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The roots of polynomials and the operator Δ_i^3 on the Hahn sequence space h

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Abstract In this paper, we define the third order generalized difference operator Δ_i^3, where

$$(\Delta_i^3 x)_k = \sum_{i=0}^{3} \frac{(-1)^i}{i+1} \binom{3}{i} x_{k-i} = x_k - \frac{3}{2} x_{k-1} + x_{k-2} - \frac{1}{4} x_{k-3},$$

and show that it is a linear bounded operator on the Hahn sequence space h. Then we study the spectrum and point spectrum of the operator Δ_i^3 on h. Furthermore, we determine the point spectrum of the adjoint of this operator. This is achieved by studying some properties of the roots of certain third order polynomials.

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1 Preliminaries, background and notation

We denote the space of all complex valued sequences by ω. Each vector subspace of ω is called a *sequence space*. The spaces of all bounded, convergent and null sequences are denoted by ℓ_∞, c and c_0, respectively. Also ϕ is the space of all sequences that terminate in zeros.

A sequence space X is called an FK–space if it is a complete linear metric space with continuous coordinates $p_n : X \to \mathbb{C}$ with $p_n(x) = x_n$ for all $x = (x_n)_{n=1}^\infty \in X$ and every $n \in \mathbb{N} = \{1,2,\dots\}$, where $\mathbb{C}$ denotes the complex field. An FK space $X \supset \phi$ is said to have AK if $X = \lim_{m\to\infty} x^{[m]}$, where $x^{[m]} = \sum_{k=1}^m x_k e^{(k)}$ $(m \in \mathbb{N})$ denotes the *m–section of the sequence* $x = (x_k)_{k=1}^\infty \in X$ and $e^{(k)} = (e_j^{(k)})_{j=1}^\infty$ for each $k \in \mathbb{N}$ is the sequence with $e_k^{(k)} = 1$ and $e_j^{(k)} = 0$ for $j \neq k$. If ϕ is dense in X, then X is called an AD space; thus AK implies AD. A normed FK space is called a BK space, that is, a BK space is a Banach sequence space with continuous coordinates, [4, pp. 272–273]. The interested reader is also referred to the text books [12,2,1,7]. The sequence spaces ℓ_∞, c and c_0 are BK-spaces with the usual supremum norm defined by $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$. By ℓ_1, ℓ_p, cs, cs_0, bs and bv, we denote the spaces of all absolutely convergent, p-absolutely convergent; where $1 < p < \infty$, convergent, convergent to zero, bounded series, and bounded variation sequences, respectively. Moreover we denote bv_0 as the sequence space $bv \cap c_0$.

The β–duals of a sequence space X is defined as

$$X^\beta = \{a = (a_k)_{k=1}^\infty \in \omega : ax = (a_k x_k)_{k=1}^\infty \in cs \text{ for all } x = (x_k)_{k=1}^\infty \in X\}.$$

Let X and Y be any two sequence spaces and $A = (a_{nk})_{n,k=1}^\infty$ be an infinite matrix of complex numbers a_{nk}, where $k,n \in \mathbb{N}$. Then, we say that A defines a *matrix transformation* from X into Y denoted by writing $A : X \to Y$, if for every sequence $x = (x_k)_{k=1}^\infty \in X$ the sequence $Ax = (A_n x)_{n=1}^\infty$, the so–called *A–transform of* x, is in Y, where

$$A_n x = \sum_{k=1}^\infty a_{nk} x_k \tag{1.1}$$

provided the series on the right side of (1.1) converges for each $n \in \mathbb{N}$. For simplicity in notation, in what follows, the summation without limits runs from 1 to ∞. By $(X : Y)$, we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X : Y)$ if and only if $A_n = (a_{nk})_{k=1}^\infty \in X^\beta$ for all $n \in \mathbb{N}$, and $Ax \in Y$ for all $x \in X$.

The Hahn sequence space h was defined by Hahn [8] and studied by many mathematicians (see [15–17]). Kirišci [9–11] compiled all results on h in his papers. In [9], he defined a new Hahn sequence space derived by the Cesàro mean. Moreover, in [11], he defined the p–Hahn sequence space h_p. Yeşilkayagil and Kirişci [20] studied the fine spectrum of forward difference operator Δ on h. More recently, Das [5] determined the fine spectrum of the lower triangular matrix $B(r,s)$ on h.

The Hahn sequence space h [8] is defined as

$$h = \left\{ x = (x_k)_{k=1}^{\infty} \in \omega : \sum_{k=1}^{\infty} k|\Delta x_k| < \infty \right\} \cap c_0, \tag{1.2}$$

where Δ denotes the forward difference operator on ω, that is, $\Delta x_k = x_k - x_{k+1}$ $(= (x_k)_{k=1}^{\infty} \in \omega)$ for all $k \in \mathbb{N}$. Hahn proved that

$$h^{\beta} = \sigma_{\infty} = \left\{ x = (x_k)_{k=1}^{\infty} \in \omega : \sup_n \frac{1}{n} \left| \sum_{k=1}^{n} x_k \right| < \infty \right\}.$$

Rao [15, Proposition 1] showed that h is a BK space with AK with respect to the norm given by $\|x\| = \sum_k k|\Delta x_k|$ for all $x \in h$, and characterizeed the classes $(h : Y)$ for $Y \in \{c_0, c, \ell_\infty, \ell_1, h\}$ [15, Popositions 6–10], in particular, [15, Proposition 10]

Theorem 1 ([15, Proposition 10]) *We have $A \in (h : h)$ if and only if*

$$\lim_{n\to\infty} a_{nk} = 0 \ \textit{for all} \ \ k = 1, 2, \ldots, \tag{1.3}$$

$$\sum_{n=1}^{\infty} n|a_{nk} - a_{n+1,k}| \ \textit{converges for all} \ \ k = 1, 2, \ldots, \tag{1.4}$$

$$\sup_k \frac{1}{k} \sum_{n=1}^{\infty} n \left| \sum_{v=1}^{k} (a_{nv} - a_{n+1,v}) \right| < \infty. \tag{1.5}$$

Remark 1 It was shown in [13, Remark 3.10] that the condition in (1.4) is redundant, so $A \in (h,h)$ is and only if the conditions in (1.3) and (1.5) are satisfied.

Let X be a Banach space. Then $\mathscr{B}(X)$ is the set of all bounded linear operators on X into itself. Also X^* denotes the continuous dual of X, that is, the space of all continuous linear functionals on X.

The adjoint operator $T^* : X^* \to X^*$ of $T \in \mathscr{B}(X)$ is defined by

$$(T^* y^*)(x) = y^*(Tx) \text{ for all } y^* \in X^* \text{ and } x \in X, \tag{1.6}$$

and $T^* \in \mathscr{B}(X^*)$.

