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On the Convergence Order of an Accelerated Simultaneous Method for Polynomial Complex Zeros

Let P be a monic polynomial of degree $n \geqq 3$,

$$P(z) = \prod_{k=1}^{n} (z - \xi_k)$$

with simple real or complex zeros $\xi_1, \dots, \xi_n$. Assume that the disjoint disks $Z_1^{(0)}, \dots, Z_n^{(0)}$, containing these zeros, have been found. The following algorithm for the simultaneous inclusion of all polynomial zeros by disks was formulated by GARGANTINI and HENRICI [3]:

$$Z_k^{(m+1)} = z_k^{(m)} - \left[\frac{P'(z_k^{(m)})}{P(z_k^{(m)})} - \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{z_k^{(m)} - Z_j^{(m)}}\right]^{-1}, \qquad k = 1, \dots, n; \qquad m = 0, 1, \dots, \tag{1}$$

where $z_k^{(m)}$ is the center of disk $Z_k^{(m)}$. The convergence order of the iterative method (1) is three.

We note that the rectangles in the complex plane can be taken in (1) instead of circular regions $Z_j^{(m)}$. In that case $z_k^{(m)}$ is the intersection of diagonals of rectangle $Z_k^{(m)}$ (that is, "the center" of the rectangle) and the upper error-bound for an approximate solution is given by a half of the diagonal (*semidiagonal*) of the inclusion rectangle, denoted, say, by $\varepsilon_k^{(m)}$. This approach provides: (i) the use of the intersection of the complex interval on the right-hand side of (1) and the previous interval approximation $Z_k^{(m)}$, in order to obtain a smaller rectangle and (ii) safe inclusion of polynomial zeros (namely, $\xi_k \in Z_k^{(m)}$ for each $k = 1, \dots, n$ and $m = 1, 2, \dots$) using rounded rectangular arithmetic. This is not always possible if circular arithmetic is used because of the presence of rounding error executing operations in finite length floating-point arithmetic. We assume that the properties of circular and rectangular interval arithmetic are familiar to the reader (see, e.g. [1], [3], [9]).

The following modification of the algorithm (1) in circular arithmetic, which uses a double Gauss-Seidel procedure, was proposed by WANG and WU [2]:

$$W_k^{(m)} = z_k^{(m)} - \left[\frac{P'(z_k^{(m)})}{P(z_k^{(m)})} - \sum_{j<k} \frac{1}{z_k^{(m)} - W_j^{(m)}} - \sum_{j>k} \frac{1}{z_k^{(m)} - Z_j^{(m)}}\right]^{-1}, \tag{2}$$

$$Z_k^{(m+1)} = z_k^{(m)} - \left[\frac{P'(z_k^{(m)})}{P(z_k^{(m)})} - \sum_{j<k} \frac{1}{z_k^{(m)} - Z_j^{(m+1)}} - \sum_{j>k} \frac{1}{z_k^{(m)} - W_j^{(m)}}\right]^{-1}, \qquad k = 1, \dots, n; \qquad m = 0, 1, \dots .$$

In the above formulas the same values of the polynomial P and its dervative P' at the point $z = z_k^{(m)}$ are used, which considerably reduces the number of operations. Besides, the algorithm (2) is suitable for programming and occupies little storage space at digital computers.

The convergence analysis of the iterative method (2) was not given in [2]. The determination of the R-order of convergence of this method is the subject of this paper. In our analysis the complex intervals $Z_j^{(m)}$ and $W_j^{(m)}$, $j = 1, \dots, n$, which appear in (2), can be either the disks or the rectangles.

Let $r_k^{(m)}$ and $\varepsilon_k^{(m)}$ be the radii (semidiagonals) of the disks (rectangles) $Z_k^{(m)}$ and $W_j^{(m)}$, respectively, and let

$$r^{(m)} = \max_{1 \leq k \leq n} r_k^{(m)}, \qquad \varrho^{(m)} = \min_{\substack{i,j\\ i\neq j}} \{|z_i^{(m)} - z_j^{(m)}| - r_j^{(m)}\}.$$

