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Trigonometric orthogonal systems and quadrature formulae[☆]

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ABSTRACT

Quadrature rules with maximal even trigonometric degree of exactness are considered. We give a brief historical survey on such quadrature rules. Special attention is given on an approach given by Turetzkii [A.H. Turetzkii, On quadrature formulae that are exact for trigonometric polynomials, East J. Approx. 11 (3) (2005) 337–359. Translation in English from Uchenye Zapiski, Vypusk 1 (149). Seria Math. Theory of Functions, Collection of papers, Izdatel'stvo Belgosuniversiteta imeni V.I. Lenina, Minsk, 1959, pp. 31–54]. The main part of the topic is orthogonal trigonometric systems on $[0, 2\pi)$ (or on $[-\pi, \pi)$) with respect to some weight functions $w(x)$. We prove that the so-called orthogonal trigonometric polynomials of semi-integer degree satisfy a five-term recurrence relation. In particular, we study some cases with symmetric weight functions. Also, we present a numerical method for constructing the corresponding quadratures of Gaussian type. Finally, we give some numerical examples. Also, we compare our method with other available methods.

1. Preliminaries

Let the weight function $w(x)$ be an integrable and nonnegative function on the interval $[0, 2\pi)$, vanishing there only on a set of a measure zero. Let us denote by $\mathcal{T}_n$, $n \in \mathbb{N}_0$, the linear space of all trigonometric polynomials of degree less than or equal to n.

Definition 1.1. We say that a quadrature rule of the following form

$$\int_0^{2\pi} f(x)w(x)\mathrm{d}x = \sum_{\nu=0}^{n} w_\nu f(x_\nu) + R_n(f),$$

where $0 \le x_0 < x_1 < \cdots < x_n < 2\pi$, has trigonometric degree of exactness equal to d if $R_n(f) = 0$ for all $f \in \mathcal{T}_d$ and there exists $g \in \mathcal{T}_{d+1}$ such that $R_n(g) \neq 0$.

We are interested in quadrature rules with maximal trigonometric degree of exactness. Such quadrature rules are known as *quadrature rules of Gaussian type*. Maximal trigonometric degree of exactness for quadrature rule with $n + 1$ nodes is n. These quadrature rules have application in numerical integration of 2π-periodic functions.

We start with a brief historical survey of available approaches for the construction of quadrature rules with maximal trigonometric degree of exactness.

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The first results on quadrature rules with maximal trigonometric degree of exactness were given in the case of an even trigonometric degree of exactness n, i.e., in the case of an odd number of nodes, in 1959 by Abram Haimovich Turetzkii (see [1]). Using his approach, nodes of quadrature rules with maximal trigonometric degree of exactness are zeros from $[0, 2\pi)$ of the so-called orthogonal trigonometric polynomial of semi-integer degree $n/2 + 1/2$ with respect to weight function w on $[0, 2\pi)$. Turetzkii's approach is described in detail in Section 2. Quadrature rules with the highest trigonometric degree of exactness for the Lebesgue measure, giving rise to the Trapezoidal Rule, were considered for the first time in [2, pp. 73–74].

Also, for an even trigonometric degree of exactness n, an approach based on ideal theory (given in [3] for algebraic cubature rules) can be applied. Using this approach, to obtain nodes of wanted quadrature rules one has to construct two quasi-orthogonal trigonometric polynomials of degree $n/2 + 1$ (orthogonal on $\mathcal{T}_{n/2-1}$), and compute $n + 1$ common zeros of these trigonometric polynomials.

A quite different approach was given by Ivan Petrovich Mysovskikh in [4,5]. He considered approximation of the following integrals

$$I(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)w(x)\mathrm{d}x,$$

by a quadrature rule like $\sum_{j=0}^{n} C_j f(x_j)$, which is exact for all $f \in \mathcal{T}_n$ for nonnegative integer n. His approach is based on reproducing kernel $S_n(a, z)$, where $a = \mathrm{e}^{\mathrm{i}x_1}$ is prescribed and $z = \mathrm{e}^{\mathrm{i}x}$. Reproducing kernel is given as follows

$$S_n(a, z) = -\frac{1}{\overline{D}_n}\begin{vmatrix} d_0 & d_{-1} & \cdots & d_{-n} & 1 \\ d_1 & d_0 & \cdots & d_{-n+1} & \overline{a} \\ \vdots & & & & \\ d_n & d_{n-1} & \cdots & d_0 & \overline{a}^n \\ 1 & z & \cdots & z^n & 0 \end{vmatrix}, \qquad \overline{D}_n = \begin{vmatrix} d_0 & d_{-1} & \cdots & d_{-n} \\ d_1 & d_0 & \cdots & d_{-n+1} \\ \vdots & & & \\ d_n & d_{n-1} & \cdots & d_0 \end{vmatrix}, \tag{1.1}$$

where d_k, $k = -n, \ldots, n$ are moments given by

$$d_k = \frac{1}{2\pi}\int_0^{2\pi} \overline{z}^k w(x)\mathrm{d}x, \quad z = \mathrm{e}^{\mathrm{i}x}, \qquad d_{-k} = \overline{d_k}, \quad k = 0, 1, \ldots, n. \tag{1.2}$$

Nodes of a quadrature rule with maximal trigonometric degree of exactness are determined by zeros of the reproducing kernel $S_n(a, z)$, given by (1.1), which is algebraic polynomial of degree n. In [4,5] the reproducing kernel is generated in expanded form, and coefficients of this polynomial are given as the quotient of two determinants of order $n + 1$.

Results given by Mysovskikh have great theoretical importance, but they are not suitable for numerical calculations because it is necessary to calculate $n + 2$ determinants of order $n + 1$, and the obtained polynomials are in expanded form.

An odd trigonometric degree of exactness n was considered in [6], where it was shown that nodes of quadrature rules with maximal trigonometric degree of exactness can be obtained as zeros of the so-called bi-orthogonal trigonometric polynomials (see also [7]). In [8] the paper [6] is completed by introducing the corresponding technical modifications in order to have a unified notation both for the odd and even degree of exactness. In [9] a system of bi-orthogonal trigonometric polynomials in Szegő's sense [7] was considered, by applying the Gram–Schmidt orthogonalization process to the trigonometric system $\{\sin \nu x, \cos \nu x\}_{\nu=0}^{n}$. Also, a connection with orthogonal polynomials on the unit circle was given.

It is known that so called n-point Szegő quadrature rules (see e.g., [10–16]) of the form

$$S_\tau(f) = \sum_{\nu=1}^{n} w_\nu f(\lambda_\nu),$$

where all weights w_ν, $\nu = 1, \ldots, n$, are positive and nodes λ_ν, $\nu = 1, \ldots, n$, are distinct and all lie on the unit circle, are characterized by the property that

$$S_\tau(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}t})w(t)\mathrm{d}t, \quad \text{for all } f \in \Lambda_{-(n-1),n-1},$$

where $w(x)$ is an integrable and nonnegative function on the interval $[-\pi, \pi)$, vanishing there only on a set of a measure zero, and $\Lambda_{-(n-1),n-1}$ denotes the set of Laurent polynomials

$$L_{n-1}(z) = \sum_{k=-(n-1)}^{n-1} c_k z^k, \quad c_k \in \mathbb{C}$$

of degree at most $n-1$. A Laurent polynomial $L_{n-1}(z)$ with $z = \mathrm{e}^{\mathrm{i}t}$ can be expressed as a trigonometric polynomial of degree at most $n - 1$. So, it follows that n-point Szegő quadrature rules integrate exactly all trigonometric polynomials of degree at most $n - 1$. The nodes of such quadrature rules are zeros of so called para-orthogonal polynomials (see [14,12,10,11,17,8] for details).

In this paper our attention is restricted to the case of quadrature rules with an even trigonometric degree of exactness, based on Turetzkii's approach. We develop this approach as an alternative method to Szegő quadratures for the construction of the mentioned quadrature rules. We investigate trigonometric orthogonal systems and corresponding quadratures of Gaussian type. The paper is organized as follows. In the Section 2 we present Turetzkii's approach, give a useful simple modification of Turetzkii's results, as well as a representation of trigonometric polynomials of semi-integer degree in terms of self-inversive algebraic polynomials. Considerations on trigonometric orthogonal systems are presented in Section 3, including some five-term recurrence relations. Section 4 is devoted to the case with symmetric weight functions. A numerical method for constructing quadratures of Gaussian type is proposed in Section 5. Finally, in Section 6 we give some numerical examples and compare our method with other methods.

2. Introduction

Let the weight function $w(x)$ be an integrable and nonnegative function on the interval $[0, 2\pi)$, vanishing there only on a set of a measure zero, and let x_ν, $\nu = 0, 1, \ldots, 2n$, be distinct points in $[0, 2\pi)$. Turetzkii in [1] considered an interpolatory quadrature rule of the form

$$\int_0^{2\pi} t(x)w(x)\mathrm{d}x = \sum_{\nu=0}^{2n} w_\nu t(x_\nu), \quad t \in \mathcal{T}_n. \tag{2.1}$$

Such a quadrature rule can be obtained from the trigonometric interpolation polynomials (cf. [18,19])

$$t_n(x) = \sum_{\nu=0}^{2n} t(x_\nu)\ell_\nu(x), \tag{2.2}$$

where

$$\ell_\nu(x) = \prod_{k=0,k\neq\nu}^{2n} \frac{\sin\frac{x-x_k}{2}}{\sin\frac{x_\nu-x_k}{2}} = \frac{A_{n+1/2}(x)}{2\sin\frac{x-x_\nu}{2}A'_{n+1/2}(x_\nu)}$$

and

$$A_{n+1/2}(x) = A\prod_{k=0}^{2n}\sin\frac{x-x_k}{2} \quad (A \text{ is a non-zero constant}). \tag{2.3}$$

Multiplying (2.2) with $w(x)$ and integrating over $[0, 2\pi)$, we obtain that the weights in the quadrature rule (2.1) are given by

$$w_\nu = \int_0^{2\pi} \ell_\nu(x)w(x)\mathrm{d}x = \int_0^{2\pi} \frac{A_{n+1/2}(x)}{2\sin\frac{x-x_\nu}{2}A'_{n+1/2}(x_\nu)}w(x)\mathrm{d}x. \tag{2.4}$$

If the nodes x_ν, $\nu = 0, 1, \ldots, 2n$, are not specified in advance, one can try to find them such that the quadrature rule (2.1) is exact for all trigonometric polynomials $t \in \mathcal{T}_{2n}$, i.e., such that quadrature rule (2.1) has trigonometric degree of exactness equal to $2n$.