If $T \in \mathscr{B}(X)$, then we write

$$\sigma(T) = \sigma(T, X) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \mathscr{B}(X)^{-1}\}$$

for the spectrum of T, where I is the identity on X and $\mathscr{B}(X)^{-1}$ is the set of all invertible operators in $\mathscr{B}(X)$, that is, the set of all $T \in \mathscr{B}(X)$ that are one to one and onto.

The *point* or *discrete spectrum* of $T \in \mathscr{B}(X)$ is the set

$$\sigma_p(T) = \sigma_p(T, X) = \{\lambda \in \mathbb{C} : Tx = \lambda x \text{ for some } x \in X \setminus \{0\}\};$$

any $\lambda \in \sigma_p(T)$ is said to be an *eigen value of T* ([12, 14, 19]).

2 The spectrum of the matrix operator Δ_i^3 on the Hahn sequence space h

The study of spectra of bounded linear operators is an important area of research in operator theory, which generalizes the notion of eigenvalues. In particular, the spectrum of difference operators has many applications in different scientific and engineering problems concerning the eigenvalues. Our study is motivated by and related to the results in [6, 18] and the references therein.

In this section, we define the matrix operator Δ_i^3 and show $\Delta_i^3 \in \mathscr{B}(h)$. Then we study the the spectrum and point spectrum of the operator Δ_i^3 on h.

The operator Δ_i^3 is defined by

$$\left(\Delta_i^3 x\right)_k = \sum_{i=0}^{3} \frac{(-1)^i}{i+1}\binom{3}{i} x_{k-i} = x_k - \frac{3}{2}x_{k-1} + x_{k-2} - \frac{1}{4}x_{k-3} \text{ for } x = (x_k)_{k=1}^{\infty} \in \omega;$$

we use the convention throughout that any term with an index ≤ 0 is equal to zero. Thus the operator $\Delta_i^3 x$ is given by the infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$, where

$$a_{nk} = \begin{cases} 1 & (k = n, n-2) \\ -\dfrac{3}{2} & (k = n-1) \\ -\dfrac{1}{4} & (k = n-3) \\ 0 & (\text{otherwise}) \end{cases} \qquad (n = 1, 2, \ldots).$$

First we show $\Delta_i^3 \in \mathscr{B}(h)$ and compute the operator norm $\|\Delta_i^3\|$.

Theorem 2 *We have $\Delta_i^3 \in \mathscr{B}(h)$ and $\|\Delta_i^3\| = 49/4$.*

Proof It is clear that Δ_i^3 is linear.

We have to show by Remark 1 that

$$\lim_{n\to\infty} a_{nk} = 0 \text{ for all } k \tag{2.1}$$

and

$$\sup_m c_m < \infty, \text{ where } c_m = \frac{1}{m}\sum_{n=1}^{\infty} n \left| \sum_{k=1}^{m} (a_{nk} - a_{n+1,k}) \right| \text{ for } m = 1, 2, \ldots. \tag{2.2}$$

The condition in (2.1) is clearly satisfied.

We are going to evaluate $\sup_m c_m$.

We put $b_{n,m} = \sum_{k=1}^{m}(a_{nk} - a_{n+1,k})$ for all n and m. If we omit the terms $b_{n,m}$ that do not contain at least one nonzero term a_{nk} or $a_{n+1,k}$, then the indices of summation n and k obviously satisfy

$$1 \leq n \leq m+3 \quad \text{and} \quad \max\{1, n-3\} \leq k \leq \min\{n+1, m\},$$

and

$$c_m = \frac{1}{m}\sum_{n=1}^{m+3} n \left| \sum_{k=\max\{n-3,1\}}^{\max\{n+1,m\}} (a_{nk} - a_{n+1,k}) \right|. \tag{2.3}$$

We put for $m = 1,2,\dots$

$$S_m^{(1)} = \sum_{n=1}^{3} \left| \sum_{k=1}^{\min\{m,n+1\}} (a_{nk} - a_{n+1,k}) \right|,$$
$$S_m^{(2)} = \sum_{n=4}^{m-1} n \left| \sum_{k=n-3}^{n+1} (a_{nk} - a_{n+1,k}) \right|,$$

and

$$S_m^{(3)} = \sum_{n=\max\{4,m\}}^{m+3} n \left| \sum_{k=n-3}^{m} (a_{nk} - a_{n+1,k}) \right|.$$

(i) First we show

$$S_m^{(2)} = 0 \quad \text{for all} \quad m = 1,2,\dots. \tag{2.4}$$

Obviously $S_m^{(2)} = 0$ for $m = 1,2,3,4$. Let $m \geq 5$. Then, for $4 \leq n \leq m-1$,

$$\begin{aligned}\left| \sum_{k=n-3}^{n+1} (a_{nk} - a_{n+1,k}) \right| &= |a_{n,n-3} - a_{n+1,n-3} + a_{n,n-2} - a_{n+1,n-2} \\ &\quad + a_{n,n-1} - a_{n+1,n-1} + a_{n,n} - a_{n+1,n} + a_{n,n+1} - a_{n+1,n+1}| \\ &= \left| -\frac{1}{4} - 0 + 1 + \frac{1}{4} - \frac{3}{2} - 1 + 1 + \frac{3}{2} + 0 - 1 \right| = 0,\end{aligned}$$

that is, (2.4) holds.

(ii) Now we show

$$S_m^{(1)} = \begin{cases} \dfrac{45}{4} & (m=1) \\ \dfrac{21}{4} & (m=2) \\ \dfrac{29}{4} & (m \geq 3). \end{cases} \tag{2.5}$$

