RANK FACTORIZATION AND MOORE-PENROSE INVERSE

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Abstract. In this paper we develop a few representations of the Moore-Penrose inverse, based on full-rank factorizations of matrices. These representations we divide into the two different classes: methods which arise from the known block decompositions and determinantal representation. In particular cases we obtain several known results.

1. Introduction

The set of $m \times n$ complex matrices of rank r is denoted by $\mathbb{C}_r^{m\times n} = \{X \in \mathbb{C}^{m\times n} : \operatorname{rank}(X) = r\}$. With $A^{|r}$ and $A_{|r}$ we denote the first r columns of A and the first r rows of A, respectively. The identity matrix of the order k is denoted by I_k, and $\mathbb{O}$ denotes the zero block of an appropriate dimensions.

We use the following useful expression for the Moore-Penrose generalized inverse $A^\dagger$, based on the full-rank factorization $A = PQ$ of A [1-2]:

$$A^\dagger = Q^\dagger P^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = Q^*(P^*AQ^*)^{-1}P^*.$$

We restate main known block decompositions [7], [16-18]. For a given matrix $A \in \mathbb{C}_r^{m\times n}$ there exist regular matrices R, G, permutation matrices E, F and unitary matrices U, V, such that:

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$(T_1)\quad RAG=\begin{bmatrix} I_r & \mathbb{O}\\ \mathbb{O} & \mathbb{O}\end{bmatrix}=N_1,\qquad (T_2)\quad RAG=\begin{bmatrix} B & \mathbb{O}\\ \mathbb{O} & \mathbb{O}\end{bmatrix}=N_2,$

$(T_3)\quad RAF=\begin{bmatrix} I_r & K\\ \mathbb{O} & \mathbb{O}\end{bmatrix}=N_3,\qquad (T_4)\quad EAG=\begin{bmatrix} I_r & \mathbb{O}\\ K & \mathbb{O}\end{bmatrix}=N_4,$

$(T_5)\quad UAG=\begin{bmatrix} I_r & \mathbb{O}\\ \mathbb{O} & \mathbb{O}\end{bmatrix}=N_1,\qquad (T_6)\quad RAV=\begin{bmatrix} I_r & \mathbb{O}\\ \mathbb{O} & \mathbb{O}\end{bmatrix}=N_1,$

$(T_7)\quad UAV=\begin{bmatrix} B & \mathbb{O}\\ \mathbb{O} & \mathbb{O}\end{bmatrix}=N_2,\qquad (T_8)\quad UAF=\begin{bmatrix} B & K\\ \mathbb{O} & \mathbb{O}\end{bmatrix}=N_5,$

$(T_9)\quad EAV=\begin{bmatrix} B & \mathbb{O}\\ K & \mathbb{O}\end{bmatrix}=N_6,$

$(T_{10})\quad EAF=\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}=N_7,$ where $\mathrm{rank}(A_{11})=\mathrm{rank}(A)$.

(T_{11}) Transformation of similarity for square matrices [11]:

$RAR^{-1}=RAFF^*R^{-1}=\begin{bmatrix} I_r & K\\ \mathbb{O} & \mathbb{O}\end{bmatrix}F^*R^{-1}=\begin{bmatrix} T_1 & T_2\\ \mathbb{O} & \mathbb{O}\end{bmatrix}.$

The block form (T_{10}) can be expressed in two different ways:

$(T_{10a})\quad EAF=\begin{bmatrix} A_{11} & A_{11}T\\ SA_{11} & SA_{11}T\end{bmatrix}$, where the multipliers S and T satisfy $T=A_{11}^{-1}A_{12},\quad S=A_{21}A_{11}^{-1}$ (see [8]);

$(T_{10b})\quad EAF=\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}=\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{21}A_{11}^{-1}A_{12}\end{bmatrix}$ (see [9]).

Block representations of the Moore-Penrose inverse is investigated in [8], [11], [15–19]. In [16], [18] the results are obtained by solving the equations (1)–(4). In [15], [8] the corresponding representations are obtained using the block decompositions (T_{10a}) and (T_{10b}) and implied full-rank factorizations.

Also, in [19] is introduced block representation of the Moore-Penrose inverse, based on $A^\dagger=A^*TA^*$, where $T\in A^*AA^*\{1\}$.

