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A SPECIAL GAUSSIAN RULE FOR TRIGONOMETRIC POLYNOMIALS

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This paper is dedicated to Professor Themistocles M. Rassias.

Abstract. Abram Haimovich Turetzkii [*Uchenye Zapiski*, **1** (149) (1959), 31–55 (translation in English in *East J. Approx.* **11** (2005), 337–359)] considered interpolatory quadrature rules which have the following form $\int_0^{2\pi} f(x)w(x)dx \approx \sum_{\nu=0}^{2n} w_\nu f(x_\nu)$, and which are exact for all trigonometric polynomials of degree less than or equal to n. Maximal trigonometric degree of exactness of such quadratures is $2n$, and such kind of quadratures are known as *quadratures of Gaussian type* or *Gaussian quadratures for trigonometric polynomials*. In this paper we prove some interesting properties of a special Gaussian quadrature with respect to the weight function $w_m(x) = 1 + \sin mx$, where m is a positive integer.

1. Introduction

This paper is dedicated to Themistocles M. Rassias for our long and fruitful collaboration in mathematical research, including the writing of our book [4], published in 1994 by World Scientific Publishing Co., jointly with Professor Dragoslav S. Mitrinović.

Let the weight function $w(x)$ be integrable and nonnegative on the interval $[0, 2\pi)$, vanishing there only on a set of a measure zero.

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By $\mathcal{T}_n$ we denote the linear space of all trigonometric polynomials of degree less than or equal to n, and by $\mathcal{T}_n^{1/2}$ we denote the linear span of the following set $\{\cos(\nu+1/2)x, \sin(\nu+1/2)x, \nu = 0,1,\ldots,n\}$. Trigonometric functions from $\mathcal{T}_n^{1/2}$, i.e., trigonometric functions of the following form

$$\sum_{\nu=0}^{n}\left[c_\nu \cos\Big(\nu+\frac{1}{2}\Big)x + d_\nu \sin\Big(\nu+\frac{1}{2}\Big)x\right], \tag{1.1}$$

where $c_\nu, d_\nu \in \mathbb{R}$, $|c_n| + |d_n| \neq 0$, are known as trigonometric polynomials of semi–integer degree $n+1/2$ (see [5], [2]). This kind of trigonometric functions are important for construction of the Gaussian quadrature rules for trigonometric polynomials.

It is obvious that

$$A_{n+1/2}(x) = A\prod_{k=0}^{2n}\sin\frac{x-x_k}{2} \qquad (A \text{ is a non-zero constant}), \tag{1.2}$$

is trigonometric polynomial of semi–integer degree $n+1/2$. Also, every trigonometric polynomial of semi-integer degree $n+1/2$ of the form (1.1) can be represented in the form (1.2) (see [5, Lemma 1]).

For any positive integer n, the quadrature rule of the Gaussian type is the following one

$$\int_0^{2\pi} t(x)w(x)dx = \sum_{\nu=0}^{2n} w_\nu t(x_\nu), \quad t \in \mathcal{T}_{2n}, \tag{1.3}$$

where weights w_ν are given by

$$w_\nu = \int_0^{2\pi}\frac{A_{n+1/2}(x)}{2\sin\dfrac{x-x_\nu}{2}A'_{n+1/2}(x_\nu)}w(x)dx, \quad \nu = 0,1,\ldots,2n, \tag{1.4}$$

and nodes are zeros of $A_{n+1/2}(x)$, which is orthogonal on $[0,2\pi)$ with respect to the weight function $w(x)$ to every trigonometric polynomial of the semi-integer degree from $\mathcal{T}_{n-1}^{1/2}$ (see [5], [2]). It is known that such orthogonal trigonometric polynomial of semi–integer degree $A_{n+1/2}(x)$ has in $[0,2\pi)$ exactly $2n+1$ distinct simple zeros (see [5, Theorem 3]) and $A_{n+1/2}$ with given leading coefficients c_n and d_n is uniquely determined (see [5, §3.]). In [2] and [3] two choices of leading coefficients were considered. For the choice $c_n = 1$, $d_n = 0$, we denote an orthogonal trigonometric polynomial of semi-integer degree by $A^C_{n+1/2}$, and for the choice $c_n = 0$ and $d_n = 1$ by $A^S_{n+1/2}$.

