PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 1, January 1994

AN ESTIMATE FOR COEFFICIENTS OF POLYNOMIALS IN L^2-NORM

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(Communicated by Andrew M. Bruckner)

ABSTRACT. Let $\mathscr{P}_n$ be the class of algebraic polynomials $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ of degree at most n and $\|P\|_{d\sigma} = (\int_{\mathbf{R}} |P(x)|^2 \, d\sigma(x))^{1/2}$, where $d\sigma(x)$ is a nonnegative measure on $\mathbb{R}$. We determine the best constant in the inequality $|a_\nu| \le C_{n,\nu}(d\sigma)\|P\|_{d\sigma}$, for $\nu = n$ and $\nu = n-1$, when $P \in \mathscr{P}_n$ and such that $P(\xi_k) = 0$, $k = 1, \dots, m$. The case $d\sigma(t) = dt$ on $[-1, 1]$ and $P(1) = 0$ was studied by Tariq. In particular, we consider the cases when the measure $d\sigma(x)$ corresponds to the classical orthogonal polynomials on the real line $\mathbb{R}$.

1. Introduction

Let $\mathscr{P}_n$ be the class of algebraic polynomials $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ of degree at most n. The first inequality of the form $|a_\nu| \le C_{n,\nu}\|P\|$ was given by Markov [3]. Namely, if $\|P\| = \|P\|_\infty = \max_{-1\le x\le 1} |P(x)|$ and $T_n(x) = \sum_{\nu=0}^{n} t_{n,\nu} x^\nu$ denotes the nth Chebyshev polynomial of the first kind, Markov proved that

$$|a_\nu| \le \begin{cases} |t_{n,\nu}| \cdot \|P\|_\infty & \text{if } n-\nu \text{ is even}, \\ |t_{n-1,\nu}| \cdot \|P\|_\infty & \text{if } n-\nu \text{ is odd}. \end{cases} \tag{1.1}$$

For $\nu = n$, (1.1) reduces to the well-known Chebyshev inequality

$$|a_n| \le 2^{n-1}\|P\|_\infty. \tag{1.2}$$

Using a restriction on the polynomial class like $P(1) = 0$ or $P(-1) = 0$, Schur [6] found the following improvement of (1.2):

$$|a_n| \le 2^{n-1} \left(\cos \frac{\pi}{4n}\right)^{2n} \|P\|_\infty.$$

This result was extended by Rahman and Schmeisser [5] for polynomials with real coefficients, which have at most $n-1$ distinct zeros in $(-1, 1)$.

Received by the editors April 24, 1992.

1991 *Mathematics Subject Classification*. Primary 26C05, 26D05, 33C45, 41A44.

Key words and phrases. Algebraic polynomials, coefficient estimates, orthogonal polynomials, classical orthogonal polynomials, nonnegative measure, norm, extremal polynomial, best constant.

The first author's research was supported in part by the Science Fund of Serbia under grant 0401F.

Similarly in the L^2-norm, Tariq [8] improved the following result of Labelle [2]:

$$|a_n| \leq \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\left(\frac{2n+1}{2}\right)^{1/2}\left(\int_{-1}^{1}|P(x)|^2\,dx\right)^{1/2}.$$

Under restriction $P(1)=0$, Tariq [8] proved that

$$|a_n| \leq \frac{n}{n+1}\cdot\frac{(2n)!}{2^n(n!)^2}\left(\frac{2n+1}{2}\right)^{1/2}\left(\int_{-1}^{1}|P(x)|^2\,dx\right)^{1/2},$$

with equality case

$$P(x) = P_n(x) - \frac{1}{n^2}\sum_{\nu=0}^{n-1}(2\nu+1)P_\nu(x). \tag{1.3}$$

Also, he obtained that

$$|a_{n-1}| \leq \frac{(n^2+2)^{1/2}}{n+1}\cdot\frac{(2n-2)!}{2^{n-1}((n-1)!)^2}\left(\frac{2n-1}{2}\right)^{1/2}\|P\|_2, \tag{1.4}$$

with equality case

$$P(x) = \frac{2n+1}{n^2+2}P_n(x) - P_{n-1}(x) + \frac{1}{n^2+2}\sum_{\nu=0}^{n-2}(2\nu+1)P_\nu(x). \tag{1.5}$$

In the absence of the hypothesis $P(1)=0$, the factor $(n^2+2)^{1/2}/(n+1)$ appearing in the right-hand side of (1.4) is to be dropped.