In order to formulate and prove his result, Turetzkii [1] considered the so-called *trigonometric polynomials of semi-integer degree* $n + 1/2$, i.e., trigonometric functions of the form

$$\sum_{\nu=0}^{n}\left[c_\nu\cos\left(\nu+\frac{1}{2}\right)x + d_\nu\sin\left(\nu+\frac{1}{2}\right)x\right], \tag{2.5}$$

where $c_\nu, d_\nu \in \mathbb{R}$, $|c_n| + |d_n| \neq 0$. Evidently, $A_{n+1/2}(x)$ in (2.3) is a trigonometric polynomial of semi-integer degree $n+1/2$. To the contrary, every trigonometric polynomial of semi-integer degree $n + 1/2$ of the form (2.5) can be represented in the form (2.3) (see [1, Lemma 1]). As it can be seen, trigonometric polynomials of semi-integer degree are defined to be from the linear span $\mathcal{T}_n^{1/2}$ of the set $\{\cos(\nu + 1/2)x, \sin(\nu + 1/2)x, \nu = 0, 1, \ldots, n\}$. Note that the dimension of $\mathcal{T}_n^{1/2}$ is $2n + 2$, while the dimension of $\mathcal{T}_n$ is $2n + 1$, since $\mathcal{T}_n$ is the span of the set $\{1, \cos x, \sin x, \ldots, \cos nx, \sin nx\}$.

For a given trigonometric polynomial of semi-integer degree $n + 1/2$

$$A_{n+1/2}(x) = \sum_{\nu=0}^{n}\left[c_\nu\cos\left(\nu+\frac{1}{2}\right)x + d_\nu\sin\left(\nu+\frac{1}{2}\right)\right], \quad |c_n| + |d_n| \neq 0,$$

every polynomial $t_{2n} \in \mathcal{T}_{2n}$ can be uniquely represented in the form

$$t_{2n}(x) = A_{n+1/2}(x)B_{n-1/2}(x) + R_n(x),$$

where $B_{n-1/2} \in \mathcal{T}_{n-1}^{1/2}$ and $R_n \in \mathcal{T}_n$. Using this auxiliary result [1, Lemma 2], Turetzkii [1] proved the following statement:

Theorem 2.1. *The quadrature formula (2.1), with coefficients w_ν, $\nu = 0, 1, \dots, 2n$, determined by (2.4), is of Gaussian type, i.e., it is exact for every $t \in \mathcal{T}_{2n}$, if and only if the nodes x_ν $(\in [0, 2\pi))$, $\nu = 0, 1, \dots, 2n$, are zeros of $A_{n+1/2}(x)$, which is orthogonal on $[0, 2\pi)$ with respect to the weight function $w(x)$ to every trigonometric polynomial of the semi-integer degree from $\mathcal{T}_{n-1}^{1/2}$.*

Remark 2.1. Some particular results connected with this problem can be found in [20] (for the constant weight $w(x) = 1$) and [21] (for symmetric weight functions with numerical results only for the weight $w(x) = 1$). Also, some results for a π-periodic weight function w on $(0, 4\pi)$ can be found in [22], with more details only for the weights $w(x) = \sin^2 x$, $w(x) = \cos^2(x)$ and $w(x) = 1$.

Remark 2.2. It is well known that for a quadrature rule with the maximal algebraic degree of exactness (Gaussian formula), its nodes are the zeros of the corresponding orthogonal (algebraic) polynomial. In the case of a quadrature with an odd maximal trigonometric degree of exactness the nodes are zeros of orthogonal trigonometric polynomials, but for an even maximal trigonometric degree of exactness the nodes are not zeros of the orthogonal trigonometric polynomial, but zeros of the orthogonal trigonometric polynomial of semi-integer degree.

Thus, in order to obtain a quadrature rule with the maximal degree in a subspace of algebraic polynomials one must consider orthogonality in the same subspace. A similar situation is with the quadrature rule with an odd maximal trigonometric degree of exactness, but in the case of an even maximal trigonometric degree of exactness one must consider the orthogonality in the subspace of trigonometric polynomials of semi-integer degree. This is not an isolated case, e.g., in a more general case of Müntz systems, the nodes of a Gaussian quadrature rule are not zeros of the corresponding orthogonal Müntz polynomial (cf. [23]).

Taking $t(x) = \left(A_{n+1/2}(x)/\sin(x - x_k)/2\right)^2$ $(\in \mathcal{T}_{2n})$ in (2.1), it is clear that the weights w_k are positive for each $k = 0, 1, \dots, 2n$ (see [1, Theorem 2]).

The point is that Theorem 2.1 requires that the trigonometric polynomial of a semi-integer degree $A_{n+1/2}$ has to be orthogonal to every element of $\mathcal{T}_{n-1}^{1/2}$, i.e., it must be

$$\int_0^{2\pi} A_{n+1/2}(x)t(x)w(x)\mathrm{d}x = 0, \quad t \in \mathcal{T}_{n-1}^{1/2}.$$

Since the dimension of $\mathcal{T}_{n-1}^{1/2}$ is $2n$ and $A_{n+1/2}$ has $2n+2$ coefficients, it can be seen that $A_{n+1/2}$ is not determined uniquely up to a multiplication constant, as in the case of algebraic polynomials. Rather, $A_{n+1/2}$ has two free constants we choose those to be c_n and d_n. The trigonometric polynomial $A_{n+1/2}$, which is orthogonal on $[0, 2\pi)$ with respect to the weight function $w(x)$ to every trigonometric polynomial of a semi-integer degree less than or equal to $n - 1/2$, with given leading coefficients c_n and d_n, is uniquely determined (see [1, Section 3]). Obviously, we cannot choose $c_n = d_n = 0$, since in that case we have not a polynomial of degree $n + 1/2$, but $n - 1/2$.

For the special choices $c_n = 1$, $d_n = 0$ and $c_n = 0$, $d_n = 1$ we denote the orthogonal polynomials of semi-integer degree by $A_{n+1/2}^C$ and $A_{n+1/2}^S$, respectively. This notation is rather natural, since $A_{n+1/2}^C$ and $A_{n+1/2}^S$ have the leading terms $\cos(n + 1/2)x$ and $\sin(n + 1/2)x$, respectively. Then every $A_{n+1/2} \in \mathcal{T}_n^{1/2}$ is a linear combination of $A_{n+1/2}^C$ and $A_{n+1/2}^S$. This also means that we are free to use zeros from $[0, 2\pi)$ of any orthogonal polynomial of semi-integer degree $A_{n+1/2} \in \mathcal{T}_n^{1/2}$ in the quadrature rule (2.1), i.e., the quadrature rule of Gaussian type is not given uniquely, due to the fact it has $4n + 2$ parameters and integrates exactly all trigonometric polynomials in the linear space $\mathcal{T}_{2n}$ of dimension $4n + 1$.

There is a simple modification of Theorem 2.1 dealing with the translations of the interval $[0, 2\pi)$.

Corollary 2.1. *For $L \in \mathbb{R}$, the quadrature formula*

$$\int_L^{2\pi+L} t(x)\widetilde{w}(x)\mathrm{d}x = \sum_{\nu=0}^{2n} \widetilde{w}_\nu t(\tau_\nu), \quad t \in \mathcal{T}_{2n}, \tag{2.6}$$

with the coefficients $\widetilde{w}_\nu$, $\nu = 0, 1, \dots, 2n$, determined by

$$\widetilde{w}_\nu = \int_L^{2\pi+L} \frac{\widetilde{A}_{n+1/2}(x)}{2\sin\frac{x-\tau_\nu}{2}\widetilde{A}'_{n+1/2}(\tau_\nu)}\widetilde{w}(x)\mathrm{d}x, \quad \widetilde{A}_{n+1/2}(x) = \widetilde{A}\prod_{k=0}^{2n}\sin\frac{x-\tau_k}{2},$$

is of Gaussian type if and only if the nodes τ_ν, $\nu = 0, 1, \dots, 2n$, are zeros from the interval $[L, 2\pi + L)$ of $\widetilde{A}_{n+1/2}(x)$, which is orthogonal on this interval with respect to the weight function $\widetilde{w}(x)$ to every trigonometric polynomial of the semi-integer degree less than or equal to $n - 1/2$.

Proof. Starting from the quadrature rule (2.1) for the weight function $w(x) = \widetilde{w}(x+L)$, $x \in [0, 2\pi)$, and introducing $x := x + L$, we get

$$\int_L^{2\pi+L} t(x-L)w(x-L)\mathrm{d}x = \sum_{k=0}^{2n} w_\nu t(x_\nu + L - L), \quad t \in \mathcal{T}_{2n}.$$

Since $t(x-L)$ is again in $\mathcal{T}_{2n}$, denoting $\widetilde{t}(x) = t(x-L)$ and $\tau_\nu = x_\nu + L$, $\nu = 0, 1, \ldots, 2n$, we obtain the following quadrature formula of Gaussian type on $[L, 2\pi + L)$

$$\int_L^{2\pi+L} \widetilde{t}(x)\widetilde{w}(x)dx = \sum_{k=0}^{2n} w_\nu \widetilde{t}(\tau_\nu), \quad \widetilde{t} \in \mathcal{T}_{2n}, \tag{2.7}$$

which is exact for any $\widetilde{t} \in \mathcal{T}_{2n}$. As we can see, only the nodes are changed from the original formula, i.e., $\tau_\nu = x_\nu + L$, $\nu = 0, 1, \ldots, 2n$.

If we denote the "nodal polynomial of the semi-integer degree" of the quadrature rule (2.7) by

$$\widetilde{A}_{n+1/2}(x) = \widetilde{A}\prod_{\nu=0}^{2n} \sin\frac{x-\tau_\nu}{2}, \quad \widetilde{A} \neq 0,$$

we see that $A_{n+1/2}(x-L) = \widetilde{A}_{n+1/2}(x)$, where $A_{n+1/2}$ is the corresponding "nodal polynomial of the semi-integer degree" of the quadrature rule (2.1). Thus, it means that the substitution $x := x + L$ into the orthogonality conditions for $A_{n+1/2}$ gets the orthogonality conditions for $\widetilde{A}_{n+1/2}$, i.e., $\widetilde{A}_{n+1/2}$ is orthogonal with respect to $\widetilde{w}(x) = w(x-L)$, $x \in [L, 2\pi + L)$, to all trigonometric polynomials of the semi-integer degree less than or equal to $n - 1/2$. Similarly, if we substitute $x := x + L$ into the integral representation for the weights w_ν, $\nu = 0, 1, \ldots, 2n$, we get exactly what is stated. $\square$

Every $t \in \mathcal{T}_n^{1/2}$ can be represented using an algebraic polynomial of degree $2n+1$. Analogous statement for trigonometric polynomials from $\mathcal{T}_n$ is proved in [24, pp. 19–20], and statement both for $\mathcal{T}_n$ and $\mathcal{T}_n^{1/2}$ is proved in a uniform terminology in [8, Theorem 1.1.1].