We obtain

$$\begin{aligned} S_1^{(1)} &= \sum_{n=1}^{3} n|a_{n1} - a_{n+1,1}| = |a_{11} - a_{21}| + 2|a_{21} - a_{22}| + 3|a_{31} - a_{41}| \\ &= \left|1 + \frac{3}{2}\right| + 2\left|-\frac{3}{2} - 1\right| + 3\left|1 + \frac{1}{4}\right| = \frac{45}{4}, \\ S_2^{(1)} &= \sum_{n=1}^{3} n \left| \sum_{k=1}^{2} (a_{nk} - a_{n+1,k} \right| \\ &= |a_{11} - a_{21} + a_{12} - a_{22}| + 2|a_{21} - a_{31} + a_{22} - a_{32}| + |a_{31} - a_{41} + a_{32} - a_{42}| \\ &= \left|1 + \frac{3}{2} + 0 - 1\right| + 2\left|-\frac{3}{2} - 1 + 1 + \frac{3}{2}\right| + 3\left|1 + \frac{1}{4} - \frac{3}{2} - 1\right| \end{aligned}$$

$$= \frac{3}{2} + 0 + 3\frac{5}{4} = \frac{21}{4},$$

and for $m \geq 3$

$$\begin{aligned}
S_m^{(1)} &= \sum_{n=1}^{2} \left| \sum_{k=1}^{2} (a_{nk} - a_{n+1,k} \right| + 3 \left| \sum_{k=1}^{3} (a_{3k} - a_{4k} \right| \\
&= |a_{11} - a_{21} + a_{12} - a_{22}| + 2|a_{21} - a_{31} + a_{22} - a_{32} + a_{23} - a_{33}| \\
&\quad + 3|a_{31} - a_{41} + a_{32} - a_{42} + a_{33} - a_{43}| \\
&= \left|1 + \frac{3}{2} + 0 - 1\right| + 2\left|-\frac{3}{2} - 1 + 1 + \frac{3}{2} + 0 - 1\right| + 3\left|1 + \frac{1}{4} - \frac{3}{2} - 1 + 1 + \frac{3}{2}\right| \\
&= \frac{3}{2} + 2 + \frac{15}{4} = \frac{29}{4},
\end{aligned}$$

that is, (2.5) holds.

(iii) Now we show

$$S_m^{(3)} = \begin{cases} 1 & (m = 1) \\ \frac{21}{4} & (m = 2) \\ \frac{25}{2} & (m = 3) \\ \frac{15m}{4} + \frac{17}{4} & (m \geq 4). \end{cases} \tag{2.6}$$

We have

$$S_1^{(3)} = 4|a_{41} - a_{51}| = 4\left|-\frac{1}{4} - 0\right| = 1,$$

$$\begin{aligned}
S_2^{(3)} &= \sum_{n=4}^{5} n \left| \sum_{k=1}^{2} (a_{nk} - a_{n+1,k}) \right| \\
&= 4|a_{41} - a_{51} + a_{42} - a_{52}| + 5|a_{51} - a_{61} + a_{52} - a_{62}| \\
&= 4\left|-\frac{1}{4} - 0 + 1 + \frac{1}{4}\right| + 5\left|0 - \frac{1}{4}\right| = 4 + \frac{5}{4} = \frac{21}{4},
\end{aligned}$$

$$\begin{aligned}
S_3^{(3)} &= \sum_{n=4}^{6} n \left| \sum_{k=n-3}^{3} (a_{nk} - a_{n+1,k}) \right| \\
&= 4\left| \sum_{k=1}^{3} (a_{4k} + a_{5k}) \right| + 5\left| \sum_{k=2}^{3} (a_{5k} - a_{6k}) \right| + 6\,|a_{63} - a_{73}| \\
&= 4|a_{41} - a_{51} + a_{42} - a_{52} + a_{43} - a_{53}| + 5|a_{52} - a_{62} + a_{53} - a_{63}| \\
&\quad + 6|a_{63} - a_{73}|
\end{aligned}$$

$$= 4\left|-\frac{1}{4}-0+1+\frac{1}{4}-\frac{3}{2}-1\right|+5\left|-\frac{1}{4}-0+1+\frac{1}{4}\right|+6\left|-\frac{1}{4}\right|$$
$$= 6+5+\frac{3}{2}=\frac{25}{2}$$

and for $m \geq 4$

$$\begin{aligned}
S_m^{(3)} &= \sum_{n=m}^{m+3} n\left|\sum_{n-3}^{m}(a_{nk}-a_{n+1,k}\right| \\
&= m|a_{m,m-3}-a_{m+1,m-3}+a_{m,m-2}-a_{m+1,m-2}+a_{m,m-1}-a_{m+1,m-1} \\
&\quad +a_{m.m}-a_{m+1,m}| \\
&+(m+1)|a_{m+1,m-2}-a_{m+2,m-2}+a_{m+1,m-1}-a_{m+2,m-1}+a_{m+1,m}-a_{m+2,m} \\
&+(m+2)|a_{m+2,m-1}-a_{m+3,m-1}+a_{m+2,m}-a_{m+3,m}| \\
&+(m+3)|a_{m+3,m}-a_{m+4,m}| \\
&= m\left|-\frac{1}{4}-0+1+\frac{1}{4}-\frac{3}{2}-1+1+\frac{3}{2}\right|+(m+1)\left|-\frac{1}{4}-0+1+\frac{1}{4}-\frac{3}{2}-1\right| \\
&+(m+2)\left|-\frac{1}{4}-0+1-\frac{1}{4}\right|+(m+3)\left|-\frac{1}{4}-0\right| \\
&= m+(m+1)\frac{3}{2}+(m+2)+\frac{m+3}{4}=\frac{15m}{4}+\frac{17}{4}.
\end{aligned}$$

Finally, it follows from (2.4), (2.5) and (2.6)

$$c_m=\frac{1}{m}(S_m^{(1)}+S_m^{(3)}=\begin{cases}\frac{45}{4}+1=\frac{49}{4} & (m=1)\\ \frac{1}{2}\left(\frac{21}{4}+\frac{21}{4}\right)=\frac{21}{4} & (m=2)\\ \frac{1}{3}\left(\frac{29}{4}+\frac{25}{2}\right)=\frac{79}{12} & (m=3)\\ \frac{1}{m}\left(\frac{29}{4}+\frac{17}{4}+\frac{15m}{4}\right)=\frac{15}{4}+\frac{23}{2m} & (m\geq 4).\end{cases}$$

Obviously

$$c_1 \geq c_m \text{ for all } m\geq 2, \text{ so that } \sup_m c_m = c_1 = \frac{49}{4}.$$

Thus the condition in (2.2) is satisfied, so $\Delta_i^3 \in \mathscr{B}(h)$, and it follws from [13, Corollary 3.15 (a)] that

$$\|\Delta_k^3\| = \sup_m c_m = \frac{49}{4}. \tag{2.7}$$

Now we determine the spectrum of $\Delta_i^3 \in \mathscr{B}(h)$.

Theorem 3 *We have*

$$\sigma(\Delta_i^3,h)\subset\left\{\lambda\in\mathbb{C}:|1-\lambda|\leq\frac{11}{4}\right\}. \tag{2.8}$$

Proof We assume $|1-\lambda| > 11/4$.