In this paper we give a short proof of a more general problem involving the L^2-norm of polynomials with respect to a nonnegative measure on the real line $\mathbb{R}$. The standard Jacobi, the generalized Laguerre, and the Hermite measures are included.

2. Main results

Let $d\sigma(x)$ be a given nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_k = \int_{\mathbb{R}} x^k\,d\sigma(x)$, $k=0,1,\dots$, exist and are finite and $\mu_0>0$. In that case, there exists a unique set of orthonormal polynomials $\pi_k(\cdot)=\pi_k(\cdot\,;d\sigma)$, $k=0,1,\dots$, defined by

$$\pi_k(x) = b_k(d\sigma)x^k + c_k(d\sigma)x^{k-1} + \text{lower degree terms}, \qquad b_k(d\sigma)>0,$$
$$(\pi_k,\pi_m)=\delta_{km}, \qquad k,m\geq 0,$$

where

$$(f,g) = \int_{\mathbb{R}} f(x)\overline{g(x)}\,d\sigma(x) \qquad (f,g\in L^2(\mathbb{R})). \tag{2.1}$$

For $P\in\mathscr{P}_n$, we define

$$\|P\|_{d\sigma} = \sqrt{(P,P)} = \left(\int_{\mathbb{R}}|P(x)|^2\,d\sigma(x)\right)^{1/2}. \tag{2.2}$$

Also, for $\xi_k\in\mathbb{C}$, $k=1,\dots,m$, we define a restricted polynomial class

$$\mathscr{P}_n(\xi_1,\dots,\xi_m) = \{P\in\mathscr{P}_n \mid P(\xi_k)=0,\ k=1,\dots,m\} \qquad (0\leq m\leq n).$$

In the case $m=0$, this class of polynomials reduces to $\mathscr{P}_n$. The case $m=n$ is trivial. If $\xi_1=\cdots=\xi_k=\xi$ $(1\leq k\leq m)$, then the restriction on polynomials at the point $x=\xi$ becomes $P(\xi)=P'(\xi)=\cdots=P^{(k-1)}(\xi)=0$.

Theorem 2.1. *If* $P \in \mathscr{P}_n(\xi_1, \dots, \xi_m)$, *then*

$$|a_n| \le b_{n-m}(d\hat{\sigma})\|P\|_{d\sigma}, \tag{2.3}$$

where the measure $d\hat{\sigma}(x)$ *is given by*

$$d\hat{\sigma}(x) = \prod_{k=1}^{m} |x - \xi_k|^2 \, d\sigma(x) \tag{2.4}$$

and the norm $\|P\|_{d\sigma}$ *is defined by* (2.2). *Inequality* (2.3) *is sharp and becomes an equality if and only if* $P(x)$ *is a constant multiple of the polynomial*

$$\pi_{n-m}(x; d\hat{\sigma}) \prod_{k=1}^{m} (x - \xi_k). \tag{2.5}$$

Proof. At first we consider the inner product (2.1). Then the polynomial $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu \in \mathscr{P}_n$ can be represented in the form $P(x) = \sum_{\nu=0}^{n} \alpha_\nu \pi_\nu(x; d\sigma)$, where $\alpha_\nu = (P, \pi_\nu)$, $\nu = 0, 1, \dots, n$, and $a_n = \alpha_n b_n(d\sigma)$. Since

$$\|P\|_{d\sigma} = \left(\sum_{\nu=0}^{n} |\alpha_\nu|^2 \right)^{1/2} \ge |\alpha_n|,$$

we have a simple estimate

$$|a_n| \le b_n(d\sigma)\|P\|_{d\sigma}, \tag{2.6}$$

with equality if and only if $P(x) = A\pi_n(x; d\sigma)$, where A is an arbitrary constant.