Lemma 2.1. *Let*

$$A_{n+1/2}(x) = \sum_{k=0}^{n}\left[c_k\cos\left(k+\frac{1}{2}\right)x + d_k\sin\left(k+\frac{1}{2}\right)x\right] \in \mathcal{T}_n^{1/2},$$

and $a_k = c_k - \mathrm{i}d_k$, $k = 0, 1, \ldots, n$. *Then* $A_{n+1/2}(x)$ *can be represented in the following form*

$$A_{n+1/2}(x) = \frac{1}{2}\mathrm{e}^{-\mathrm{i}(n+1/2)x}Q_{2n+1}(\mathrm{e}^{\mathrm{i}x}),$$

where $Q_{2n+1}(z)$ *is an algebraic polynomial of degree* $2n+1$, *given by*

$$Q_{2n+1}(z) = \bar{a}_n + \bar{a}_{n-1}z + \cdots + \bar{a}_1 z^{n-1} + \bar{a}_0 z^n + a_0 z^{n+1} + \cdots + a_{n-1}z^{2n} + a_n z^{2n+1}.$$

It can be easily concluded that $Q_{2n+1}(z) = z^{2n+1}\overline{Q_{2n+1}(1/\bar{z})}$, i.e., the polynomial $Q_{2n+1}(z)$ is self-inversive (see [7] and [24, p. 16]). If z is a zero of the polynomial $Q_{2n+1}(z)$, then $x = \arg(z)$ is a zero of $A_{n+1/2}(x)$.

If $A_{n+1/2}(x)$ is orthogonal on $[0, 2\pi)$ with respect to the weight function $w(x)$ to every trigonometric polynomial of the semi-integer degree less than or equal to $n - 1/2$, then it has in $[0, 2\pi)$ exactly $2n + 1$ distinct simple zeros (see [1, Theorem 3]). Thus, the corresponding algebraic polynomial $Q_{2n+1}(z)$ has $2n + 1$ distinct zeros on the unit circle $|z| = 1$.

3. Trigonometric orthogonal systems

We use the following notation for the expanded forms of $A^C_{n+1/2}$ and $A^S_{n+1/2}$,

$$A^C_{n+1/2}(x) = \cos\left(n+\frac{1}{2}\right)x + \sum_{\nu=0}^{n-1}\left[c_\nu^{(n)}\cos\left(\nu+\frac{1}{2}\right)x + d_\nu^{(n)}\sin\left(\nu+\frac{1}{2}\right)x\right], \tag{3.1}$$

$$A^S_{n+1/2}(x) = \sin\left(n+\frac{1}{2}\right)x + \sum_{\nu=0}^{n-1}\left[f_\nu^{(n)}\cos\left(\nu+\frac{1}{2}\right)x + g_\nu^{(n)}\sin\left(\nu+\frac{1}{2}\right)x\right]. \tag{3.2}$$

For a given weight function w, we introduce the inner product of the functions f and g by

$$(f, g) = \int_0^{2\pi} f(x)g(x)w(x)\mathrm{d}x. \tag{3.3}$$

Orthogonality of trigonometric polynomials of a semi-integer degree on $[0, 2\pi)$, with respect to the inner product (3.3), can be reduced to the orthogonality of nearly polynomial functions on the real line.

The following result will be needed in Section 4.

Theorem 3.1. *Let w be a weight function on $[0, 2\pi)$, with a sequence of orthogonal trigonometric polynomials of the semi-integer degree*

$$A_{n+1/2}(x) = \sum_{\nu=0}^{n} \left[c_\nu^{(n)} \cos\left(\nu + \frac{1}{2}\right) x + d_\nu^{(n)} \sin\left(\nu + \frac{1}{2}\right) x \right], \quad |c_n^{(n)}| + |d_n^{(n)}| \neq 0.$$

Then, for every $n \in \mathbb{N}$, $0 \leq k \leq n-1$, we have

$$\int_{-1}^{1} \left[\sum_{\nu=0}^{n} \left(c_\nu^{(n)} T_{2\nu+1}(x) + d_\nu^{(n)} \sqrt{1-x^2}\, U_{2\nu}(x) \right) \right] \left[\sum_{\nu=0}^{k} \left(c_\nu^{(k)} T_{2\nu+1}(x) + d_\nu^{(k)} \sqrt{1-x^2}\, U_{2\nu}(x) \right) \right] \frac{w(2\arccos x)}{\sqrt{1-x^2}} \mathrm{d}x = 0,$$

where T_ν and U_ν, $\nu \in \mathbb{N}_0$, are Chebyshev polynomials of the first and second kind, respectively.

Proof. The orthogonality relation for $A_{n+1/2}$ states

$$\int_0^{2\pi} A_{n+1/2}(x) A_{k+1/2}(x) w(x) \mathrm{d}x = 0, \quad n \neq k.$$

If we introduce $x := 2\arccos x$, using representations for Chebyshev polynomials $T_n(x) = \cos(n \arccos x)$ and $\sqrt{1-x^2}\, U_n(x) = \sin((n+1)\arccos x)$, $n \in \mathbb{N}_0$, we get what is stated. $\square$

For $\nu, \mu \in \mathbb{N}_0$, we define

$$\begin{aligned} I_\nu^C &= (A_{\nu+1/2}^C, A_{\nu+1/2}^C), & J_{\nu,\mu}^C &= (2\cos x\, A_{\nu+1/2}^C, A_{\mu+1/2}^C), \\ I_\nu^S &= (A_{\nu+1/2}^S, A_{\nu+1/2}^S), & J_{\nu,\mu}^S &= (2\cos x\, A_{\nu+1/2}^S, A_{\mu+1/2}^S), \\ I_\nu &= (A_{\nu+1/2}^C, A_{\nu+1/2}^S), & J_{\nu,\mu} &= (2\cos x\, A_{\nu+1/2}^C, A_{\mu+1/2}^S). \end{aligned}$$

Theorem 3.2. *The trigonometric polynomials of semi-integer degree $A_{k+1/2}^C(x)$ and $A_{k+1/2}^S(x)$, $k \geq 1$, satisfy the following five-term recurrence relations:*

$$A_{k+1/2}^C(x) = (2\cos x - \alpha_k^{(1)}) A_{k-1/2}^C(x) - \beta_k^{(1)} A_{k-1/2}^S(x) - \alpha_k^{(2)} A_{k-3/2}^C(x) - \beta_k^{(2)} A_{k-3/2}^S(x), \tag{3.4}$$

$$A_{k+1/2}^S(x) = (2\cos x - \delta_k^{(1)}) A_{k-1/2}^S(x) - \gamma_k^{(1)} A_{k-1/2}^C(x) - \delta_k^{(2)} A_{k-3/2}^S(x) - \gamma_k^{(2)} A_{k-3/2}^C(x), \tag{3.5}$$

where the coefficients $\alpha_k^{(j)}, \beta_k^{(j)}, \gamma_k^{(j)}, \delta_k^{(j)}$, $k \geq 1$, $j = 1, 2$, are solutions of the following systems of linear equations

$$\begin{aligned} J_{k-1,k-j}^C &= \alpha_k^{(j)} I_{k-j}^C + \beta_k^{(j)} I_{k-j}, & J_{k-1,k-j} &= \alpha_k^{(j)} I_{k-j} + \beta_k^{(j)} I_{k-j}^S, \\ J_{k-1,k-j} &= \gamma_k^{(j)} I_{k-j}^C + \delta_k^{(j)} I_{k-j}, & J_{k-1,k-j}^S &= \gamma_k^{(j)} I_{k-j} + \delta_k^{(j)} I_{k-j}^S, \end{aligned}$$

with $\alpha_1^{(2)} = \beta_1^{(2)} = \gamma_1^{(2)} = \delta_1^{(2)} = 0$.

Proof. According to the linear independence of polynomials $A_{\nu+1/2}^C(x)$ and $A_{\nu+1/2}^S(x)$, $\nu = 0, 1, \ldots, k$, the expression $2\cos x\, A_{k-1+1/2}^C(x)$ can be represented in the following form

$$2\cos x\, A_{k-1+1/2}^C(x) = A_{k+1/2}^C(x) + \sum_{\nu=0}^{k-1} \left(\alpha_k^{(k-\nu)} A_{\nu+1/2}^C(x) + \beta_k^{(k-\nu)} A_{\nu+1/2}^S(x) \right).$$

Multiplying both sides of this equality by $w(x) A_{\nu+1/2}^C(x)$ and $w(x) A_{\nu+1/2}^S(x)$ for $\nu = 0, 1, \ldots, k-3$, and integrating over $[0, 2\pi)$, according to the orthogonality, we obtain the following homogeneous systems of linear equations

$$\alpha_k^{(k-\nu)} I_\nu^C + \beta_k^{(k-\nu)} I_\nu = 0, \quad \alpha_k^{(k-\nu)} I_\nu + \beta_k^{(k-\nu)} I_\nu^S = 0,$$

with the unknown coefficients $\alpha_k^{(k-\nu)}, \beta_k^{(k-\nu)}$, $\nu = 0, 1, \ldots, k-3$. The determinants of the previous systems are equal to

$$D_\nu = \left(\int_0^{2\pi} \left(A_{\nu+1/2}^C(x) \right)^2 w(x) \mathrm{d}x \right) \left(\int_0^{2\pi} \left(A_{\nu+1/2}^S(x) \right)^2 w(x) \mathrm{d}x \right) - \left(\int_0^{2\pi} A_{\nu+1/2}^C(x) A_{\nu+1/2}^S(x) w(x) \mathrm{d}x \right)^2, \quad \nu = 0, 1, \ldots, k-3.$$

In order to prove that $D_\nu \neq 0$, $\nu = 0, 1, \ldots, k-3$, we need the well-known Cauchy–Schwarz–Bunjakowsky integral inequality (see [25, p. 45]):

$$\left(\int_a^b f(x)g(x)\mathrm{d}x\right)^2 \leq \left(\int_a^b f(x)^2\mathrm{d}x\right)\left(\int_a^b g(x)^2\mathrm{d}x\right), \quad f, g \in L^2[a,b],$$

which reduces to an equality if and only if the functions f and g are linearly dependent.

According to linear independence of $A^C_{\nu+1/2}(x)$ and $A^S_{\nu+1/2}(x)$, we get that $D_\nu \neq 0$, $\nu = 0, 1, \ldots, k-3$. Therefore, the previous homogeneous systems have only trivial solutions, i.e., $\alpha_k^{(k-\nu)} = \beta_k^{(k-\nu)} = 0$, $\nu = 0, 1, \ldots, k-3$.

Thus, the previous recurrence relation reduces to the following form

$$2\cos x\, A^C_{k-1+1/2}(x) = A^C_{k+1/2}(x) + \alpha_k^{(1)} A^C_{k-1+1/2}(x) + \beta_k^{(1)} A^S_{k-1+1/2}(x) + \alpha_k^{(2)} A^C_{k-2+1/2}(x) + \beta_k^{(2)} A^S_{k-2+1/2}(x),$$

i.e., we obtain the recurrence relation (3.4).

Multiplying both sides of the previous recurrence relation with functions $w(x)A^C_{k-j+1/2}(x)$ and $w(x)A^S_{k-j+1/2}(x)$, $j = 1, 2$, and integrating on $[0, 2\pi)$, we obtain the following system of linear equations with unknown coefficients $\alpha_k^{(j)}$, $\beta_k^{(j)}$, $j = 1, 2$,

$$J^C_{k-1,k-j} = \alpha_k^{(j)} I^C_{k-j} + \beta_k^{(j)} I_{k-j}, \qquad J_{k-1,k-j} = \alpha_k^{(j)} I_{k-j} + \beta_k^{(j)} I^S_{k-j}, \quad j = 1, 2,$$

which also, by the same arguments, has the unique solution.