Assume that

$$\varrho^{(0)} > 3(n-1)\, r^{(0)} \tag{3}$$

is valid. Applying similar considerations as in [4] (see, also [6], [7], [8]) and mathematical induction, after extentive but elementary evaluation we derive the following relations under the condition (3):

$$\varepsilon_k^{(m)} < \frac{q^2}{n-1} r_k^{(m)2} \left(\sum_{j=1}^{k-1} \varepsilon_j^{(m)} + \sum_{j=k+1}^{n} r_j^{(m)}\right), \qquad r_k^{(m+1)} < \frac{q^2}{n-1} r_k^{(m)2} \left(\sum_{j=1}^{k-1} r_j^{(m+1)} + \sum_{j=k+1}^{n} \varepsilon_j^{(m)}\right),$$
$$k = 1, \dots, n; \qquad m = 0, 1, \dots, \tag{4}$$

where $q = q(\varrho^{(0)}, r^{(0)}, n)$ is a real positive constant which depends only on the distribution of initial complex intervals, their radii (semidiagonals) and the polynomial degree n. Under the condition (3) it can be proved that

$$q < \frac{3(n-1)}{\varrho^{(0)}}. \tag{5}$$

Substituting $h_k^{(m)} = q r_k^{(m)}$ and $\hat{h}_k^{(m)} = q \varepsilon_k^{(m)}$ in (4), we find

$$\hat{h}_k^{(m)} < \frac{h_k^{(m)2}}{n-1} \left(\sum_{j=1}^{k-1} \hat{h}_j^{(m)} + \sum_{j=k+1}^{n} h_j^{(m)}\right), \qquad h_k^{(m+1)} < \frac{h_k^{(m)2}}{n-1} \left(\sum_{j=1}^{k-1} h_j^{(m+1)} + \sum_{j=k+1}^{n} \hat{h}_j^{(m)}\right),$$
$$k = 1, \dots, n; \qquad m = 0, 1, \dots . \tag{6}$$

Let $h = \max\limits_{1 \leqq k \leqq n} h_k^{(0)}$. By (3) and (5) we find $h_k^{(0)} \leqq h = qr_k^{(0)} < 1$. According to this and (6) we conclude that the sequences $(h_k^{(m)})$, $k = 1, \dots, n$, and, consequently, $(r_k^{(m)})$, converge to zero. Therefore, the iterative process (2) is *convergent* under the condition (3).

Further, we can write

$$h_k^{(m+1)} \leqq h^{u_k^{(m+1)}}, \qquad k = 1, \dots, n; \qquad m = 0, 1 \dots,$$

where the components $u_1^{(m+1)}, u_2^{(m+1)}, \dots, u_n^{(m+1)}$ of the vector $\boldsymbol{u}^{(m+1)} = [u_1^{(m+1)}, u_2^{(m+1)}, \dots, u_n^{(m+1)}]^{\mathrm{T}}$ can be evaluated by

$$\boldsymbol{u}^{(m+1)} = A_n \boldsymbol{u}^{(m)}, \qquad m = 0, 1, \dots,$$

starting from $\boldsymbol{u}^{(0)} = [1\ 1\ \dots\ 1]^{\mathrm{T}}$. The matrix A_n has the form

$$A_3 = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}, \qquad A_n = \begin{bmatrix} 2 & 2 & 1 & & & & & & \\ & 2 & 2 & 1 & & \mathbf{0} & & & \\ & & 2 & 2 & 1 & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & \mathbf{0} & & & \cdot & \cdot & \cdot & \\ & & & & & & 2 & 2 & 1 \\ 2 & 1 & 0 & & & & 0 & 2 & 2 \\ 2 & 2 & 1 & 1 & & & 0 & 0 & 2 \end{bmatrix}, \qquad n \geqq 4.$$

Denote the R-order of convergence of the iterative process IP with the limit point $\xi = [\xi_1, \xi_2, \dots, \xi_n]^{\mathrm{T}}$ by $O_R(\mathrm{IP}, \xi)$ (c. f. Ortega and Rheinboldt [5]). Using the same consideration as in [1, Ch. 8], it can be proved that the lower bound of $O_R((2), \xi)$ is given by

$$O_R((2), \xi) \geqq \varrho(A_n),$$

where $\varrho(A_n)$ is the spectral radius of the matrix A_n.

In order to determine $\varrho(A_n)$, we applied the well-known power method for finding the dominant eigenvalue. The values of $\varrho(A_n)$ are shown in Table 1.

Table 1

n	3	4	5	6	7	8	9	10	15	20
$\varrho(A_n)$	6.373	5.938	5.696	5.557	5.464	5.397	5.348	5.309	5.198	5.146

Remark: The presented results relative to the R-order of convergence can be also applied in the case of "ordinary" iterative methods of type (2), where the point approximations (complex numbers) $z_k^{(m)}$ and $w_k^{(m)}$ to the zeros ξ_k are taken instead of the complex intervals $Z_k^{(m)}$ and $W_k^{(m)}$, $k = 1, \dots, n$. This is obvious because of $|z_k^{(m)} - \xi_k| = O(r_k^{(m)})$.

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