Suppose now that $P \in \mathscr{P}_n(\xi_1, \dots, \xi_m)$. Then we can write

$$P(x) = Q(x) \prod_{k=1}^{m} (x - \xi_k), \tag{2.7}$$

where $Q \in \mathscr{P}_{n-m}$. Applying the inequality (2.6) with the measure $d\hat{\sigma}(x)$ given by (2.4) to the polynomial Q, we find

$$|a_n| \le b_{n-m}(d\hat{\sigma})\|Q\|_{d\hat{\sigma}},$$

because the leading coefficient of the polynomial Q is equal to a_n. Since

$$\|Q\|_{d\hat{\sigma}}^2 = \int_{\mathbb{R}} |Q(x)|^2 \, d\hat{\sigma}(x) = \int_{\mathbb{R}} |P(x)|^2 \, d\sigma(x) = \|P\|_{d\sigma}^2,$$

we obtain inequality (2.3), with equality if and only if $P(x)$ is a constant multiple of the polynomial (2.5). □

Theorem 2.2. *Let* $P \in \mathscr{P}_n(\xi_1, \dots, \xi_m)$, $s_m = \sum_{k=1}^{m} \xi_k$, *and let the measure* $d\hat{\sigma}(x)$ *and the norm* $\|P\|_{d\sigma}$ *be given by* (2.4) *and* (2.2), *respectively. Then*

$$|a_{n-1}| \le \sqrt{(\hat{c}_{n-m} - s_m \hat{b}_{n-m})^2 + \hat{b}_{n-m-1}^2}\, \|P\|_{d\sigma}, \tag{2.8}$$

where $\hat{b}_\nu = b_\nu(d\hat{\sigma})$ *and* $\hat{c}_\nu = c_\nu(d\hat{\sigma})$ *are the coefficients in the orthonormal polynomial* $\hat{\pi}_\nu(\cdot) = \pi_\nu(\cdot; d\hat{\sigma})$.

The extremal polynomial in (2.8) *is a constant multiple of the polynomial*

$$((\hat{c}_{n-m} - s_m \hat{b}_{n-m})\hat{\pi}_{n-m}(x) + \hat{b}_{n-m-1}\hat{\pi}_{n-m-1}(x)) \prod_{k=1}^{m} (x - \xi_k).$$

Proof. Similar to the proof of Theorem 2.1, we consider the inner product (2.1) and any polynomial $P \in \mathscr{P}_n$. Then we have

$$
\begin{aligned}
a_n &= \alpha_n b_n(d\sigma) = (P, b_n(d\sigma)\pi_n), \\
a_{n-1} &= \alpha_n c_n(d\sigma) + \alpha_{n-1} b_{n-1}(d\sigma) = (P, c_n(d\sigma)\pi_n + b_{n-1}(d\sigma)\pi_{n-1}),
\end{aligned} \tag{2.9}
$$

where $\pi_\nu(\cdot) = \pi_\nu(\cdot\,; d\sigma)$.

Using now (2.7), where $Q(x) = a'_{n-m}x^{n-m} + a'_{n-m-1}x^{n-m-1} + \cdots$, we find that

$$
a'_{n-m} = a_n, \qquad a'_{n-m-1} = a_{n-1} + s_m a_n, \tag{2.10}
$$

where $s_m = \sum_{k=1}^m \xi_k$. Now, the corresponding equalities (2.9) for polynomial Q in the measure $d\hat{\sigma}$, defined by (2.5), become

$$
\begin{aligned}
a'_{n-m} &= (Q, \hat{b}_{n-m}\hat{\pi}_{n-m}), \\
a'_{n-m-1} &= (Q, \hat{c}_{n-m}\hat{\pi}_{n-m} + \hat{b}_{n-m-1}\hat{\pi}_{n-m-1}),
\end{aligned}
$$

where we put $\hat{\pi}_\nu(\cdot) = \pi_\nu(\cdot, d\hat{\sigma})$.