The Gaussian quadrature (1.3) is not given uniquely since it is possible to use any orthogonal polynomial of semi-integer degree $A_{n+1/2} \in \mathcal{T}_n^{1/2}$. In [2] a numerical method for constructing the Gaussian quadratures using $A^C_{n+1/2}$ is given. That method is based on five–term recurrence relations for $A^C_{n+1/2}$ and $A^S_{n+1/2}$. The main problem in that method is the calculation of five–term recurrence coefficients. Explicit formulas for five–term recurrence coefficients for some weight functions are given in [3].

In this paper, in Section 2, we prove some interesting properties for Gaussian quadratures (1.3) with respect to the weight function $w_m(x) = 1+\sin mx$, $m \in \mathbb{N}$, and in Section 3 we give a numerical example.

2. Main results

As an example, the nodes x_ν and weight w_ν, $\nu = 0, 1, \ldots, 2n$, for the Gaussian quadrature for $n = 25$ and $w_{15}(x) = 1 + \sin 15x$ are given in [2]. In that case we have seen some kind of symmetry, since $w_\nu = w_{\nu+17j}$ and $x_{\nu+17j} = x_\nu + 2j\pi/3$, $j = 1, 2$, $\nu = 0, 1, \ldots, 16$. This symmetry is not an isolated case, namely Gaussian quadratures (1.3) with respect to the weight functions $w_m(x) = 1+\sin mx$, $m \in \mathbb{N}$, have interesting properties, which are presented in Theorem 2.1. If we want to construct a Gaussian quadrature rule with $2n_1 + 1$ nodes for the weight function $w_{m_1}(x) = 1 + \sin m_1 x$ where $\gcd(m_1, 2n_1 + 1) = d \neq 1$ we can obtain nodes and weights for such quadrature directly from nodes and weights of the Gaussian quadrature rule with $(2n_1 + 1)/d$ nodes with respect to the weight $w_{m_1/d}$. In a given example in [2] we have $n_1 = 25$, $m_1 = 15$ and $\gcd(15, 51) = 3$.

Theorem 2.1. *Let denote by x_ν, w_ν, $\nu = 1, \ldots, 2n + 1$, the nodes and weights for the Gaussian quadrature rule with respect to the weight function $w_m(x) = 1 + \sin mx$. Then*

$$\widehat{x}_{j(2n+1)+\nu} = \frac{x_\nu}{q} + \frac{2j\pi}{q}, \ \widehat{w}_{j(2n+1)+\nu} = \frac{w_\nu}{q},$$

for $j = 0, 1, \ldots, q - 1$, $\nu = 1, \ldots, 2n + 1$, are nodes and weights for the Gaussian quadrature rule with $(2n+1)q$ nodes with respect to the weight function $w_{mq}(x) = 1 + \sin mqx$, where q is an odd positive integer.

Proof. According to orthogonality conditions for trigonometric polynomials of semi–integer degree with respect to the weight function $w_m(x) = 1 + \sin mx$, we obtain

$$\int_0^{2\pi} A^C_{n+1/2}(x) \cos(k + 1/2)x(1 + \sin mx)dx = 0, \quad 0 \le k \le n - 1,$$

$$\int_0^{2\pi} A^C_{n+1/2}(x) \sin(k + 1/2)x(1 + \sin mx)dx = 0, \quad 0 \le k \le n - 1.$$