Analogously one can obtain the recurrence relation (3.5) for $A^S_{k+1/2}(x)$. $\square$

Lemma 3.1. *For $n \geq 1$, the following equations*

$$J^C_{n,n-1} = I^C_n, \qquad J^S_{n,n-1} = I^S_n, \qquad J_{n,n-1} = J_{n-1,n} = I_n$$

hold.

Proof. Using the recurrence relations (3.4) and (3.5) and orthogonality conditions, we get

$$I^C_n = (A^C_{n+1/2}, A^C_{n+1/2}) = (2\cos x\, A^C_{n-1+1/2}, A^C_{n+1/2}) = J^C_{n,n-1},$$

and, similarly $I^S_n = J^S_{n,n-1}$. Finally,

$$I_n = (A^C_{n+1/2}, A^S_{n+1/2}) = (2\cos x\, A^C_{n-1+1/2}, A^S_{n+1/2}) = J_{n-1,n}. \quad \square$$

Thus, for computations of the recurrence coefficients we need the following integrals: $I^C_n, I^S_n, I_n, J^C_{n,n}, J^S_{n,n}$, and $J_{n,n}$. Therefore, we denote

$$J^C_n = J^C_{n,n}, \qquad J^S_n = J^S_{n,n}, \qquad J_n = J_{n,n}.$$

Now, Theorem 3.2. and Lemma 3.1. give the following corollary.

Corollary 3.1. *The recurrence coefficients in (3.4) and (3.5) can be calculated by the following formulae*

$$\begin{aligned}
\alpha_k^{(1)} &= \frac{I^S_{k-1}J^C_{k-1} - I_{k-1}J_{k-1}}{D_{k-1}}, & \alpha_k^{(2)} &= \frac{I^C_{k-1}I^S_{k-2} - I_{k-1}I_{k-2}}{D_{k-2}},\\
\beta_k^{(1)} &= \frac{I^C_{k-1}J_{k-1} - I_{k-1}J^C_{k-1}}{D_{k-1}}, & \beta_k^{(2)} &= \frac{I_{k-1}I^C_{k-2} - I^C_{k-1}I_{k-2}}{D_{k-2}},\\
\gamma_k^{(1)} &= \frac{I^S_{k-1}J_{k-1} - I_{k-1}J^S_{k-1}}{D_{k-1}}, & \gamma_k^{(2)} &= \frac{I_{k-1}I^S_{k-2} - I^S_{k-1}I_{k-2}}{D_{k-2}},\\
\delta_k^{(1)} &= \frac{I^C_{k-1}J^S_{k-1} - I_{k-1}J_{k-1}}{D_{k-1}}, & \delta_k^{(2)} &= \frac{I^S_{k-1}I^C_{k-2} - I_{k-1}I_{k-2}}{D_{k-2}},
\end{aligned} \tag{3.6}$$

where $D_{k-j} = I^C_{k-j}I^S_{k-j} - I^2_{k-j}$, $j = 1, 2$, for $k \geq 1$, except $\alpha_1^{(2)} = \beta_1^{(2)} = \gamma_1^{(2)} = \delta_1^{(2)} = 0$.

Let $\widetilde{A}^C_{n+1/2}(x)$ and $\widetilde{A}^S_{n+1/2}(x)$ be trigonometric polynomials of semi-integer degree $n+1/2$, orthogonal with respect to the weight function $\widetilde{w}$ on $[L, 2\pi + L)$, with expansions

$$\widetilde{A}^C_{n+1/2}(x) = \cos\left(n+\frac{1}{2}\right)x + \sum_{\nu=0}^{n-1}\left[\widetilde{c}_\nu^{(n)}\cos\left(\nu+\frac{1}{2}\right)x + \widetilde{d}_\nu^{(n)}\sin\left(\nu+\frac{1}{2}\right)x\right], \tag{3.7}$$

$$\widetilde{A}^S_{n+1/2}(x) = \sin\left(n+\frac{1}{2}\right)x + \sum_{\nu=0}^{n-1}\left[\widetilde{f}_\nu^{(n)}\cos\left(\nu+\frac{1}{2}\right)x + \widetilde{g}_\nu^{(n)}\sin\left(\nu+\frac{1}{2}\right)x\right]. \tag{3.8}$$

The existence of such polynomials is proved in Corollary 2.1.

Theorem 3.3. *The trigonometric polynomials of semi-integer degree $\widetilde{A}^C_{n+1/2}(x)$ and $\widetilde{A}^S_{n+1/2}(x)$, orthogonal on $[-\pi,\pi)$ with respect to a weight function $\widetilde{w}(x)$, can be represented as*

$$\widetilde{A}^C_{n+1/2}(x) = (-1)^n A^S_{n+1/2}(x+\pi), \qquad \widetilde{A}^S_{n+1/2}(x) = (-1)^{n-1} A^C_{n+1/2}(x+\pi),$$

where $A^C_{n+1/2}(x)$ and $A^S_{n+1/2}(x)$ are the corresponding trigonometric polynomials of semi-integer degree orthogonal on $[0,2\pi)$ with respect to the weight function $w(x) = \widetilde{w}(x-\pi)$.

If we denote the five-term recurrence coefficients for the weight $\widetilde{w}$ by $\widetilde{\alpha}^{(j)}_k$, $\widetilde{\beta}^{(j)}_k$, $\widetilde{\gamma}^{(j)}_k$, $\widetilde{\delta}^{(j)}_k$, $k \in \mathbb{N}$, $j = 1, 2$, then

$$\widetilde{\alpha}^{(j)}_k = (-1)^j \delta^{(j)}_k, \qquad \widetilde{\beta}^{(j)}_k = (-1)^{j-1}\gamma^{(j)}_k, \qquad \widetilde{\gamma}^{(j)}_k = (-1)^{j-1}\beta^{(j)}_k, \qquad \widetilde{\delta}^{(j)}_k = (-1)^j \alpha^{(j)}_k,$$

where $\alpha^{(j)}_k$, $\beta^{(j)}_k$, $\gamma^{(j)}_k$, $\delta^{(j)}_k$, $k \in \mathbb{N}$, $j = 1, 2$, are the corresponding five-term recurrence coefficients for the weight $w(x)$.

Proof. Using the identity $\widetilde{A}_{n+1/2}(x) = A_{n+1/2}(x-L)$, from the proof of Corollary 2.1. we have that

$$(-1)^{n-1}A^C_{n+1/2}(x+\pi) = \sin\left(n+\frac{1}{2}\right)x + \sum_{\nu=0}^{n-1}\left[(-1)^{n+\nu}c^{(n)}_\nu \sin\left(\nu+\frac{1}{2}\right)x + (-1)^{n+\nu-1}d^{(n)}_\nu \cos\left(\nu+\frac{1}{2}\right)x\right]$$

and

$$(-1)^{n}A^S_{n+1/2}(x+\pi) = \cos\left(n+\frac{1}{2}\right)x + \sum_{\nu=0}^{n-1}\left[(-1)^{n+\nu-1}f^{(n)}_\nu \sin\left(\nu+\frac{1}{2}\right)x + (-1)^{n+\nu}g^{(n)}_\nu \cos\left(\nu+\frac{1}{2}\right)x\right]$$

are trigonometric polynomials of semi-integer degree orthogonal with respect to $\widetilde{w}(x) = w(x+\pi)$, $x \in [-\pi,\pi)$. Thus, we only have to identify $\widetilde{A}^C_{n+1/2}(x)$ and $\widetilde{A}^S_{n+1/2}(x)$ from (3.7) and (3.8), respectively, in order to finish the proof.

If we put

$$A^C_{n+1/2}(x) = (-1)^{n-1}\widetilde{A}^S_{n+1/2}(x-\pi) \quad \text{and} \quad A^S_{n+1/2}(x) = (-1)^{n}\widetilde{A}^C_{n+1/2}(x-\pi)$$

in the five-term recurrence relations (3.4) and (3.5), with the substitution $x := x-\pi$, we get what is stated.

Finally, it is easy to see that

$$\widetilde{c}^{(n)}_\nu = (-1)^{n+\nu}g^{(n)}_\nu, \qquad \widetilde{d}^{(n)}_\nu = (-1)^{n+\nu-1}f^{(n)}_\nu, \qquad \widetilde{f}^{(n)}_\nu = (-1)^{n+\nu-1}d^{(n)}_\nu, \qquad \widetilde{g}^{(n)}_\nu = (-1)^{n+\nu}c^{(n)}_\nu. \quad \square$$

Remark 3.1. A five-term recurrence relation for a system of orthonormal sequence of trigonometric polynomials with respect to a finite positive Borel measure μ on $[0,2\pi]$, with infinite points of increasing, was given in [9]. The orthogonal basis $\{P_n(\theta)\}$ of trigonometric polynomials, i.e., a sequence satisfying $\langle P_n(\theta), P_m(\theta)\rangle_\mu = \int_0^{2\pi} P_n(\theta)P_m(\theta)\,d\mu(\theta) = k_n\delta_{n,m}$, where $k_n \neq 0$ for all nonnegative integers n, was obtained by applying the Gram–Schmidt method to the trigonometric system $\{\sin\nu x, \cos\nu x\}_{\nu=0}^n$. Trigonometric polynomials $P_n(\theta)$ were normalized in such a way that the leading coefficients of these polynomials in $\sin n\theta$ and $\cos n\theta$ were 2^{1-n} and $P_0(\theta) = 1$. For the corresponding orthonormal sequence $\{p_n(\theta)\}$ there exist three sequences of coefficients $\{a_n\}_{n\ge 2}$, $\{b_n\}_{n\ge 0}$ and $\{c_n\}_{n\ge 1}$ such that for $n \ge 0$

$$\cos\theta\, p_n(\theta) = a_{n+2}p_{n+2}(\theta) + c_{n+1}p_{n+1}(\theta) + b_n p_n(\theta) + c_n p_{n-1}(\theta) + a_n p_{n-2}(\theta),$$

with initial conditions

$$p_{-2}(\theta) = p_{-1}(\theta) = 0, \qquad p_0(\theta) = \frac{1}{\sqrt{m_0}}, \qquad p_1(\theta) = \sqrt{\frac{m_0}{m_0m_2 - m_1^2}}\left(\sin\theta - \frac{m_1}{m_0}\right),$$

where $m_0 = \langle 1, 1\rangle_\mu$, $m_1 = \langle 1, \sin\theta\rangle_\mu$ and $m_2 = \langle \sin\theta, \sin\theta\rangle_\mu$ (see [9, Theorem 2.2]).

4. Symmetric weights

In this section we consider an interesting case of symmetric weights, i.e., the case when $w(x) = w(2\pi - x)$.