First, we observe that since $\Delta_i^3 - \lambda I$ is a triangle, its inverse $B = (b_{nk})_{n,k=1}^{\infty} = (\Delta_i^3 - \lambda I)^{-1}$ exists. An application of [3, Theorem 2] yields

$$B = \begin{bmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{3}{2(1-\lambda)^2} & \frac{1}{1-\lambda} & 0 & 0 & 0 & 0 & \cdots \\ \frac{5+4\lambda}{4(1-\lambda)^3} & \frac{3}{2(1-\lambda)^2} & \frac{1}{1-\lambda} & 0 & 0 & 0 & \cdots \\ \frac{2\lambda^2+20\lambda+5}{8(1-\lambda)^4} & \frac{5+4\lambda}{4(1-\lambda)^3} & \frac{3}{2(1-\lambda)^2} & \frac{1}{1-\lambda} & 0 & 0 & \cdots \\ \frac{28\lambda^2+52\lambda+1}{16(1-\lambda)^5} & \frac{2\lambda^2+20\lambda+5}{8(1-\lambda)^4} & \frac{5+4\lambda}{4(1-\lambda)^3} & \frac{3}{2(1-\lambda)^2} & \frac{1}{1-\lambda} & 0 & \cdots \\ \frac{16\lambda^3+150\lambda^2+84\lambda-7}{32(1-\lambda)^6} & \frac{28\lambda^2+52\lambda+1}{16(1-\lambda)^5} & \frac{2\lambda^2+20\lambda+5}{8(1-\lambda)^4} & \frac{5+4\lambda}{4(1-\lambda)^3} & \frac{3}{2(1-\lambda)^2} & \frac{1}{1-\lambda} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We obtain the general expression of elements b_{nk}, in terms of $\xi = 1/(1-\lambda)$, for all $k \le n$ by the following calculation as

$$\begin{aligned} b_{n,n} &= \frac{1}{1-\lambda} = \xi, \\ b_{n,n-1} &= \frac{3}{2(1-\lambda)^2} = \frac{3}{2}\xi^2, \\ b_{n,n-2} &= \frac{5+4\lambda}{4(1-\lambda)^3} = \frac{1}{4}\xi^2(9\xi-4), \\ b_{n,n-3} &= \frac{2\lambda^2+20\lambda+5}{8(1-\lambda)^4} = \frac{1}{8}\xi^2\left(27\xi^2-24\xi+2\right), \\ b_{n,n-4} &= \frac{28\lambda^2+52\lambda+1}{16(1-\lambda)^5} = \frac{1}{16}\xi^3\left(81\xi^2-108\xi+28\right), \\ b_{n,n-5} &= \frac{16\lambda^3+150\lambda^2+84\lambda-7}{32(1-\lambda)^6} = \frac{1}{32}\xi^3\left(243\xi^3-432\xi^2+198\xi-16\right), \\ b_{n,n-6} &= \frac{4\lambda^4+192\lambda^3+480\lambda^2+68\lambda-15}{64(1-\lambda)^7} \\ &= \frac{1}{64}\xi^3\left(729\xi^4-1620\xi^3+1080\xi^2-208\xi+4\right), \\ b_{n,n-7} &= \frac{3(44\lambda^4+368\lambda^3+342\lambda^2-20\lambda-5)}{128(1-\lambda)^8} \\ &= \frac{3}{128}\xi^4\left(729\xi^4-1944\xi^3+1710\xi^2-544\xi+44\right), \\ &\vdots \end{aligned}$$

$$b_{n,k} = \frac{1}{1-\lambda}\left(\frac{3}{2}b_{n-1,k} - b_{n-2,k} + \frac{1}{4}b_{n-3,k}\right) \quad \text{for all } k \le n-1, \quad \text{etc.}$$

We obtain by the computation above

$$b_{nk} = \begin{cases} \dfrac{1}{1-\lambda} & (k=n), \\ \dfrac{1}{1-\lambda}\left(\dfrac{3}{2}b_{n-1,k} - b_{n-2,k} + \dfrac{1}{4}b_{n-3,k}\right) & (0 \le k \le n-1), \\ 0 & (k > n). \end{cases} \tag{2.9}$$

Now we prove $B \in \mathscr{B}(h)$. We have to show by Remark 1 that

$$\lim_{n\to\infty} b_{nk} = 0 \ \text{ for all } \ k \tag{2.10}$$

and

$$\sup_m c_m < \infty, \ \text{ where } \ c_m = \frac{1}{m}\sum_{n=1}^{\infty} n\left|\sum_{k=1}^{m}(b_{nk} - b_{n+1,k})\right| \ \text{ for } \ m = 1,2,\ldots. \tag{2.11}$$

First, we observe that the recursion formula (2.9) yields

$$b_{n+1,k+1} = b_{nk} \ \text{ for } 1 \le k \le n \ \text{ and } n = 1,2,\ldots. \tag{2.12}$$

So in order to show (2.10), it suffices to show

$$\lim_{n\to\infty} \alpha_n = 0, \ \text{ where } \ \alpha_n = b_{n1} \text{ for all } n.$$

We apply the recursion formula

$$\begin{aligned} |\alpha_n| &\le \frac{1}{|1-\lambda|}\left(\frac{3}{2} + 1 + \frac{1}{4}\right) \cdot \max\{|\alpha_{n-1}|, |\alpha_{n-2}|, |\alpha_{n-3}|\} \\ &= \frac{11}{4} \cdot \frac{1}{|1-\lambda|} \cdot |\alpha_{n_1}|, \text{ where } |\alpha_{n_1}| = \max\{|\alpha_{n-1}|, |\alpha_{n-2}|, |\alpha_{n-3}|\} \end{aligned}$$

Writing

$$|\alpha_{n_{l+1}}| = \max\{|\alpha_{n_l-1}|, |\alpha_{n_l-2}|, |\alpha_{n_l-3}|\} \text{ for } l \ge 1,$$

and $|\alpha| = \max\{|\alpha_3|, |\alpha_2|, |\alpha_1|\}$, we obtain by repeated application of the recursion formula

$$|\alpha_n| \le \left(\frac{11}{4} \cdot \frac{1}{|1-\lambda|}\right)^{n/3} \cdot |\alpha|.$$

Finally, we put

$$\rho = \sqrt[3]{\frac{11}{4} \cdot \frac{1}{|1-\lambda|}} < 1,$$

and obtain

$$|\alpha_n| \le \rho^n |\alpha| \ \text{ for } \ n \ge 1,$$

hence

$$0 \leq \lim_{n\to\infty} |\alpha_n| \leq |\alpha| \lim_{n\to\infty} \rho^n = 0.$$

Thus we have shown (2.10).

Now we show (2.11).