Block decomposition (T_{11}) is investigated in [11], but only for square matrices and the group inverse.

The notion *determinantal representation* of the Moore-Penrose inverse of A means representation of elements of $A^\dagger$ in terms of minors of A. Determinantal representation of the Moore-Penrose inverse is examined in [1–2], [4–6], [12–14]. For the sake of completeness, we restate here several notations and the main result. For an $m\times n$ matrix A let $\alpha=\{\alpha_1,\dots,\alpha_r\}$ and $\beta=\{\beta_1,\dots,\beta_r\}$ be subsets of $\{1,\dots,m\}$ and $\{1,\dots,n\}$, respectively.

Then $A\begin{pmatrix}\alpha_1 \dots \alpha_r \\ \beta_1 \dots \beta_r\end{pmatrix} = |\, A^{\alpha}_{\beta}\,|$ denotes the minor of A determined by the rows indexed by α and the columns indexed by β, and A^{α}_{β} represents the corresponding submatrix. Also, the algebraic complement of A^{α}_{β} is defined by

$$\frac{\partial}{\partial a_{ij}}|\, A^{\alpha}_{\beta}\,| = A_{ij}\begin{pmatrix}\alpha_1 \dots \alpha_{p-1}\; i\; \alpha_{p+1} \dots \alpha_r \\ \beta_1 \dots \beta_{q-1}\; j\; \beta_{q+1} \dots \beta_r\end{pmatrix} = (-1)^{p+q} A\begin{pmatrix}\alpha_1 \dots \alpha_{p-1}\; \alpha_{p+1} \dots \alpha_r \\ \beta_1 \dots \beta_{q-1}\; \beta_{q+1} \dots \beta_r\end{pmatrix}.$$

Adjoint matrix of a square matrix B is denoted by $\operatorname{adj}(B)$, and its determinant by $|B|$.

Determinantal representation of full rank matrices is introuced in [1], and for full-rank matrices in [4–6]. In [12–14] is introduced an elegant derivation for determinantal representation of the Moore-Penrose inverse, using a full-rank factorization and known results for full-rank matrices. Main result of these papers is:

Proposition 1.1. *The (i,j)th element of the Moore-Penrose inverse $G = (g_{ij})$ of $A \in \mathbb{C}_r^{m\times n}$ is given by*

$$g_{ij} = \frac{\sum\limits_{\alpha: j\in\alpha\,;\,\beta: i\in\beta} |\, \overline{A}^{\alpha}_{\beta}\,|\, \dfrac{\partial}{\partial a_{ji}} |\, A^{\alpha}_{\beta}\,|}{\sum\limits_{\gamma,\delta} |\, \overline{A}^{\gamma}_{\delta}\,|\,|\, A^{\gamma}_{\delta}\,|}.$$

In this paper we investigate two different representations of the Moore-Penrose inverse. The first class of representations is a continuation of the papers [8] and [15]. In other words, from the presented block factorizations of matrices find corresponding full-rank decompositions $A = PQ$, and then apply $A^\dagger = Q^*(P^*AQ^*)^{-1}P^*$. In the second representation, $A^\dagger$ is represented in terms of minors of the matrix A. In this paper we describe an elegant proof of the well-known determinantal representation of the Moore-Penrose inverse. Main advantages of described block representations are their simply derivation and computation and possibility of natural generalization. Determinantal representation of the Moore-Penrose inverse can be implemented only for small dimensions of matrices ($n \le 10$).

2. Block representation

Theorem 2.1. *The Moore-Penrose inverse of a given matrix $A \in \mathbb{C}_r^{m\times n}$ can be represented as follows, where block representations (G_i) correspond to the block decompositions (T_i), $i \in \{1, \dots, 9, 10a, 10b, 11\}$:*

$$(G_1) \quad A^\dagger = \left(G^{-1}{}_{|r}\right)^* \left(\left(R^{-1^{|r}}\right)^* A \left(G^{-1}{}_{|r}\right)^*\right)^{-1} \left(R^{-1^{|r}}\right)^*,$$

(G_2) $\quad A^\dagger = \left(G^{-1}{}_{|r}\right)^* \left(\left(R^{-1|r} B\right)^* A \left(G^{-1}{}_{|r}\right)^*\right)^{-1} \left(R^{-1|r} B\right)^*,$