Lemma 4.1. *If the weight function satisfies $w(x) = w(2\pi - x)$, then we have $\beta^{(j)}_k = 0$, $\gamma^{(j)}_k = 0$, $j = 1, 2$, $k \in \mathbb{N}$, and $d^{(n)}_k = 0$, $f^{(n)}_k = 0$, $k \in \{0, 1, \ldots, n\}$, $n \in \mathbb{N}$.*

Proof. If we apply the well-known Gram–Schmidt orthogonalization procedure to the basis of $\mathcal{T}_n^{1/2}$,

$$\cos\left(0+\frac{1}{2}\right)x, \sin\left(0+\frac{1}{2}\right)x, \ldots, \cos\left(n+\frac{1}{2}\right)x, \sin\left(n+\frac{1}{2}\right)x,$$

with respect to the inner product (3.3), we conclude that the obtained system of orthogonal functions can be represented by two sequences of functions f_ν and g_ν, $\nu = 0, 1, \ldots, n$, where f_ν depends only on cos-functions and g_ν depends only on sin-functions, because for $k, \nu \in \mathbb{N}_0$, we have

$$\begin{aligned}\int_0^{2\pi} \cos\left(k+\frac{1}{2}\right)x \sin\left(\nu+\frac{1}{2}\right)x w(x)\mathrm{d}x &= \int_0^{\pi} \cos\left(k+\frac{1}{2}\right)x \sin\left(\nu+\frac{1}{2}\right)x w(x)\mathrm{d}x \\ &\quad + \int_\pi^0 \cos\left(k+\frac{1}{2}\right)(2\pi-x) \sin\left(\nu+\frac{1}{2}\right)(2\pi-x) w(2\pi-x)(-\mathrm{d}x) \\ &= \int_0^{\pi} \cos\left(k+\frac{1}{2}\right)x \sin\left(\nu+\frac{1}{2}\right)x w(x)\mathrm{d}x - \int_0^{\pi} \cos\left(k+\frac{1}{2}\right)x \sin\left(\nu+\frac{1}{2}\right)x w(2\pi-x)\mathrm{d}x = 0.\end{aligned}$$

That system of functions, since it is unique, must be equal to the orthogonal trigonometric polynomials of semi-integer degree $A^C_{k+1/2}$ and $A^S_{k+1/2}$, $k = 0, 1, \ldots, n$, i.e., $A^C_{k+1/2}$ depends only on cos-functions and $A^S_{k+1/2}$ depends only on sin-functions, which means $d_\nu^{(k)} = 0$, and $f_\nu^{(k)} = 0$, for all $\nu \in \{0, 1, \ldots, k\}$, $k = 0, 1, \ldots, n$. Hence, our system of five-term recurrence relations degenerates into two independent three-term recurrence relations, i.e.,

$$\beta_k^{(j)} = 0, \qquad \gamma_k^{(j)} = 0, \quad j = 1, 2, k \in \mathbb{N}. \quad \square$$

Thus, the recurrence relations (3.4) and (3.5) reduce to the three-term recurrence relations and trigonometric polynomials of semi-integer degree (3.1) and (3.2) reduce to

$$A^C_{n+1/2}(x) = \sum_{\nu=0}^{n} c_\nu^{(n)} \cos\left(\nu+\frac{1}{2}\right)x, \quad c_n^{(n)} = 1 \tag{4.1}$$

and

$$A^S_{n+1/2}(x) = \sum_{\nu=0}^{n} g_\nu^{(n)} \sin\left(\nu+\frac{1}{2}\right)x, \quad g_n^{(n)} = 1, \tag{4.2}$$

respectively.

Now, we get an immediate result:

Lemma 4.2. *For every $\nu \in \mathbb{N}_0$, we have*

$$A^C_{\nu+1/2}(\pi) = 0, \quad A^S_{\nu+1/2}(0) = 0.$$

Using Theorem 3.1. we can reduce the problem of symmetric weights to algebraic polynomials.

Theorem 4.1. *For a weight function $w(x)$, $x \in (0, 2\pi)$, with the property $w(x) = w(2\pi - x)$, we have the following equations*

$$\int_{-1}^{1} C_{2n+1}(x)C_{2k+1}(x)\frac{w(2\arccos x)}{\sqrt{1-x^2}}\mathrm{d}x = 0, \quad C_{2n+1}(x) = \sum_{\nu=0}^{n} c_\nu^{(n)} T_{2\nu+1}(x)$$

and

$$\int_{-1}^{1} S_{2n}(x)S_{2k}(x)\sqrt{1-x^2}\,w(2\arccos x)\mathrm{d}x = 0, \quad S_{2n}(x) = \sum_{\nu=0}^{n} g_\nu^{(n)} U_{2\nu}(x),$$

for all $0 \le k \le n-1$, $n \in \mathbb{N}$. The polynomials $C_{2n+1}(x)$ and $S_{2n}(x)$ satisfy the following three-term recurrence relations

$$\begin{aligned} C_{2n+1}(x) &= (4x^2 - 2 - \alpha_n^{(1)})C_{2n-1}(x) - \alpha_n^{(2)} C_{2n-3}(x), \quad n \in \mathbb{N}, \alpha_1^{(2)} = 0, \\ S_{2n}(x) &= (4x^2 - 2 - \delta_n^{(1)})S_{2n-2}(x) - \delta_n^{(2)} S_{2n-4}(x), \quad n \in \mathbb{N}, \delta_1^{(2)} = 0, \end{aligned}$$

with $C_1(x) = x$ and $S_0(x) = 1$.

Proof. We apply Theorem 3.1. to the sequences $A^C_{n+1/2}$ and $A^S_{n+1/2}$, $n \in \mathbb{N}$, and keep in mind that those are given by the expansions (4.1) and (4.2), respectively.

By substitution $x := 2\arccos x$ into the three-term recurrence for C_{2n+1}, $n \in \mathbb{N}$, and applying $\cos(2\arccos x) = 2x^2 - 1$, we get what is stated. A similar proof can be done for the sequence S_{2n}, $n \in \mathbb{N}$. $\square$

If the weight function $w(x)$, $x \in (0, 2\pi)$, satisfies $w(x) = w(2\pi - x)$, then the weight function $\widetilde{w}(x) = w(x+\pi)$, $x \in (-\pi, \pi)$, satisfies $\widetilde{w}(x) = \widetilde{w}(-x)$, $x \in (-\pi, \pi)$, i.e., the function $\widetilde{w}$ is an even function on its domain. Hence, using Lemma 4.1. for $\widetilde{A}^C_{n+1/2}$ and $\widetilde{A}^S_{n+1/2}$, $n \in \mathbb{N}$, the expansions given in (3.7) and (3.8) for $\nu = 0, 1, \ldots, n-1$ satisfy $\widetilde{d}_\nu^{(n)} = 0$ and $\widetilde{f}_\nu^{(n)} = 0$, respectively, if the weight function $\widetilde{w}$ is even on $(-\pi, \pi)$.

Theorem 4.2. *For an even weight function $\widetilde{w}(x)$, $x \in (-\pi, \pi)$, for all $0 \leq k \leq n-1$, $n \in \mathbb{N}$, we have*

$$\int_{-1}^{1} \widetilde{C}_n(x)\widetilde{C}_k(x)\sqrt{\frac{1+x}{1-x}}\widetilde{w}(\arccos x)\mathrm{d}x = 0, \quad \widetilde{C}_n(x) = \sum_{\nu=0}^{n} \widetilde{c}_\nu^{(n)}(T_\nu(x) - (1-x)U_{\nu-1}(x))$$

and

$$\int_{-1}^{1} \widetilde{S}_n(x)\widetilde{S}_k(x)\sqrt{\frac{1-x}{1+x}}\widetilde{w}(\arccos x)\mathrm{d}x = 0, \quad \widetilde{S}_n(x) = \sum_{\nu=0}^{n} \widetilde{g}_\nu^{(n)}(T_\nu(x) + (1+x)U_{\nu-1}(x)).$$

The polynomials $\widetilde{C}_n$ and $\widetilde{S}_n$, $n \in \mathbb{N}$, satisfy the following three term-recurrence relations

$$\widetilde{C}_n(x) = (2x - \widetilde{\alpha}_n^{(1)})\widetilde{C}_{n-1}(x) - \widetilde{\alpha}_n^{(2)}\widetilde{C}_{n-2}(x), \quad \widetilde{\alpha}_1^{(2)} = 0, \quad \widetilde{C}_0 = 1,$$
$$\widetilde{S}_n(x) = (2x - \widetilde{\delta}_n^{(1)})\widetilde{S}_{n-1}(x) - \widetilde{\delta}_n^{(2)}\widetilde{S}_{n-2}(x), \quad \widetilde{\delta}_1^{(2)} = 0, \quad \widetilde{S}_0 = 1.$$

Proof. Since $\widetilde{w}(x)$ is an even function, from the orthogonality conditions for $\widetilde{A}_{n+1/2}^C$, we conclude that

$$\int_0^{\pi} \widetilde{A}_{n+1/2}^C(x)\widetilde{A}_{k+1/2}^C(x)\widetilde{w}(x)dx = 0, \quad n, k \in \mathbb{N}, n > k.$$

Applying the substitution $x := \arccos x$, we get

$$\int_{-1}^{1} \widetilde{A}_{n+1/2}^C(\arccos x)\widetilde{A}_{k+1/2}^C(\arccos x)\frac{\widetilde{w}(\arccos x)}{\sqrt{1-x^2}}\mathrm{d}x = 0. \tag{4.3}$$

It is easy to see that

$$\cos\left(\left(k+\frac{1}{2}\right)\arccos x\right) = \sqrt{\frac{1+x}{2}}T_k(x) - \sqrt{\frac{1-x}{2}}\sqrt{1-x^2}U_{k-1}(x)$$

and then

$$\widetilde{A}_{n+1/2}^C(\arccos x) = \sqrt{\frac{1+x}{2}}\sum_{\nu=0}^{n}\widetilde{c}_\nu^{(n)}\left(T_\nu(x) - (1-x)U_{\nu-1}(x)\right).$$

Substituting the obtained formulae in (4.3), after some elementary transformations, we get the first assertion.