We put

$$S_m^{(1)} = \sum_{n=1}^{m-1} n \left| \sum_{k=1}^{n+1} (b_{nk} - b_{n+1,k}) \right| \quad \text{and} \quad S_m^{(2)} = \sum_{n=m}^{\infty} n \left| \sum_{k=1}^{m} (b_{nk} - b_{n+1,k}) \right|.$$

We obtain for $m = 1$, $S_1^{(1)} = 0$ and by Part (2.10)

$$S_1^{(2)} = \sum_{n=1}^{\infty} n|b_{n1} - b_{n+1,1}| \leq |\alpha| \left(\sum_{n=1}^{\infty} n\rho^n + \sum_{n=1}^{\infty} n\rho^{n+1} \right) = |\alpha|(1+\rho)S,$$

where

$$S = \sum_{n=1}^{\infty} n\rho^n < \infty.$$

Furthermore, if $m \geq 2$, we obtain by (2.12), and since $b_{n,n+1} = 0$,

$$\begin{aligned}
S_m^{(1)} &= \sum_{n=1}^{m-1} n \left| \sum_{k=1}^{n+1} (b_{nk} - b_{n+1,k}) \right| \\
&= \sum_{n=1}^{m-1} n \left| b_{n1} - b_{n+1,1} + b_{n2} - b_{n+1,2} + \cdots + b_{n,n} - b_{n+1,n} + b_{n,n+1} - b_{n+1,n+1} \right| \\
&= \sum_{n=1}^{m-1} n|b_{n+1,1}| < |\alpha| \sum_{n=1}^{m-1} n\rho^{n+1}
\end{aligned}$$

and

$$\begin{aligned}
S_m^{(2)} &= \sum_{n=m}^{\infty} n \left| \sum_{k=1}^{m} (b_{nk} - b_{n+1,k}) \right| \\
&= \sum_{n=m}^{\infty} n|b_{n+1,1} - b_{n,m}| < |\alpha| \sum_{n=m}^{\infty} n\rho^{n+1} + \sum_{n=m}^{\infty} n|b_{n-m+1,1}| \\
&< |\alpha| \left(\sum_{n=m}^{\infty} n\rho^{n+1} + \sum_{n=1}^{\infty} (n+m)\rho^n \right),
\end{aligned}$$

and so

$$S_m^{(1)} + S_m^{(2)} = |\alpha| \left((\rho+1)S + m \sum_{n=1}^{\infty} \rho^n \right).$$

Consequently, we have

$$\sup_m c_m = \sup_m \frac{1}{m} \left(S_m^{(1)} + S_m^{(2)} \right) < \infty.$$

Thus we have also shown (2.11).

Now we show that the point spectrum of the operator Δ_i^3 on the Hahn space is equal to the empty set.

Theorem 4 *We have*

$$\sigma_p(\Delta_i^3, h) = \emptyset. \tag{2.13}$$

Proof We assume that λ is an eigenvalue of the operator Δ_i^3. Then there exist non–zero eigenvectors $x \in h$ such that $\Delta_i^3 x = \lambda x$, that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \cdots \\ -\frac{3}{2} & 1 & 0 & 0 & 0 & 0 \cdots \\ 1 & -\frac{3}{2} & 1 & 0 & 0 & 0 \cdots \\ -\frac{1}{4} & 1 & -\frac{3}{2} & 1 & 0 & 0 \cdots \\ 0 & -\frac{1}{4} & 1 & -\frac{3}{2} & 1 & 0 \cdots \\ 0 & 0 & -\frac{1}{4} & 1 & -\frac{3}{2} & 1 \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \\ \vdots \end{bmatrix}.$$

This yields

$$\begin{aligned} x_1 &= \lambda x_1, \\ -\frac{3}{2}x_1 + x_2 &= \lambda x_2, \\ x_1 - \frac{3}{2}x_2 + x_3 &= \lambda x_3, \\ -\frac{1}{4}x_1 + x_2 - \frac{3}{2}x_3 + x_4 &= \lambda x_4, \\ &\vdots \\ -\frac{1}{4}x_{n-3} + x_{n-2} - \frac{3}{2}x_{n-1} + x_n &= \lambda x_n, \\ &\vdots \end{aligned}$$

Let $k \in \mathbb{N}$ is the smallest index for which $x_k \neq 0$. Then $x_k = \lambda x_k$ implies $\lambda = 1$ and then $x_{k+1} = (-3/2)x_k + x_{k+1}$, that is, $x_k = 0$, a contradiction.

If $T \in B(h)$ given by a matrix A, that is, if $T(x) = Ax$ for all $x \in h$, then it is known [19, IV. 8, Problem 7, p. 233] that the adjoint operator $T^* : h^* \to h^*$ is defined by the transpose A^t of the matrix A.We note that the continuous dual h^* of h is isometrically isomorphic to ([13, Propostion 2.3]) σ_∞.

3 Roots of polynomials

We want to estimate the point spectrum of T^*. To be able to achieve this we need to study the roots of the polynomial

$$P(z) = z^3 - 4z^2 + 6z + 4(-1+\lambda) = 0. \tag{3.1}$$

We put $\lambda = 1 + re^{\mathrm{i}\theta}$ $\big(r = |\lambda - 1| \geq 0,\ \theta \in [0, 2\pi)\big)$. The roots of any polynomial are continuous functions of its coefficients; in our case they are a continuous function of the constant term $a_0 = 4re^{\mathrm{i}\theta}$ and in this problem we are only interested in the roots z_k of the polynomial

$$P(z) = z^3 - 4z^2 + 6z + 4re^{\mathrm{i}\theta}. \tag{3.2}$$

If $r = 0$, then $z_1 = 0$ and the other two roots are then the complex numbers $2 \pm \mathrm{i}\sqrt{2}$. If $\theta = 0$ or $\theta = \pi$, that is, $a_0 = \pm 4r$ real, then z_1 will be always real,

$$z_1 = z_1(a_0) = \frac{4}{3} + \frac{\sqrt[3]{3\sqrt{3}\sqrt{27a_0^2 + 176a_0 + 288} - 27a_0 - 88}}{3\sqrt[3]{2}} - \frac{2\sqrt[3]{2}}{3\sqrt[3]{3\sqrt{3}\sqrt{27a_0^2 + 176a_0 + 288} - 27a_0 - 88}},$$

and the other two roots are conjugate complex with absolute value greater than 1. For such a value z_1, which is a decreasing function with respect to a_0, we have $z_1(-3) = 1$ and $z_1(11) = -1$ (of course, $z_1(0) = 0$), that is, $|z_1| \leq 1$ if $-3 \leq a_0 \leq 11$ (for $\theta = 0$ we must have $r \leq 11/4$, and for $\theta = \pi$, $r \leq 3/4$).