According to (2.10), we obtain $a_{n-1} = a'_{n-m-1} - s_m a'_{n-m} = (Q, W_{n-m})$, where

$$
W_{n-m}(x) = (\hat{c}_{n-m} - s_m\hat{b}_{n-m})\hat{\pi}_{n-m}(x) + \hat{b}_{n-m-1}\hat{\pi}_{n-m-1}(x).
$$

Then, using the Cauchy inequality, we get

$$
|a_{n-1}| \le C_{n,n-1}\|Q\|_{d\hat{\sigma}} = C_{n,n-1}\|P\|_{d\sigma},
$$

where

$$
C_{n,n-1} = \|W_{n-m}\|_{d\hat{\sigma}} = \sqrt{(\hat{c}_{n-m} - s_m\hat{b}_{n-m})^2 + \hat{b}^2_{n-m-1}}.
$$

The extremal polynomial is $x \mapsto W_{n-m}(x)\prod_{k=1}^m (x - \xi_k)$. $\square$

In the next section we consider examples with the measures of the classical orthogonal polynomials (Jacobi, generalized Laguerre, and Hermite polynomials).

3. Special cases

At first we observe the following Jacobi case: $d\sigma(x) = (1-x)^\alpha(1+x)^\beta\,dx$, $\alpha, \beta > -1$. Let $\{P_n^{(\alpha,\beta)}\}$ be the sequence of the Jacobi polynomials orthogonal with respect to the measure $d\sigma(x)$ on $(-1, 1)$. For such polynomials we have (cf. Szegö [7])

$$
(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n [(1-x)^{n+\alpha}(1+x)^{n+\beta}].
$$

Their leading coefficients are given by

$$
k_n^{(\alpha,\beta)} = \frac{(n+\alpha+\beta+1)_n}{2^n n!} \qquad (P_n^{(\alpha,\beta)}(x) = k_n^{(\alpha,\beta)}x^n + \text{lower degree terms}).
$$

The leading coefficients in the corresponding orthonormal polynomials are given by

$$b_n(d\sigma) = b_n^{(\alpha,\beta)} = \frac{(n+\alpha+\beta+1)_n}{2^n n!}\left(\frac{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right)^{1/2}.$$

Also, we need the coefficients

$$c_n(d\sigma) = c_n^{(\alpha,\beta)} = r_n^{(\alpha,\beta)} b_n^{(\alpha,\beta)}, \quad v_n^{(\alpha,\beta)} = \frac{b_{n-1}^{(\alpha,\beta)}}{b_n^{(\alpha,\beta)}}, \quad \tau_n^{(\alpha,\beta)} = \frac{k_n^{(\alpha,\beta)}}{k_{n-1}^{(\alpha,\beta)}}(v_n^{(\alpha,\beta)})^2,$$

where

$$r_n^{(\alpha,\beta)} = \frac{(\alpha-\beta)n}{2n+\alpha+\beta}, \qquad v_n^{(\alpha,\beta)} = \left(\frac{4n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2((2n+\alpha+\beta)^2-1)}\right)^{1/2},$$

and

$$\tau_n^{(\alpha,\beta)} = \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.$$

Corollary 3.1. *Under restrictions* $P^{(i)}(1) = 0$ $(i = 0, \dots, k-1)$ *and* $P^{(i)}(-1) = 0$ $(i = 0, \dots, m-k-1)$, *we have that*

$$|a_n| \le b_{n-m}^{(\alpha+2k,\beta+2m-2k)}\|P\|_{d\sigma}, \tag{3.1}$$

with equality if and only if $P(x)$ *is a constant multiple of the polynomial*

$$(x-1)^k(x+1)^{m-k}P_{n-m}^{(\alpha+2k,\beta+2m-2k)}(x).$$

Proof. Since the restrictions on polynomials are given only in the points $x = 1$ and $x = -1$, the new measure $d\hat{\sigma}(x)$ is again the Jacobi measure

$$d\hat{\sigma}(x) = (1-x)^{\alpha+2k}(1+x)^{\beta+2m-2k}\,dx,$$

so the result follows immediately from Theorem 2.1. □

Similarly, from Theorem 2.2 follows

Corollary 3.2. *Under the same restrictions as in Corollary* 3.1, *we have that*

$$|a_{n-1}| \le \hat{b}_{n-m}\sqrt{(\hat{r}_{n-m}-s_m)^2+\hat{v}_{n-m}^2}\,\|P\|_{d\sigma}, \tag{3.2}$$

where

$$\hat{b}_{n-m} = b_{n-m}^{(\alpha+2k,\beta+2m-2k)}, \quad \hat{r}_{n-m} = r_{n-m}^{(\alpha+2k,\beta+2m-2k)}, \quad \hat{v}_{n-m} = v_{n-m}^{(\alpha+2k,\beta+2m-2k)}.$$