(G_3) $\quad A^\dagger = F \begin{bmatrix} I_r \\ K^* \end{bmatrix} \left(\left(R^{-1|r}\right)^* AF \begin{bmatrix} I_r \\ K^* \end{bmatrix}\right)^{-1} \left(R^{-1|r}\right)^*,$

(G_4) $\quad A^\dagger = \left(G^{-1}{}_{|r}\right)^* \left([\, I_r, \quad K^* \,]\, EA \left(G^{-1}{}_{|r}\right)^*\right)^{-1} [\, I_r, \quad K^* \,]\, E,$

(G_5) $\quad A^\dagger = \left(G^{-1}{}_{|r}\right)^* \left(U_{|r} A \left(G^{-1}{}_{|r}\right)^*\right)^{-1} U_{|r},$

(G_6) $\quad A^\dagger = V^{|r} \left(\left(R^{-1|r}\right)^* AV^{|r}\right)^{-1} \left(R^{-1|r}\right)^*,$

(G_7) $\quad A^\dagger = V^{|r} \left(B^* U_{|r} AV^{|r}\right)^{-1} B^* U_{|r},$

(G_8) $\quad A^\dagger = F \begin{bmatrix} B^* \\ K^* \end{bmatrix} \left(U_{|r} AF \begin{bmatrix} B^* \\ K^* \end{bmatrix}\right)^{-1} U_{|r},$

(G_9) $\quad A^\dagger = V^{|r} \left([\, B^*, \quad K^* \,]\, EAV^{|r}\right)^{-1} [\, B^*, \quad K^* \,]\, E,$

$$(G_{10a}) \quad A^\dagger = F \begin{bmatrix} I_r \\ T^* \end{bmatrix} \left(A_{11}^* \,[\, I_r, \quad S^* \,]\, EAF \begin{bmatrix} I_r \\ T^* \end{bmatrix}\right)^{-1} A_{11}^* \,[\, I_r, \quad S^* \,]\, E$$
$$= F \begin{bmatrix} I_r \\ T^* \end{bmatrix} (I_r + TT^*)^{-1} A_{11}^{-1} (I_r + S^* S)^{-1} \,[\, I_r, \quad S^* \,]\, E,$$

$$(G_{10b}) \; A^\dagger = F \begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix} \left((A_{11}^*)^{-1} [A_{11}^*, \quad A_{21}^*] EAF \begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix}\right)^{-1} (A_{11}^*)^{-1} [A_{11}^*, \quad A_{21}^*] E$$
$$= F \begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix} (A_{11} A_{11}^* + A_{12} A_{12}^*)^{-1} A_{11} (A_{11}^* A_{11} + A_{21}^* A_{21})^{-1} [A_{11}^*, \quad A_{21}^*] E,$$

$$(G_{11}) \quad A^\dagger = R^* \begin{bmatrix} I_r \\ \left(T_1^{-1} T_2\right)^* \end{bmatrix} \left(\left(R^{-1|r} T_1\right)^* AR^* \begin{bmatrix} I_r \\ \left(T_1^{-1} T_2\right)^* \end{bmatrix}\right)^{-1} \left(R^{-1|r} T_1\right)^*.$$

Proof. (G_1) Starting from (T_1), we obtain

$$A = R^{-1} \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} G^{-1} = R^{-1} \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} [\, I_r, \quad \mathbb{O} \,]\, G^{-1},$$

which implies

$$P = R^{-1} \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} = R^{-1|r}, \qquad Q = [\, I_r, \quad \mathbb{O} \,]\, G^{-1} = G^{-1}{}_{|r}.$$

Now, we get

$$A^\dagger = Q^* \left(P^* AQ^*\right)^{-1} P^* = \left(G^{-1}{}_{|r}\right)^* \left(\left(R^{-1|r}\right)^* A \left(G^{-1}{}_{|r}\right)^*\right)^{-1} \left(R^{-1|r}\right)^*.$$

The other block decompositions can be obtained in a similar way.