If we introduce $x := qx$, we have

$$\int_0^{2\pi/q} A^C_{n+1/2}(qx) \cos(k + 1/2)qx(1 + \sin mqx)dx = 0, \quad 0 \le k \le n - 1,$$

$$\int_0^{2\pi/q} A^C_{n+1/2}(qx) \sin(k + 1/2)qx(1 + \sin mqx)dx = 0, \quad 0 \le k \le n - 1.$$

Substituting $t = x + 2j\pi/q$, $j = 1, \ldots, q - 1$, we get

$$\int_{\frac{2j\pi}{q}}^{\frac{2(j+1)\pi}{q}} A^C_{n+1/2}(qt - 2j\pi) \cos(k + 1/2)(qt - 2j\pi)(1 + \sin m(qt - 2j\pi))dt = 0,$$

$$\int_{\frac{2j\pi}{q}}^{\frac{2(j+1)\pi}{q}} A^C_{n+1/2}(qt-2j\pi)\sin(k+1/2)(qt-2j\pi)(1+\sin m(qt-2j\pi))dt$$
$$= 0,$$

for $0 \le k \le n-1$, i.e.,

$$\int_{2j\pi/q}^{2(j+1)\pi/q} A^C_{n+1/2}(qt)\cos(k+1/2)qt(1+\sin mqt)dt = 0,$$
$$\int_{2j\pi/q}^{2(j+1)\pi/q} A^C_{n+1/2}(qt)\sin(k+1/2)qt(1+\sin mqt)dt = 0,$$

because of

$$\cos(k+1/2)(qt-2j\pi) = (-1)^j\cos(k+1/2)qt$$

and

$$\sin(k+1/2)(qt-2j\pi) = (-1)^j\sin(k+1/2)qt,$$

for $j = 1,\dots,q-1$.

Thus, we have

$$\int_0^{2\pi} A^C_{n+1/2}(qx)\cos(k+1/2)qx(1+\sin mqx)dx = 0, \quad 0 \le k \le n-1,$$
$$\int_0^{2\pi} A^C_{n+1/2}(qx)\sin(k+1/2)qx(1+\sin mqx)dx = 0, \quad 0 \le k \le n-1.$$

Let denote by $\mathcal{T}^q$ the linear span of the following functions

$$\cos(k+1/2)qx = \cos(kq+(q-1)/2+1/2)x, \quad k = 0,1,\dots,n,$$
$$\sin(k+1/2)qx = \sin(kq+(q-1)/2+1/2)x, \quad k = 0,1,\dots,n.$$

Obviously, $\mathcal{T}^q \subseteq \mathcal{T}^{1/2}_{nq+(q-1)/2}$. Using integrals

$$\int_0^{2\pi}\cos(\nu+1/2)x\cos(\ell+1/2)xw_{qm}(x)dx = \pi\delta_{\nu,\ell},$$
$$\int_0^{2\pi}\sin(\nu+1/2)x\sin(\ell+1/2)xw_{qm}(x)dx = \pi\delta_{\nu,\ell},$$
$$\int_0^{2\pi}\cos(\nu+1/2)x\sin(\ell+1/2)xw_{qm}(x)dx$$
$$= \frac{\pi}{2}(\delta_{\nu,\ell-qm}+\delta_{\nu,qm-\ell-1}-\delta_{\nu,\ell+qm}),$$

for $\nu = kq+(q-1)/2$, we see that integrals might not be equal zero only under condition $\ell = iq+(q-1)/2$, $i \in \mathbb{Z}$. With this argument we obtain the following orthogonality $\mathcal{T}^q \perp (\mathcal{T}^{1/2}_{nq+(q-1)/2} \ominus \mathcal{T}^q)$ with respect to the inner product

$$(f,q) = \int_0^{2\pi} f(x)g(x)w_{mq}(x)dx.$$

Then, $A^C_{n+1/2}(qx)$ is orthogonal on $[0,2\pi)$ to all trigonometric polynomials of semi–integer degree $t \in \mathcal{T}^{1/2}_{nq+(q-1)/2-1}$ with respect to the weight function $w_{mq}(x)$.