The second assertion can be proved in the same way using $\widetilde{A}_{n+1/2}^S$ and applying

$$\sin\left(\left(k+\frac{1}{2}\right)\arccos x\right) = \sqrt{\frac{1-x}{2}}T_k(x) + \sqrt{\frac{1+x}{2}}\sqrt{1-x^2}U_{k-1}(x).$$

For the proof of the corresponding recurrence relations just take the recurrence relations for $\widetilde{A}_{n+1/2}^C$ and $\widetilde{A}_{n+1/2}^S$ and put $x := \arccos x$. □

5. Numerical construction of quadrature rules of Gaussian type

In this section we present a method for constructing quadrature rules of Gaussian type. As we mentioned in Section 2, for any positive integer n, the quadrature rule of Gaussian type is the following one

$$\int_0^{2\pi} t(x)w(x)\mathrm{d}x = \sum_{\nu=0}^{2n} w_\nu t(x_\nu), \quad t \in \mathcal{T}_{2n}, \tag{5.1}$$

where the nodes x_ν, $\nu = 0, 1, \ldots, 2n$, are zeros of $A_{n+1/2}$ and weights w_ν are given by

$$w_\nu = \int_0^{2\pi} \frac{A_{n+1/2}(x)}{2\sin\frac{x-x_\nu}{2}A'_{n+1/2}(x_\nu)}w(x)\mathrm{d}x, \quad \nu = 0, 1, \ldots, 2n. \tag{5.2}$$

We choose to use $A_{n+1/2}^C$, i.e., the polynomial of semi-integer degree with leading cosine function (of course, any other $A_{n+1/2}$ can be used instead). As usual, the algorithm has two parts, dealing with the computation of nodes and weights. The construction of nodes is independent from the construction of weights and to the contrary, weights can be computed only when nodes are given.

The construction of weights can be performed using formula (5.2), provided we can calculate the integral and $A_{n+1/2}^C$ efficiently. A calculation of the integral can be performed using the Gauss–Legendre quadrature rule provided the weight function is smooth enough and the semi-integer degree of $A^C_{n+1/2}$ is not so big. In practice, this degree should not excite 101/2, if the Gauss–Legendre quadrature rule is applied in double precision arithmetics. For non-smooth weight functions, problems with instabilities in (3.6) are equivalent with the instabilities for the three-term recurrence coefficients for algebraic orthogonal polynomials. Provided τ_k, σ_k, $k = 1, \dots, N$, are the nodes and weights of the Gauss–Legendre quadrature rule respectively, we have

$$w_\nu \approx \frac{\pi}{2(A^C_{n+1/2})'(x_\nu)} \sum_{k=1}^{N} \sigma_k \frac{(wA^C_{n+1/2})(\pi\tau_k + \pi)}{\cos\frac{\pi\tau_k - x_\nu}{2}}, \quad \nu = 0, 1, \dots, 2n.$$

Here we emphasize that the possibility $\cos(\pi\tau_k - x_\nu)/2 = 0$ for some $k \in \{1, \dots, N\}$ and $\nu \in \{0, 1, \dots, 2n\}$ is not an essential problem. Once $\{x_k\}_{k=1}^N$ is computed, if $\cos(\pi\tau_k - x_\nu)/2 = 0$ for some $k \in \{1, \dots, N\}$ and $\nu \in \{0, 1, \dots, 2n\}$ we can always choose a Gauss–Legendre rule with $N + 1$ instead of N points and start again. As we can see, the computation requires the values of $A^C_{n+1/2}(x)$ at various points on $[0, 2\pi)$. It turns out that $A^C_{n+1/2}(x)$ can be computed using five-term recurrences, given in (3.4) and (3.5), in a numerically stable way, i.e., using double precision arithmetics one gets nearly double precision results.

The derivative of $A^C_{n+1/2}(x)$ can be computed using the same recurrence relations. Namely, differentiating the recurrence relations (3.4) and (3.5), it is easy to obtain recurrence relations for derivatives of $A^C_{n+1/2}(x)$ and $A^S_{n+1/2}(x)$.

This approach for calculating values of $A^C_{n+1/2}(x)$ and values of its derivative requires the five-term recurrence coefficients. These coefficients can be computed using formulae (3.6), where integrals are approximated using the Gauss–Legendre quadrature rule. However, this approach still suffers from the fact that it can be used for a computation of a small number of recurrence coefficients. For examples given below we did not use this procedure; rather, we used analytical expressions for the recurrence coefficients, which we recently derived (see [26]).

There is still another way for computing weights, based on the fact that the polynomial $A_{n+1/2}(x)/\sin\frac{x-x_\nu}{2}$ ($\nu = 0, 1, \dots, 2n$) under integral in (5.2) is a trigonometric polynomial of degree n. In order to calculate exactly this integral (up to rounding errors) we need a quadrature rule of Gaussian type with much less number of nodes; precisely with only $2[(n+1)/2]+1$ nodes. One may consider the construction of the sequence of $m = [\log_2 n]+1$ quadrature rules of Gaussian type (5.1) exact on the sequence of spaces $\mathcal{T}_{2n_k}$, where nonnegative integers n_k, $k = 1, \dots, m$, are determined by $n_m = n$, $n_k = [(n_{k+1}+1)/2]$, $k = m-1, \dots, 1$. With this approach, only for weights of quadrature rule of Gaussian type with $2n_1+1$ nodes, we use the Gauss–Legendre quadrature rule, and for the weights of all other quadratures in considered sequence we use previously obtained quadrature.

For constructing the nodes x_ν, $\nu = 0, 1, \dots, 2n$, we use the algebraic polynomial $Q_{2n+1}(z)$ introduced in Lemma 2.1. According to the representation

$$A_{n+1/2}(x) = \frac{e^{-i(n+1/2)x}}{2} Q_{2n+1}(e^{ix}), \tag{5.3}$$

we conclude that the zeros of $Q_{2n+1}(z)$ on the unit circle correspond to the zeros of $A_{n+1/2}(x)$ on the interval $[0, 2\pi)$. Since $A_{n+1/2}(x)$ has $2n + 1$ distinct zeros in $[0, 2\pi)$, the algebraic polynomial $Q_{2n+1}(z)$ has $2n + 1$ simple zeros on the unit circle. Using $A^C_{n+1/2}(x)$, the algebraic polynomial $Q_{2n+1}(z)$ has the form

$$Q_{2n+1}(z) = 1 + \overline{a}^{(n)}_{n-1} z + \cdots + \overline{a}^{(n)}_1 z^{n-1} + \overline{a}^{(n)}_0 z^n + a^{(n)}_0 z^{n+1} + \cdots + z^{2n+1},$$

where $a^{(n)}_\nu = c^{(n)}_\nu - i d^{(n)}_\nu$, $\nu = 0, 1, \dots, n-1$ (see Lemma 2.1.).

At first, we determine the zeros z_ν, $\nu = 0, 1, \dots, 2n$, of the algebraic polynomial $Q_{2n+1}(z)$ by the following simultaneous iterative process

$$z^{(k+1)}_\nu = z^{(k)}_\nu - \frac{Q_{2n+1}(z^{(k)}_\nu)}{P'_k(z^{(k)}_\nu)}, \quad \nu = 0, 1, \dots, 2n;\ k = 0, 1, \dots, \tag{5.4}$$

where $P_k(z) = \prod_{\nu=0}^{2n}(z - z^{(k)}_\nu)$. The starting values must be mutually different, i.e., $z^{(0)}_i \neq z^{(0)}_j$, $i \neq j$. This iterative process converges quadratically, because it is equivalent to the Newton–Kantorovich method applied to the system of Viète formulae (cf. [27]).

For the iterative process (5.4) we need to calculate the values $Q_{2n+1}(z^{(k)}_\nu)$, where $z^{(k)}_\nu \in \mathbb{C}$ in general. As we know, we have (5.3) with $x \in [0, 2\pi)$. Since in (5.3) all functions are analytic, the relation holds true for all $x \in \mathbb{C}$ by the principle of analytic continuation (see [28]). As we mentioned before, the values of $A_{n+1/2}(x)$ can be computed accurately using the five-term recurrence relations. Thus, it holds also for the values of the polynomial Q_{2n+1}. The problem is that we have to compute Q_{2n+1} at $z^{(k)}_\nu \in \mathbb{C}$, i.e., we have to compute $A_{n+1/2}$ at the point $-i\,\mathrm{Log}(z^{(k)}_\nu)$. It might appear that for different branches of the Log function we might get different results for $Q_{2n+1}(z^{(k)}_\nu)$. Actually, this is not the case. Namely, let $z \neq 0$, and $-i\,\mathrm{Log}\, z = -i\log z + 2k\pi$ for some $k \in \mathbb{Z}$, where $\log 1 = 0$. Then, for $x = -i\log z$,

$$\begin{aligned} Q_{2n+1}(z) &= 2e^{i(n+1/2)(x+2k\pi)} A_{n+1/2}(x + 2k\pi) \\ &= 2(-1)^k e^{i(n+1/2)x}(-1)^k A_{n+1/2}(x) = 2e^{i(n+1/2)x} A_{n+1/2}(x), \end{aligned}$$

i.e., the computed value using any branch of the Log function is the same. In the case $z_\nu^{(k)} = 0$, we easily obtain $Q_{2n+1}(0) = 1$, and we do not perform computation using $A_{n+1/2}$. Thus, we conclude

$$Q_{2n+1}(z) = \begin{cases} 1, & z = 0, \\ 2\mathrm{e}^{(n+1/2)\log z} A_{n+1/2}(-\mathrm{i}\log z), & z \neq 0. \end{cases}$$

The computation of Q_{2n+1} can be ill-conditioned if performed in this way, provided iterations $z_\nu^{(k)}$ are not close enough to the unit circle. So, one should have the starting values as good as possible.

Knowing zeros z_ν, $\nu = 0, 1, \ldots, 2n$, of the algebraic polynomial Q_{2n+1}, the zeros x_ν, $\nu = 0, 1, \ldots, 2n$, of $A^C_{n+1/2}$ can be obtained as

$$x_\nu = \arg z_\nu \in [0, 2\pi), \quad \nu = 0, 1, \ldots, 2n.$$

All computations are performed in double precision arithmetic (16 decimal digits mantissa) in Mathematica, using the corresponding software package described in [29].

The main problem in the previous iterative process is a choice of the starting iterations $z_\nu^{(0)}$, $\nu = 0, 1, \ldots, 2n$. The following auxiliary result is very useful for setting good starting values in the iterative process (5.4).

Lemma 5.1. *There exists some $\ell \in \mathbb{Z}$ such that for the zeros x_ν, $\nu = 0, 1, \ldots, 2n$, of $A^C_{n+1/2}$ we have*

$$x_0 + x_1 + \cdots + x_{2n} = (2\ell + 1)\pi, \quad \ell \in \mathbb{Z}.$$

Proof. This assertion follows directly from the Viète formula:

$$z_0 z_1 \cdots z_{2n} = \mathrm{e}^{\mathrm{i}x_0}\mathrm{e}^{\mathrm{i}x_1}\cdots\mathrm{e}^{\mathrm{i}x_{2n}} = \mathrm{e}^{\mathrm{i}(x_0+x_1+\cdots+x_{2n})} = -1. \quad \square$$

Knowing coefficients of five-term recurrence relations, for good chosen starting values in iterative process (5.4), using described procedure we obtain zeros of $A^C_{n+1/2}$ in $O(n^2)$ floating point operations.

As we saw before, the corresponding weights (5.2) can be calculated exactly using quadrature rule of Gaussian type with $2[(n+1)/2] + 1$ nodes. Since $\sum_{k=0}^{[\log_2 n]} (n/2^k)^2 = (4 - 2^{-2[\log_2 n]})n^2/3$, it is easy to see that the total cost of our algorithm for construction of mentioned $m = [\log_2 n] + 1$ quadrature rules of Gaussian type (5.1) is also $O(n^2)$ operations.