(G_5) Block decomposition (T_5) implies

$$A = U^* \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} G^{-1} = U^* \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} [\, I_r, \quad \mathbb{O} \,]\, G^{-1},$$

which means

$$P = U^* \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} = U^{*|r}, \qquad Q = [\, I_r, \quad \mathbb{O} \,]\, G^{-1} = G^{-1}{}_{|r}.$$

(G_7) It is easy to see that (T_7) implies

$$A = U^* \begin{bmatrix} B & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} V^* = U^* \begin{bmatrix} B \\ \mathbb{O} \end{bmatrix} [\, I_r, \quad \mathbb{O} \,]\, V^*.$$

Thus,

$$P = U^* \begin{bmatrix} B \\ \mathbb{O} \end{bmatrix} = U^{*|r} B, \qquad Q = [\, I_r, \quad \mathbb{O} \,]\, V^* = V^*{}_{|r},$$

which means

$$P^* = B^* U_{|r}, \qquad Q^* = V^{|r}.$$

(G_{10a}) From (T_{10a}) we obtain

$$A = E^* \begin{bmatrix} A_{11} & A_{11}T \\ SA_{11} & SA_{11}T \end{bmatrix} F^* = E^* \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11} [\, I_r, \quad T \,]\, F^*,$$

which implies, for example, the following full rank factorization of A:

$$P = E^* \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11}, \quad Q = [\, I_r, \quad T \,]\, F^*.$$

Now,

$$A^\dagger = F \begin{bmatrix} I_r, \\ T^* \end{bmatrix} \left(A_{11}^* [\, I_r, \quad S^* \,]\, EAF \begin{bmatrix} I_r, \\ T^* \end{bmatrix} \right)^{-1} A_{11}^* [\, I_r, \quad S^* \,]\, E.$$

The proof can be completed using

$$EAF = \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11} [\, I_r, \quad T \,]\,. \qquad \square$$

Remarks 2.1. (i) A convenient method for finding the matrices S, T and A_{11}^{-1}, required in (T_{10a}) was introduced in [8], and it was based on the following extended Gauss-Jordan transformation:

$$\begin{bmatrix} A_{11} & A_{12} & I \\ A_{21} & A_{22} & \mathbb{O} \end{bmatrix} \to \begin{bmatrix} I & T & A_{11}^{-1} \\ \mathbb{O} & \mathbb{O} & -S \end{bmatrix}.$$

(ii) In [3] it was used the following full-rank factorization of A, derived from $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$:

$$P = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}, \qquad Q = [\, I_r, \quad A_{11}^{-1}A_{12} \,].$$

3. Determinantal representation

In the following definition are generalized concepts of determinant, *algebraic complement*, adjoint matrix and *determinantal representation* of generalized inverses. (see also [14].)

Definition 3.1. Let A be $m \times n$ matrix of rank r.

(i) The generalized determinant of A, denoted by $N_r(A)$, is equal to

$$N_r(A) = \sum_{\alpha,\beta} |\, \overline{A}^{\alpha}_{\beta} \,|\, |\, A^{\alpha}_{\beta} \,|\,,$$

(ii) *Generalized algebraic complement* of A corresponding to a_{ij} is

$$A^{\dagger}_{ij} = \sum_{\alpha: j\in\alpha;\beta: i\in\beta} |\, \overline{A}^{\alpha}_{\beta} \,|\, \frac{\partial}{\partial a_{ji}} \,|\, A^{\alpha}_{\beta} \,|\,.$$

(iii) Generalized adjoint matrix of A, denoted by $\operatorname{adj}^{\dagger}(A)$ is the matrix whose elements are $A^{\dagger}_{ij}$.

For full-rank matrix A the following results can be proved:

Lemma 3.1. [14] *If A is an $m \times n$ matrix of full-rank, then:*

(i) $N_r(A) = \begin{cases} |\, AA^* \,|, & r = m \\ |\, R^*A \,|, & r = n. \end{cases}$

(ii) $A^{\dagger}_{ij} = \begin{cases} (A^* \operatorname{adj}(AA^*))_{ij}\,, & r = m \\ (\operatorname{adj}(A^*A)A^*)_{ij}\,, & r = n. \end{cases}$

(iii) $A^{\dagger} = \begin{cases} A^*(AA^*)^{-1}, & r = m \\ (A^*A)^{-1}A^*, & r = n. \end{cases}$

(iv) $\operatorname{adj}^{\dagger}(A) = \begin{cases} A^* \operatorname{adj}(AA^*), & r = m \\ \operatorname{adj}(A^*A)A^*, & r = n. \end{cases}$

Main properties of the *generalized adjoint matrix*, *generalized algebraic complement* and *generalized determinant* are investigated in [14].