Since $A_{n+1/2}^C(x)$ can be represented as

$$A_{n+1/2}^C(x) = A \prod_{\nu=1}^{2n+1} \sin \frac{x - x_\nu}{2}, \quad A \neq 0,$$

we get

$$A_{n+1/2}^C(qx) = A \prod_{\nu=1}^{2n+1} \sin \frac{qx - x_\nu}{2} = A \prod_{\nu=1}^{2n+1} \sin \frac{q(x - x_\nu/q)}{2},$$

so it is obvious that $\widehat{x}_{j(2n+1)+\nu}$, $j = 0, 1, \ldots, q-1$, $\nu = 1, \ldots, 2n+1$, are zeros of $A_{n+1/2}^C(qx)$.

Formulas for weight coefficients $\widehat{w}_{j(2n+1)+\nu}$, $j = 0, 1, \ldots, q-1$, $\nu = 1, \ldots, 2n+1$, can be easily obtained using (1.4). □

3. Numerical example

We determine the nodes and weights for a Gaussian quadrature with 75 nodes and the weight function $w_{45}(x) = 1 + \sin 45x$.

Since $\gcd(45, 75) = 15$, according to Theorem 2.1, we can obtain parameters for this Gaussian quadrature directly from the nodes and weights of a Gaussian quadrature with 5 nodes and the weight function $w_3(x) = 1 + \sin 3x$.

The parameters x_ν and weight w_ν, $\nu = 0, 1, \ldots, 2n$, for $n = 2$ and $w(x) = 1 + \sin 3x$ are given in Table 1. (Numbers in parentheses indicate decimal exponents.) We calculate these parameters using a numerical method described in [2] with explicit formulas for five-term recurrence coefficients given in [3]. All computations are performed in double precision arithmetic (16 decimal digits mantissa) in Mathematica, using the corresponding software package described in [1].

Now it is easy to obtain the nodes $\widehat{x}_\nu$ and the weights $\widehat{w}_\nu$, $\nu = 0, 1, \ldots, 74$, of the Gaussian quadrature for the $w_{45}(x) = 1 + \sin 45x$. The nodes $\widehat{x}_\nu$, $\nu = 0, 1, \ldots, 4$, and the weights $\widehat{w}_k$, $k = 0, 1, \ldots, 74$, are given in Table 2. The other nodes x_k, $k = 5, 6, \ldots, 74$, are given by

$$\widehat{x}_{5j+\nu} = \widehat{x}_\nu + \frac{2j\pi}{15}, \quad j = 1, \ldots, 14, \ \nu = 0, 1, \ldots, 4.$$

Table 1. Nodes x_ν and weights w_ν, $\nu = 0, 1, \ldots, 4$, for $w_3(x) = 1 + \sin 3x$

ν	x_ν	w_ν
0	5.717322718269771(−1)	1.700722884785417
1	2.062284642241416	9.534684415662144(−1)
2	2.933402103642034	1.316125839231413
3	4.554306420567181	1.586274703648100
4	5.586237829671358	7.265934379484426(−1)

TABLE 2. Nodes $\widehat{x}_\nu$ and weights $\widehat{w}_{5j+\nu}$, $\nu = 0,1,\ldots,4$, $j = 0,1,\ldots,14$, for $n = 37$ and $w_{45}(x) = 1 + \sin 45x$

ν	$\widehat{x}_\nu$	$\widehat{w}_{5j+\nu}$, $j = 0,1,\ldots,14$
0	0.3811548478846514(−1)	0.1133815256523611
1	0.1374856428160944	0.6356456277108096(−1)
2	0.1955601402428023	0.8774172261542751(−1)
3	0.3036204280378120	0.1057516469098733
4	0.3724158553114239	0.4843956252989617(−1)

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