For complex $a_0 = 4re^{\mathrm{i}\theta}$, the behaviour of the absolute value of the root z_1 is shown in the picture for $0 \leq \theta < 2\pi$, and for some characteristic values of r $(= 1/4, 1/2, 3/4)$, as well as for $r > 3/4$, where we see that root is out of the unit circle (Figure 1).

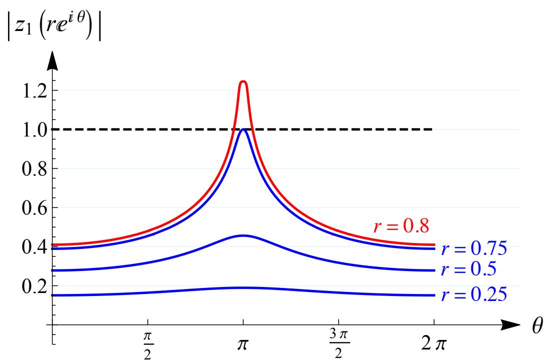

Fig. 1 Behaviour of the absolute value of the root z_1

Hence, if we search the ball in the λ-plane with center in 1 and where one root of the polynomial has absolute value less then 1, then it is a ball $|\lambda - 1| \leq 3/4$ (only for the value $\lambda = 1 - 3/4 = 1/4$ of the root $z_1 = z_1(4 \times 3/4e^{\mathrm{i}\pi}) = z_1(-3) = 1$).

But, the domain of the values of λ for which the root z_1 has absolute value < 1 is the interior of the curve coloured in red in Figure 2 (left). For any complex number λ on the boundary, the absolute values of roots are equal to 1. The red circle line corresponds to the equality $|z_1(-4(1-\lambda))| = 1$. The green circle in the same figure was earlier identified with its center in $(1,0)$: $|\lambda - 1| \leq 3/4$. The largest circle line with $|z_1| < 1$ is the blue circle line $|\lambda - 2| \leq 7/4$ in the same figure. There are two real points on the circle line where $|z_1| = 1$ with $\lambda = 1/4$ and $\lambda = 15/4$ corresponding to the free terms $a_0 = -3$ and $a_0 = 11$, that is, when $(r,\theta) = (3/4, \pi)$ and $(r,\theta) = (11/4, 0)$, respectively.

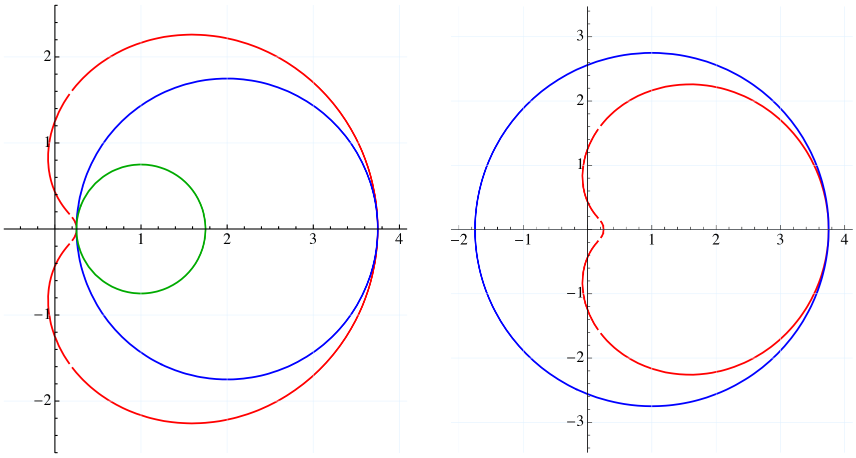

Fig. 2 λ–plane

We know from Theorem 3 that the inclusion in (2.8) holds, that is,

$$\sigma(\Delta_i^3, h) \subset \left\{ \lambda \in \mathbb{C} : |1-\lambda| \leq \frac{11}{4} \right\}.$$

Hence, we consider the interior of the blue circle and the exterior of the red curve which is similar to a cardioide in Figure 2 (right). Let us denote this region by D and let $\lambda = x + iy$. By the symmetry of our problem it is enough to consider, for instance, only the upper part, where $\mathrm{Im}(\lambda) = y > 0$, which we denote by D_+.

On the real line, one root is real

$$z_1(x) = \frac{1}{3}\left(4 + \sqrt[3]{2}\sqrt[3]{3\sqrt{3}\sqrt{27x^2 - 10x + 1} - 27x + 5} - \frac{2^{2/3}}{\sqrt[3]{3\sqrt{3}\sqrt{27x^2 - 10x + 1} - 27x + 5}}\right),$$

and the other two roots are conjugate complex with equal absolute values. The graphs $x \mapsto z_1(x)$ (red) and $x \mapsto |z_2(x)| = |\bar{z}_2(x)|$ (green) are given in Figure 3. We see that for $x \in [1/4, 15/4]$, we have $|z_1(x)| \leq 1$.The abolute value of the other two conjugate complex roots is greater than 1. What is the intersection of the red and green lines, that is, the solution of the equation $z_1(x) = |z_2(x)|$? We obtain by direct computation $x = 5/32$. Then

$$z_1(5/32) = \frac{3}{2}, \quad z_{2,3}(5/32) = \frac{5 \pm \mathrm{i}\sqrt{11}}{4},$$

with absolute value $3/2$.

Clearly, if $x < 5/32$, then the zero $z_1(x)$ is greater than the absolute value of the other two roots. For $x = 5/32$, all three roots have the same absolute value. The problem is in the interval $x \in (5/32, 1/4]$, where two of the conjuate complex roots have the same absolute value which is strictly greater than $z_1(x)$.

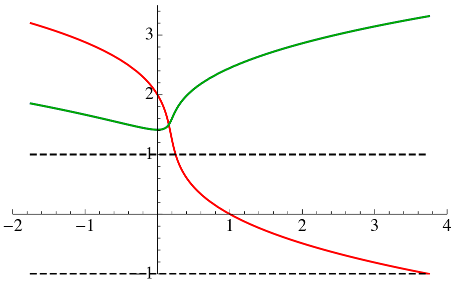

Fig. 3 Graphs of $x \mapsto z_1(x)$ (red) and $x \mapsto |z_2(x)| = |\bar{z}_2(x)|$ (green)

We have to complete our study with the case of $y > 0$, that is, when $\lambda \in D_+$. In this case, all three zeros are complex and the absolute value of one of them is always greater than that of the others. For fixed $x \in (-7/4, 15/4)$, we can see the behaviour of the absolute values of the roots (in three different colours) in the interval

$$y \in \left(0, \frac{1}{4}\sqrt{-16x^2 + 32x + 105}\right).$$

The graphs of the absolute values of the roots for different y for some typical values of x are represented in the next figures. Perhaps this could be proved in a simpler way without those graphs? Of course, in research, we can only make a conclusion that follows from numerical calculations, which is legitimate!