Finally, we address a question of the construction of quadrature rules for an even weight function $w(x) = w(-x)$, $x \in (-\pi, \pi)$. Using Theorem 4.2, we have

$$\widetilde{A}^C_{n+1/2}(\arccos x) = \sqrt{\frac{1+x}{2}}\,\widetilde{C}_n(x), \tag{5.5}$$

and algebraic polynomials satisfy the following three-term recurrence relation

$$\widetilde{C}_n(x) = (2x - \widetilde{\alpha}_n^{(1)})\widetilde{C}_{n-1}(x) - \widetilde{\alpha}_n^{(2)}\widetilde{C}_{n-2}(x), \qquad \widetilde{\alpha}_1^{(2)} = 0, \qquad \widetilde{C}_0 = 1. \tag{5.6}$$

This means that we can calculate the zeros of $\widetilde{C}_n$, i.e., the zeros of $\widetilde{A}_{n+1/2}$, using QR-algorithm (see [30–33]). The weights w_ν, $\nu = 0, 1, \ldots, 2n$, of the quadrature rule (2.6) can be also constructed using QR-algorithm.

Lemma 5.2. *Let $\widetilde{w}$ be an even weight function on $(-\pi, \pi)$. Let x_ν and ω_ν, $\nu = 1, \ldots, n$, be nodes and weights of the n-point Gaussian quadrature rule for the following weight function $\widetilde{w}(\arccos x)\sqrt{(1+x)/(1-x)}$, $x \in (-1, 1)$, constructed for algebraic polynomials. Then, for the quadrature rule of Gaussian type (2.6) with respect to the weight function $\widetilde{w}$ on $(-\pi, \pi)$, we have*

$$\widetilde{w}_{2n-\nu-1} = \widetilde{w}_\nu = \frac{\omega_{\nu+1}}{1 + x_{\nu+1}}, \quad \nu = 0, 1, \ldots, n-1, \qquad \widetilde{w}_{2n} = \int_{-\pi}^{\pi} \widetilde{w}(x)\mathrm{d}x - \sum_{\nu=0}^{2n-1} \widetilde{w}_\nu,$$
$$\tau_{2n-\nu-1} = -\tau_\nu = \arccos x_{\nu+1}, \quad \nu = 0, 1, \ldots, n-1, \qquad \tau_{2n} = \pi.$$

Proof. The Gaussian quadrature rule for algebraic polynomials can be constructed using the three-term recurrence (5.6). The recursion coefficients (for monic orthogonal polynomials) are given by $\widetilde{\alpha}_\nu^{(1)}/2$ and $\widetilde{\alpha}_\nu^{(2)}/4$, $\nu \in \mathbb{N}$. Using QR-algorithm we get the nodes x_ν, $\nu = 1, \ldots, n$, and applying $x := \arccos x$ we get the zeros of $\widetilde{A}^C_{n+1/2}$, given by $\tau_{2n-\nu} = -\tau_\nu = -\arccos x_\nu$, $\nu = 0, 1, \ldots, 2n-1$. Using Lemma 4.2. and Theorem 3.3. we obtain $\tau_{2n} = \pi$.

It is well known that the weights in Gaussian quadrature rules can be constructed using Shohat formula (see [34,35]). In our case we have

$$\omega_\nu = \mu_0 \left(\sum_{k=0}^{n-1} \left(\frac{\widetilde{C}_k(x_\nu)}{\prod_{j=2}^{k} \alpha_j^{(2)}} \right)^2 \right)^{-1},$$

where

$$\mu_0 = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,\widetilde{w}(\arccos x)\mathrm{d}x.$$

Applying (5.5) we get

$$\omega_\nu = \frac{(1+x_\nu)\mu_0}{2\sum_{k=0}^{n-1}\left(\frac{\widetilde{A}^C_{k+1/2}(\tau_{2n-\nu-1})}{\prod_{j=2}^{k}\alpha_j^{(2)}}\right)^2}, \quad \nu = 1,\ldots,n.$$

According to Corollary 2.1. the weights of the quadrature rule are not affected by the translation of the support. Using a result from [1], the weights w_ν for the weight function $w(x) = w(2\pi - x)$, $x \in [0, 2\pi)$, are given by

$$w_{2n-\nu} = \widetilde{w}_{2n-\nu} = \frac{\mu_0}{2\sum_{k=0}^{n-1}\left(\frac{A^S_{k+1/2}(\tau_{2n-\nu}+\pi)}{\prod_{j=2}^{k}\alpha_j^{(2)}}\right)^2} = \frac{\mu_0}{2\sum_{k=0}^{n-1}\left(\frac{\widetilde{A}^S_{k+1/2}(\tau_{2n-\nu})}{\prod_{j=2}^{k}\widetilde{\alpha}_j^{(2)}}\right)^2},$$

for $\nu = 0, 1, \ldots, n-1$, where we use the fact that nodes are transformed by an additive law from Corollary 2.1, and where

$$\mu_0 = \int_0^{2\pi} \sin^2\frac{x}{2}\,w(x)\mathrm{d}x = \int_{-\pi}^{\pi}\cos^2\frac{x}{2}\,\widetilde{w}(x)\mathrm{d}x = \int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,\widetilde{w}(\arccos x)\mathrm{d}x.$$

Combining formulae for the weights we obtain

$$\widetilde{w}_{2n-\nu} = \frac{\omega_{\nu+1}}{1+x_{\nu+1}}, \quad \nu = 0, 1, \ldots, n-1.$$

Since this formula is symmetric, we have $\widetilde{w}_\nu = \widetilde{w}_{2n-\nu-1}$, $\nu = 0, 1, \ldots, n-1$. Finally, it must be

$$\sum_{\nu=0}^{2n}\widetilde{w}_\nu = \int_{-\pi}^{\pi}\widetilde{w}(x)\mathrm{d}x,$$

and, therefore,

$$w_{2n} = \int_{-\pi}^{\pi}\widetilde{w}(x)\mathrm{d}x - \sum_{\nu=0}^{2n-1}\widetilde{w}_\nu. \quad \square$$

Completely similar arguments can be applied to prove the following result.

Lemma 5.3. *Let $\widetilde{w}$ be an even weight function on $(-\pi, \pi)$. Let x_ν and ω_ν, $\nu = 1, \ldots, n$, be nodes and weights for the n-point Gaussian quadrature rule for the following weight function $\widetilde{w}(\arccos x)\sqrt{(1-x)/(1+x)}$, $x \in (-1, 1)$, constructed for algebraic polynomials. Then, for the quadrature rule of Gaussian type (2.6) with respect to the weight function $\widetilde{w}$ on $(-\pi, \pi)$, we have*

$$\widetilde{w}_{2n-\nu} = \widetilde{w}_\nu = \frac{\omega_{\nu+1}}{1-x_{\nu+1}}, \quad \nu = 0, 1, \ldots, n-1, \qquad \widetilde{w}_n = \int_{-\pi}^{\pi}\widetilde{w}(x)\mathrm{d}x - \sum_{\nu=0}^{2n-1}\widetilde{w}_\nu,$$

$$\tau_{2n-\nu} = -\tau_\nu = \arccos x_{\nu+1}, \quad \nu = 0, 1, \ldots, n-1, \qquad \tau_n = 0.$$

Remark 5.1. Interesting connection with Gaussian quadrature rule with respect to a weight function $\sigma(x)$ on the interval $[-1, 1]$ and Szegő quadrature rule with respect to symmetric weight function $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$ on $[-\pi, \pi]$ was given in [36].

6. Numerical examples

In this section we give two numerical examples.

Example 6.1. Let $w(x) = 1 + \sin mx$, where m is an odd integer.

According to the explicit formulae given in [1,26], for $m \geq 3$ and for all positive integers $n \leq [m/2]$, the trigonometric polynomial of semi-integer degree $A^C_{n+1/2}(x)$ is given by $A^C_{n+1/2}(x) = \cos(n + 1/2)x$. Thus, the nodes x_ν and the weights w_ν,

Table 6.1
Nodes x_ν, $\nu = 0, 1, \ldots, 16$, and weights w_ν, $\nu = 0, 1, \ldots, 50$, for $n = 25$ and $w(x) = 1 + \sin 15x$

ν	x_ν	$w_{17j+\nu}, j = 0, 1, 2$
0	0.0734401134707617	0.1849824504539084
1	0.1720803992707313	0.1537846831517915
2	0.2993059533314362	0.0307792282514791
3	0.4461644342899088	0.1401732745185891
4	0.5445227530917630	0.1898081115080525
5	0.6414418782341709	0.0778455112386129
6	0.8337565680606713	0.0908209422996504
7	0.9305122558894531	0.1925521119018577
8	1.0284562970240495	0.1271338202783832
9	1.1941155620071260	0.0374793000746708
10	1.3057227567822522	0.1667479588549361
11	1.4044808463174123	0.1773988199716636
12	1.5068361968932778	0.0494873674658285
13	1.6866505393455060	0.1148661358623812
14	1.7841265106107811	0.1937788694612881
15	1.8811643708168992	0.1029645966854189
16	2.0695809349059611	0.0637919204146833

$\nu = 0, 1, \ldots, 2n$, can be obtained explicitly

$$x_\nu = \frac{2\nu + 1}{2n + 1}\pi, \quad w_\nu = \frac{2\pi}{2n + 1}, \quad \nu = 0, 1, \ldots, 2n. \tag{6.1}$$

It is easy to see that the sum of zeros x_ν, $\nu = 0, 1, \ldots, 2n$, given by (6.1), is equal to $(2n + 1)\pi$.

For $n > [m/2]$ we calculate the nodes using the iterative process (5.4). For the weights we start with a quadrature rule of Gaussian type, such that the nodes and the weights are known explicitly, and we use the procedure described in Section 5, dealing with the construction of a sequence of quadrature rules of Gaussian type.

In our numerical experiments, at first, we select the starting values $x_\nu^{(0)} \in (0, 2\pi)$ and then set the starting iteration $z_\nu^{(0)} = \mathrm{e}^{\mathrm{i}x_\nu^{(0)}}$, $\nu = 0, 1, \ldots, 2n$. We choose the starting values $x_\nu^{(0)}$ such that their sum is equal to $(2n + 1)\pi$. In addition, for some values $n > [m/2] + 1$, we use the zeros of $A_{n-1/2}^C(x)$ to generate the starting values $x_\nu^{(0)}$, $\nu = 0, 1, \ldots, 2n$. Based on the several numerical experiments we observed that for $n > [m/2] + 1$ the zeros have some kind of interlacing property (obviously all polynomials $A_{k+1/2}^C(x)$, $k = 0, 1, \ldots, [m/2]$, have one common zero $x_k = \pi$).