The equality is attained in (3.2) *if and only if* $P(x)$ *is a constant multiple of the polynomial*

$$(x-1)^k(x+1)^{m-k}((\hat{r}_{n-m}-s_m)P_{n-m}^{(\alpha+2k,\beta+2m-2k)}(x)+\hat{\tau}_{n-m}P_{n-m-1}^{(\alpha+2k,\beta+2m-2k)}(x)),$$

where $\hat{\tau}_{n-m} = \tau_{n-m}^{(\alpha+2k,\beta+2m-2k)}$.

The case when $\alpha = \beta = 0$ and $m = k = 1$ (i.e., only with restriction $P(1) = 0$) was investigated by Tariq [8]. In that case, the best constant in (3.1) is $b_{n-1}^{(2,0)}$, and the extremal polynomial is given by $x \mapsto (x-1)P_{n-1}^{(2,0)}(x)$, which is equivalent to (1.3). Since

$$\hat{r}_{n-1} = r_{n-1}^{(2,0)} = 1 - \frac{1}{n}, \qquad \hat{v}_{n-1} = v_{n-1}^{(2,0)} = \frac{n^2-1}{n\sqrt{4n^2-1}},$$
$$\hat{\tau}_{n-1} = \tau_{n-1}^{(2,0)} = \frac{n^2-1}{n(2n+1)},$$

inequality (3.2) reduces then to (1.4), with an extremal polynomial

$$x \mapsto (x-1)\left(P_{n-1}^{(2,0)}(x) - \frac{n^2-1}{2n+1}P_{n-2}^{(2,0)}(x)\right),$$

which is equivalent to (1.5) up to a constant factor.

In a general case for polynomials $P \in \mathscr{P}_n(\xi_1, \dots, \xi_m)$, the problem reduces to the generalized Jacobi measure

$$d\hat{\sigma}(x) = (1-x)^\alpha(1+x)^\beta \prod_{k=1}^{m} |x-\xi_k|^2.$$

Consider now the generalized Laguerre measure $d\sigma(x) = x^\alpha e^{-x}\,dx$, $\alpha > -1$, on $(0, +\infty)$. With $L_n^{(\alpha)}(x)$ we denote the generalized Laguerre polynomial. The leading coefficient $b_n(d\sigma) = b_n^{(\alpha)}$ in the corresponding orthonormal polynomial is given by $b_n^{(\alpha)} = 1/\sqrt{n!\Gamma(n+\alpha+1)}$.

As a direct corollary of Theorem 2.1 we have

Corollary 3.3. *Under restriction* $P^{(i)}(0) = 0$ $(i = 0, \dots, m-1)$, *we have that*

$$|a_n| \le \frac{\|P\|_{d\sigma}}{\sqrt{(n-m)!\Gamma(n+\alpha+m+1)}},$$

with equality if and only if $P(x) = Ax^m L_{n-m}^{(\alpha+2m)}(x)$, *where* A *is an arbitrary constant.*

Similar to the above, in the Hermite case $d\sigma(x) = e^{-x^2}\,dx$ on $(-\infty, +\infty)$, the problem reduces to the generalized Hermite measure

$$d\hat{\sigma}(x) = e^{-x^2} \prod_{k=1}^{m} |x-\xi_k|^2\,dx.$$

Some considerations on polynomials under the restriction $LP = 0$, where L is a given functional from $\mathscr{P}_n$ to $\mathbb{C}$, was given by Milovanović and Marinković [4] using a method of Giroux and Rahman [1].

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