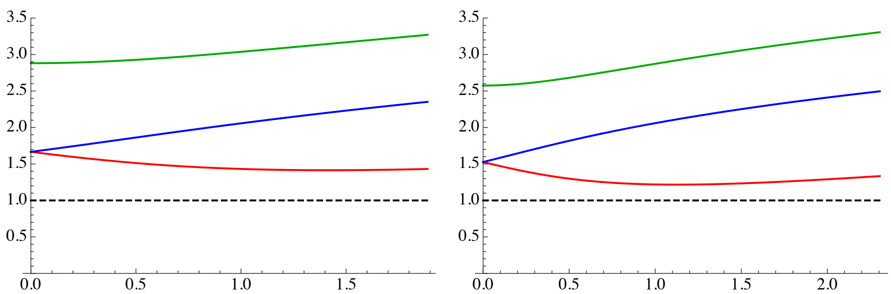

Fig. 4 Graphs for $x = -1$ (left) and $x = -0.5$ (right)

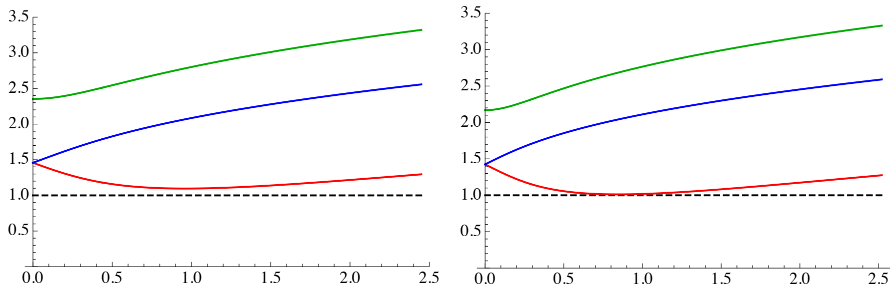

Fig. 5 Graphs for $x = -0.25$ (left) and $x = -0.1$ (right)

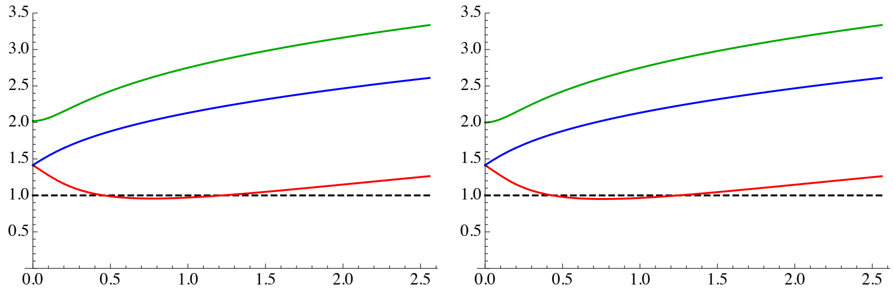

Fig. 6 Graphs for $x = -0.01$ (left) and $x = 0$ (right)

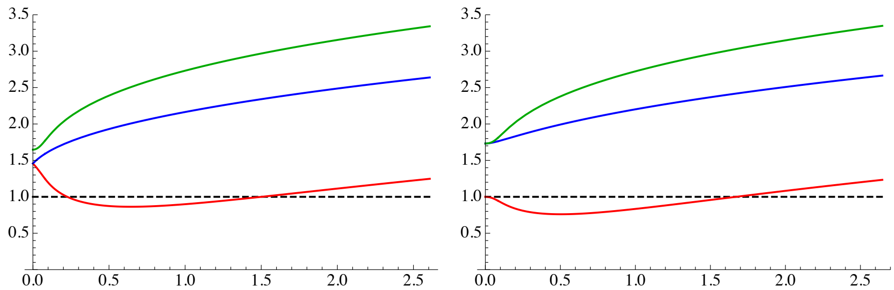

Fig. 7 Graphs for $x = 0.125$ (left) and $x = 0.25$ (right)

4 Some more results concerning the spectra

Let G denote the set bounded by the red curve in Figure 2 including the boundary.

Theorem 5 *The point spectrum of the $(\Delta_i^3)^*$ operator on h^* is*

$$G = \sigma_p((\Delta_i^3)^*, h^*). \tag{4.1}$$

Proof We assume that λ is an eigenvalue of the operator $(\Delta_i^3)^*$, then there exist a non–zero eigenvector $x \in \sigma_\infty$ such that $(\Delta_i^3)^* x = (\Delta_i^3)^T x = \lambda x$, where $(\Delta_i^3)^*$ is

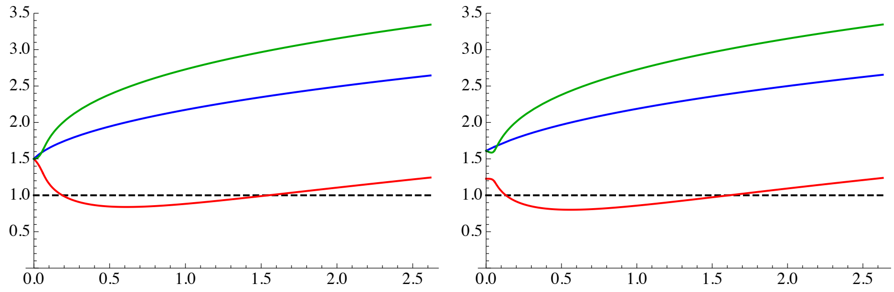

Fig. 8 Graphs for $x = 5/32$ (left) and $x = 13/64$ (right). We remark that here the "blue root" greater than the "green one" for small values of y, and this happens in a "hole" for $x \in (5/32, 1/4]$ (see the earlier situation on the real line)

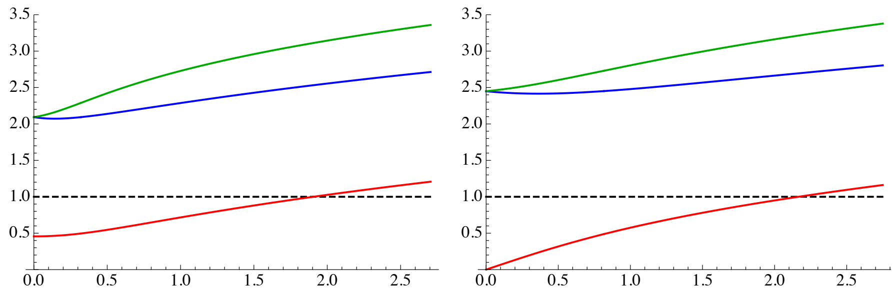

Fig. 9 Graphs for $x = 0.5$ (left) and $x = 1$ (right)