If $\tau_0^{(n)} < \tau_1^{(n)} < \cdots < \tau_{2n}^{(n)}$ are zeros of $A_{n+1/2}^C(x)$, $n > [m/2] + 1$, in order to calculate zeros of $A_{n+3/2}^C(x)$ we select the starting values $x_\nu^{(0)} \in (0, 2\pi)$, $\nu = 0, 1, \ldots, 2n + 2$, in such a way that the following inequalities hold

$$x_0^{(0)} < \tau_0^{(n)} < x_1^{(0)} < \tau_1^{(n)} < \cdots < \tau_n^{(n)} < x_{n+1}^{(0)} < x_{n+2}^{(0)} < \tau_{n+1}^{(n)} < x_{n+3}^{(0)} < \cdots < \tau_{2n}^{(n)} < x_{2n+2}^{(0)}.$$

We use explicit formulae in order to get the recurrence coefficients in (3.4) and (3.5). In numerical experiments we observe that the problem of finding starting values is more difficult for small m (e.g. $m = 1, 3, 5$). The simplest starting values are

$$x_\nu^{(0)} = \frac{2\nu + 1}{2n + 1}\pi, \quad \nu = 0, 1, \ldots, 2n \tag{6.2}$$

(starting nodes are equidistant in $(0, 2\pi)$ and sum of them is equal to $(2n + 1)\pi$). We have applied the iterative process (5.4) with the starting iteration $z_\nu^{(0)} = \mathrm{e}^{\mathrm{i}x_\nu^{(0)}}$, $\nu = 0, 1, \ldots, 2n$, where $x_\nu^{(0)}$ are given by (6.2) in the following cases:

- case $m = 9$ for $5 \le n \le 45$;
- case $m = 15$ for $8 \le n \le 25$ and for $n = 30(5)85$;
- case $m = 75$ for $n \in \{38, 39, 40\}$ (the number of iterations is equal to 6) and for $n = 45(5)100$ (for all of these values n the number of iterations is 5).

But, in the case $m = 1$, $m = 3$, and $m = 5$, we can use the starting values (6.2) only for $n < 10$, $n < 20$, and $n < 30$, respectively. For bigger values of n we use zeros of $A_{n-1/2}^C$ in order to generate the starting iteration.

In Table 6.1, the nodes x_ν and weights w_ν, $\nu = 0, 1, \ldots, 2n$, for $n = 25$ and $w(x) = 1 + \sin 15x$ are given. In this case we have some symmetry, since $w_\nu = w_{\nu+17j}$, $j = 1, 2$, $\nu = 0, 1, \ldots, 16$, and $x_{\nu+17j} = x_\nu + 2j\pi/3$, $j = 1, 2$, $\nu = 0, 1, \ldots, 16$, so that only x_ν, w_ν, $\nu = 0, 1, \ldots, 16$, are presented in Table 6.1.

Example 6.2. Let $w(x) = 1 + \sin mx$, where m is an even integer.

According to [1,26], for $m \ge 4$, for all positive integers $n \le m/2 - 1$, we have $A_{n+1/2}^C(x) = \cos(n + 1/2)x$, and nodes x_ν and weights w_ν, $\nu = 0, 1, \ldots, 2n$, are given by (6.1).

For $n \ge m/2$ we calculate the nodes using the iterative process (5.4).

Table 6.2
Nodes x_ν, $\nu = 0, 1, \dots, 2n$, for $n = 25$ and $w(x) = 1 + \sin 50x$

ν	x_ν	$x_{17+\nu}$	$x_{34+\nu}$
0	0.0437696975461690	2.1636406417855051	4.2651012450237054
1	0.1680572310428011	2.2883880600363662	4.3852058050373930
2	0.2924577772609259	2.4130974373388425	4.5046039583815650
3	0.4169517048517318	2.5377596507423607	4.6234496920089237
4	0.5415228156521995	2.6623641411606281	4.7420276283431659
5	0.6661575921351178	2.7868985020797753	4.8606971330055078
6	0.7908446090617998	2.9113479584697332	4.9797911584537416
7	0.9155740710944654	3.0356947016184854	5.0995317482730982
8	1.0403374458729844	3.1599170377010471	5.2200072319479234
9	1.1651271687302412	3.2839883015926769	5.3412019414901779
10	1.2899364005225395	3.4078754885382263	5.4630423375637033
11	1.4147588240731420	3.5315375783972615	5.5854341400227252
12	1.5395884676639467	3.6549236000621025	5.7082840158088338
13	1.6644195460588000	3.7779706681729619	5.8315092331359442
14	1.7892463108602127	3.9006026336522674	5.9550403615334283
15	1.9140629026986649	4.0227307996524905	6.0788207018859129
16	2.0388631978902007	4.1442595261254562	6.2028045110215764

Table 6.3
Weights w_ν, $\nu = 0, 1, \dots, 2n$, for $n = 25$ and $w(x) = 1 + \sin 50x$

ν	w_ν	$w_{17+\nu}$	$w_{34+\nu}$
0	0.1242234407383922	0.1247636888477994	0.1204763708301496
1	0.1243476420018499	0.1247298161762916	0.1197370793194431
2	0.1244501944630916	0.1246874233622537	0.1190846784234882
3	0.1245349734569849	0.1246352503882133	0.1186551985157037
4	0.1246050049375511	0.1245716712995453	0.1185624353256949
5	0.1246626514561804	0.1244945990189778	0.1188336003611944
6	0.1247097559001120	0.1244013597149017	0.1193917984733010
7	0.1247477509830184	0.1242885289281262	0.1201035034380715
8	0.1247777417982283	0.1241517217768391	0.1208440219991488
9	0.1248005674244240	0.1239853332159593	0.1215325114340890
10	0.1248168462168162	0.1237822367135557	0.1221322173565282
11	0.1248270082283731	0.1235334819122534	0.1226356842093022
12	0.1248313172236255	0.1232281062294575	0.1230502424417207
13	0.1248298839460249	0.1228533330493070	0.1233886346224002
14	0.1248226716326329	0.1223957327593081	0.1236642040515870
15	0.1248094941871655	0.1218444226955629	0.1238888869081770
16	0.1247900068753692	0.1211979506978345	0.1240726312135605

The situation with the starting iteration is similar as in Example 6.1. We tested the following cases using the starting values (6.2):

- case $m = 10$ for $5 \le n \le 50$;
- case $m = 50$ for $n = 25(5)100$ (for $n = 25$ the number of iterations is 6, and for $n = 30(5)100$ the number of iterations is 5);
- case $m = 100$ for $n = 50(5)100$ (for $n = 50$ and $n = 55$ the number of iterations is 6, and for $n = 60(5)100$ the number of iterations is 5).

In the case $m = 4$ we can use the starting values (6.2) for $n \le 22$, and in the case $m = 2$ we can use the starting values (6.2) only for $n \le 2$. Thus, for bigger values of n we generate the starting iteration using zeros of $A^C_{n-1/2}$.

The nodes x_ν and the weights w_ν, $\nu = 0, 1, \dots, 2n$, for $n = 25$ and $w(x) = 1 + \sin 50x$, are presented in Tables 6.2 and 6.3, respectively.

Remark 6.1. In order to use zeros of $A^C_{n-1/2}$ to generate the starting values for computing zeros of $A^C_{n+1/2}$, we may consider the construction of the sequence of the quadrature rules of Gaussian type (5.1) exact on the sequence of spaces $\mathcal{T}_{2k}$, $1 \le k \le n$. Then the total cost of our algorithm is $O(n^3)$ operations.

Finally, we compare our method with the other methods.

At first, we consider Mysovskih's results given in [4,5]. It has been already said that his theoretical results are not suitable for numerical calculations. Notice that if moments (1.2) are known, we can calculate $\int_0^{2\pi} f(x)w(x)\mathrm{d}x$, $f \in \mathcal{T}_n$, so in the case of an even trigonometric degree of exactness it is possible to use these moments for computation of recurrence coefficients by (3.6), since the product of two trigonometric polynomials of semi-integer degree is trigonometric polynomial.

For the case of an even trigonometric degree of exactness, using approach based on ideal theory (see [3]), nodes can be obtained as common zeros of two quasi-orthogonal trigonometric polynomials, instead of one trigonometric polynomials of semi-integer degree, i.e., of one algebraic polynomial in our method.

Nodes of Szegő quadrature rules are eigenvalues of $n \times n$ unitary upper Hessenberg matrix $H_n(\tau)$, determined by parameter τ from unit circle and the so-called Schur parameters $\gamma_1, \ldots, \gamma_{n-1}$, and weights are determined by the square of the first component of the corresponding eigenvector of unit length (see e.g., [12,10,11]). Matrix $H_n(\tau)$ is given by $H_n(\tau) = \Delta_n^{-1/2}\widehat{H}_n(\tau)\Delta_n^{1/2}$, where

$$\widehat{H}_n(\tau) = \begin{bmatrix} -\overline{\gamma}_0\gamma_1 & -\overline{\gamma}_0\gamma_2 & \cdots & -\overline{\gamma}_0\gamma_{n-1} & -\overline{\gamma}_0\tau \\ 1-|\gamma_1|^2 & -\overline{\gamma}_1\gamma_2 & \cdots & -\overline{\gamma}_1\gamma_{n-1} & -\overline{\gamma}_1\tau \\ 0 & 1-|\gamma_2|^2 & \cdots & -\overline{\gamma}_2\gamma_{n-1} & -\overline{\gamma}_2\tau \\ \vdots & & & & \\ 0 & 0 & \cdots & 1-|\gamma_{n-1}|^2 & -\overline{\gamma}_{n-1}\tau \end{bmatrix},$$

$\gamma_0 = 1$, $\Delta_n = \mathrm{diag}[\delta_0, \delta_1, \ldots, \delta_{n-1}]$, $\delta_0 = 1$, and $\delta_j = \delta_{j-1}(1 - |\gamma_j|^2)$, $j = 1, \ldots, n-1$. There are several algorithms for the eigen decomposition of such kind of matrices (see [37–39]). In [40], it is shown that Hessenberg matrices can be compactly represented, and working with such representation, computation of eigensystem can be performed very efficiently (see [37, 39–46]). The total cost of these methods are $O(n^3)$ operations, and sometimes can be reduced down to $O(n^2)$ operations. There is still another way to compute nodes of the Szegö quadrature rules using five diagonal matrices presented in [15].

Our method is restricted only to quadrature rules with an odd number of nodes (an even trigonometric degree of exactness). As it was said in Remark 2.2, for the case of an even number of nodes, quadratures can be considered in similar way using trigonometric polynomials instead of trigonometric polynomials of semi-integer degree. We use recurrence relations to obtain wanted orthogonal systems in order to escape numerical non-stability which is characteristic for Gram–Schmidt method. Also, recurrence relations provide a stable way for computation of values of trigonometric polynomials of semi-integer degree in some fixed points in contrast to using expanded forms. This method is a simulation of the development of Gaussian quadrature rules for algebraic polynomials. We demonstrated how in the case of symmetric weight function the quadrature rules of Gaussian type can be constructed using orthogonal polynomials on the real line. Also, our method can be extended to the quadrature rules with multiple nodes with maximal trigonometric degree of exactness (such quadratures for constant weight function were considered in [47,48]).

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