm V = 0.1(0.1)0.9$ when *d* =1 (left) and for $\pm V = 0.3(0.2)0.9$ when *d* =1/10 (right)

The corresponding results for second Love's integral equation can be obtained in a similar way. The solutions for different values of the distance d ($d=1$, $d=1/10$, and $d=1/100$) are presented in Figure 5.2 (right). A problem in approximation can appear when $d \to 0^+$ (cf. [19]). Namely, in that case we have

$$(Kf)(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{d}{d^2+(x-y)^2} f(y)\,\mathrm{d}y \to f(x),\; d \to 0^+$$

which means that for $-1 < x < 1$, the solution $f(x)$ of second Love's equation is nearly equal to ½, but at the endpoints $f(\pm 1) \approx 3/4$. Thus, in this case with small parameter d some difficulties in approximation, especially by polynomials, have appeared. An efficient procedure for a very small value of the parameter d in this equation has recently been introduced by Pastore in [20].

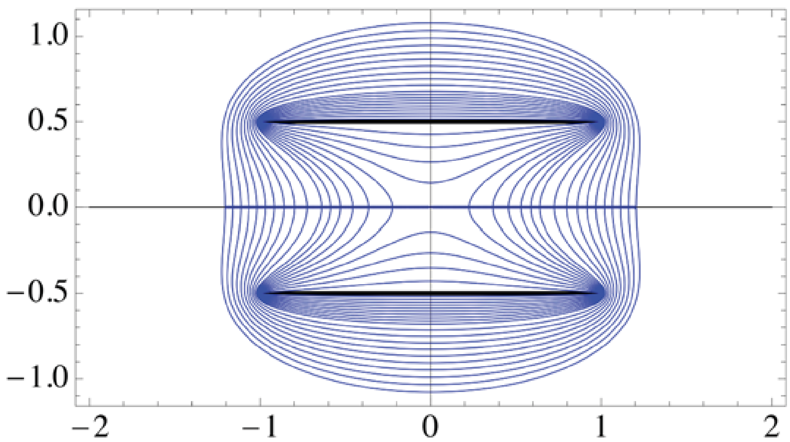

Fig. 5.4. Equipotential lines for V=0.70(0.02)0.88 and V=0.90(0.01)0.99 when d =1

Finally, in Figure 5.4 we presented equipotential lines for the electrostaic system when the potentials of the both plates are V=+1 and d=1.

ACKNOWLEDGEMENTS

The author was supported in parts by the Serbian Ministry of Education, Science and Technological Development (No. #174015).

REFERENCES

[1] Love E.R., The electrostatic field of two equal circular co-axial conducting disks. Quart. J. Mech. Appl. Math. **2** (1949), pp. 428-451.

[2] Love E.R., The potential due to a circular parallel plate condenser. Mathematika **37** (1990), pp. 217-231.

[3] Norgren M., Jonsson B.L.G., The capacitance of the circular parallel plate capacitor obtained by solving the Love integral equation using an analytic expansion of the kernel. Progress In Electromagnetics Research, PIER 97, 2009, pp. 357-372.

[4] Hutson V., The circular plate condenser at small separations. Proc. Cambridge Philos. Soc. **59** (1963), pp. 211-224.

[5] Chew W.C., Kong J.A., Asymptotic formula for the capacitance of two oppositely charged discs. Math. Proc. Cambridge Philos. Soc. **89**, no. 2 (1981), pp. 373-384.

[6] Carlson G.T., Illman B.L., The circular disk parallel plate capacitor. Amer. J. Phys. **62**, No. 12 (1994), pp. 1099-1105.

[7] Reich, E., E.R. Love's integral equation for the circular plate condenser. ANZIAM **46** (2004), pp. 85-93.

[8] Kumar A.S., An analytical solution to applied mathematics-related Love's equation using the Boubaker polynomials expansion scheme. J. Franklin Inst. **347** (2010), pp. 1755-1761.

[9] Boubaker K., Boubaker polynomials expansion scheme (BPES) solution to Boltzmann diffusion equation in the case of strongly anisotropic neutral particles forward-backward scattering. Annals of Nuclear Energy **38** (2011), pp. 1715-1717.

[10] Mastroianni G., Milovanović G.V., Interpolation Problems - Basic Theory and Applications. Spriner Verlag, Berlin - Heidelberg - New York, 2008.

[11] Mastroianni G., Milovanović G.V., Well-conditioned matrices for numerical treatment of Fredholm integral equations of the second kind. Numer. Linear Algebra Appl. **16** (2009), pp. 995-1011.

[12] Milovanović G.V., Joksimović D., Properties of Boubaker polynomials and an application to Love's integral equation. Appl. Math. Comput. **224** (2013), pp. 74-87.

[13] Atkinson K.E., The Numerical Solution of Integral Equations of the Second Kind. Cambridge Cambridge University Press, Cambridge, 1997.

[14] Kythe P.K., Puri P., Computational Methods for Linear Integral Equations. Birkhäuser, Boston - Basel - Berlin, 2002.

[15] Prössdorf S., Silbermann B., Numerical Analysis for Integral and Related Operator Equations. Birkhäuser Verlag, Basel, 1991.

[16] B. Karem Ben Mahmoud, Temperature 3D profiling in cryogenic cylindrical devices using Boubaker polynomials expansion scheme (BPES), Cryogenics 49 (2009), 217-220.

[17] Elliott D., A Chebyshev series method for the numerical solution of Fredholm integral equations, Computer J. 6 (1) (19630, 102-112.

[18] Fox L., Goodwin E.T., The numerical solution of non-singular linear integral equations, Philos. Trans. R. Soc. Lond. Ser. A 245 (1953), pp. 501-534.

[19] Monegato G., Palamara Orsi A., Product formulas for Fredholm integral equations with rational kernel functions, In: Numerical Integration, III Oberwolfach, 1987, Internat. Schriftenreihe Numer. Math., Vol. 85, pp. 140-156, Birkhäuser, Basel, 1988.

[20] Pastore P., The numerical treatment of Love's integral equation having very small parameter, J. Comput. Appl. Math. **236**, no. 6 (2011), pp. 1267-1281.