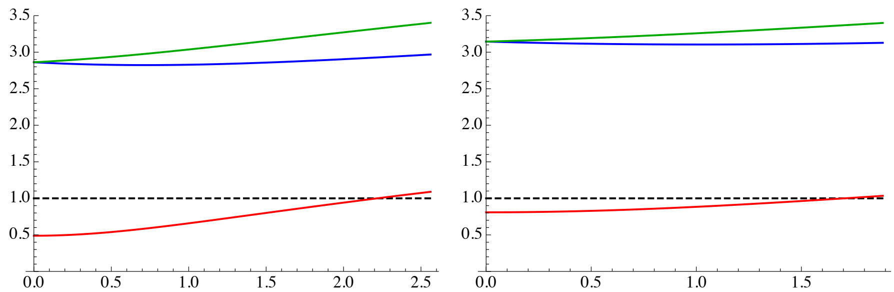

Fig. 10 Graphs for $x = 2$ (left) and $x = 3$ (right)

represented by the matrix

$$\begin{bmatrix} 1 & -\frac{3}{2} & 1 & -\frac{1}{4} & 0 & 0 & \cdots \\ 0 & 1 & -\frac{3}{2} & 1 & -\frac{1}{4} & 0 & \cdots \\ 0 & 0 & 1 & -\frac{3}{2} & 1 & -\frac{1}{4} & \cdots \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & -\frac{3}{2} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \\ \vdots \end{bmatrix}. \tag{4.2}$$

Then we have the following system of equations

$$\begin{aligned}
x_1 - \frac{3}{2}x_2 + x_3 - \frac{1}{4}x_4 &= \lambda x_1,\\
x_2 - \frac{3}{2}x_3 + x_4 - \frac{1}{4}x_5 &= \lambda x_2,\\
x_3 - \frac{3}{2}x_4 + x_5 - \frac{1}{4}x_6 &= \lambda x_3,\\
&\vdots\\
x_n - \frac{3}{2}x_{n+1} + x_{n+2} - \frac{1}{4}x_{n+3} &= \lambda x_n,\\
&\vdots
\end{aligned}$$

We have to solve the following difference equation

$$x_{n+3} - 4x_{n+2} + 6x_{n+1} - 4(1-\lambda)x_n, \;\; n \geq 1. \tag{4.3}$$

For this we consider the roots of the polynomial in (3.2).
If $\lambda \in G$, then we obtain from Section 3 that the polynomial in (3.2) has a root z_1 with $|z_1| \leq 1$. Now we put $x_n = z_1^n$ for each n. Then x_n satisfies the recursion formula (4.3) and it follows that

$$\frac{1}{n}\left|\sum_{k=1}^{n} x_k\right| \leq 1,$$

that is, $x = (x_n) \in \sigma_\infty$. This shows

$$G \subset \sigma_p((\Delta_i^3)^*, h^*). \tag{4.4}$$

To show the ocnverse inclusion of (4.4), we assume $\lambda \notin G$. The the absolute values of the roots z_1, z_2 and z_3 of our polynomial (3.2) are greater than 1. The general form of x_n in the recursion formula (4.3) is given by

$$x_n = c_1 z_1^n + c_2 z_2^n + c_3 z_3^n,$$

where c_1, c_2 and c_3 are complex constants. We write $z_k = \rho \exp(i\Theta_k)$ for $k = 1,2,3$ and obtain

$$\sum_{k=1}^{n} x_k = \sum_{j=1}^{3} c_j \sum_{k=1}^{n} z_j^k = \sum_{j=1}^{3} c_j \frac{z_j(1-z_j^n)}{1-z_j}.$$

Now we write $\xi = \max\{|z_1|, |z_2|, |z_3|\}$. We note

$$\lim_{n\to\infty} \frac{\xi^n}{n} = \infty,$$

thus

$$\sup_n \left|\frac{1}{n}\sum_{k=1}^{n} x_k\right| = \sup_n \frac{\xi^n}{n}\left|\sum_{k=1}^{n}\frac{x_k}{\xi^n}\right| = \sup_n \frac{\xi^n}{n}\left|\frac{1}{\xi^n}\sum_{j=1}^{3} c_j \frac{z_j(1-z_j^n)}{1-z_j}\right| = \infty.$$

Consequently we have $x \notin \sigma_\infty$.

We know from Theorems 3 and 5 that

$$G \subset \sigma(\Delta_i^3, h) \subset \left\{ \lambda \in \mathbb{C} : |1-\lambda| \leq \frac{11}{4} \right\}.$$

Now we establish the converse inclusion and obtain altogether the following result.

Theorem 6 *We have*

$$G = \sigma(\Delta_i^3, h). \tag{4.5}$$

Proof We assume $\lambda \notin G$. Then we have the recursion formula

$$b_{nk} = \frac{1}{1-\lambda} \left(\frac{1}{3} b_{n-1,k} - b_{n-2,k} + \frac{1}{4} b_{n-3,k} \right) \text{ for } k \leq n-1. \tag{4.6}$$

To solve this, we set $\alpha_n = b_{n,1}$ and consider its characteristic equation

$$\mu^3 = \frac{1}{1-\lambda} \left(\frac{3}{2}\mu^2 - \mu + \frac{1}{4} \right). \tag{4.7}$$

If all the roots μ_1, μ_2, μ_3 are distinct, then the solution of the homogeneous linear difference equation (4.7) is

$$\alpha_n = C_1 \mu_1^n + C_2 \mu_2^n + C_3 \mu_3^n. \tag{4.8}$$

If $\mu_1 = \mu_2 \neq \mu_2$, then we obtain

$$\alpha_n = C_1 \mu_1^n + C_2 n \mu_1^n + C_3 \mu_3^n, \tag{4.9}$$

and trivially, if $\mu_1 = \mu_2 = \mu_3$, then

$$\alpha_n = C_1 \mu_1^n + C_2 n \mu_1^n + C_3 n^2 \mu_1^n. \tag{4.10}$$

We note that μ_i is a root of (4.7) if and only if $\mu_i = 1/z_i$ for sme root z_i of (3.1). Since $\lambda \notin G$, it follows that $|\mu_i| = 1/|z_i| < 1$.

Now it is obvious that $B \in (h,h)$ in any of the cases (4.8), (4.9) and (4.10), and so $\lambda \notin \sigma(\Delta_i^3)$.

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