Lemma 3.2. [14] *If $A = PQ$ is a full-rank factorization of an $m \times n$ matrix A of rank r, then*

(i) $\mathrm{adj}^\dagger(Q) \cdot \mathrm{adj}^\dagger(P) = \mathrm{adj}^\dagger(A)$;
(ii) $N_r(Q) \cdot N_r(P) = N_r(P) \cdot N_r(Q) = N_r(A)$;

From Lemma 3.1 and Lemma 3.2 we obtain an elegant proof for the determinantal representation of the Moore-Penrose inverse.

Theorem 3.1. *Let A be an $m \times n$ matrix of rank r, and $A = PQ$ be its full-rank factorization. The Moore-Penrose inverse of A posseses the followig determinantal representation:*

$$a_{ij}^\dagger = \frac{\sum\limits_{\alpha:j\in\alpha\,;\,\beta:i\in\beta} |\,\overline{A}_\beta^\alpha\,| \dfrac{\partial}{\partial a_{ji}} |\,A_\beta^\alpha\,|}{\sum\limits_{\gamma,\delta} |\,\overline{A}_\delta^\gamma\,|\,|\,A_\delta^\gamma\,|}.$$

Proof. Using $A^\dagger = Q^\dagger P^\dagger$ [3] and the results of Lemma 3.1 and Lemma 3.2, we obtain:

$$A^\dagger = Q^\dagger P^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = \frac{Q^*\,\mathrm{adj}(QQ^*)}{|\,QQ^*\,|}\,\frac{\mathrm{adj}(P^*P)P^*}{|\,P^*P\,|} =$$

$$= \frac{\mathrm{adj}^\dagger(Q)\,\mathrm{adj}^\dagger(P)}{N_r(Q)N_r(P)} = \frac{\mathrm{adj}^\dagger(A)}{N_r(A)}. \qquad \square$$

4. Examples

Example 4.1. Block decomposition (T_1) can be obtained by applying transformation (T_3) two times:

$$R_1AF_1 = \begin{bmatrix} I_r & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_3,$$

$$R_2N_3^TF_2 = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1.$$

Then, the regular matrices R, G can be computed as follows:

$$N_1 = N_1^T = F_2^TN_3R_2^T = F_2^TR_1AF_1R_2^T \Rightarrow R = F_2^TR_1, \quad G = F_1R_2^T.$$

For the matrix $A = \begin{pmatrix} -1 & 1 & 3 & 5 & 7 \\ 1 & 0 & -2 & 0 & 4 \\ 1 & 1 & -1 & 5 & 15 \\ -1 & 2 & 4 & 10 & 18 \end{pmatrix}$ we obtain

$$R_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix}, \quad F_1 = I_5,$$

$$R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 \\ 0 & -5 & 0 & 1 & 0 \\ -4 & -11 & 0 & 0 & 1 \end{pmatrix}, \quad F_2 = I_4.$$

From $R = R_1$, $G = R_2^T$, we get

$$R^{-1|2} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 1 & 1 \\ -1 & 2 \end{pmatrix}, \quad G^{-1}{}_{|2} = \begin{pmatrix} 1 & 0 & -2 & 0 & 4 \\ 0 & 1 & 1 & 5 & 11 \end{pmatrix}.$$

Using formula (G_1), we obtain

$$A^{\dagger} = \begin{pmatrix} -\frac{169}{6720} & \frac{67}{2240} & \frac{233}{6720} & -\frac{137}{6720} \\ \frac{1}{128} & -\frac{1}{128} & -\frac{1}{128} & \frac{1}{128} \\ \frac{781}{13440} & -\frac{330}{4480} & -\frac{1037}{13440} & \frac{653}{13440} \\ \frac{5}{128} & -\frac{5}{128} & -\frac{5}{128} & \frac{5}{128} \\ -\frac{197}{13440} & \frac{151}{4480} & \frac{709}{13440} & \frac{59}{13440} \end{pmatrix}.$$

Example 4.2. For the matrix $A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & -3 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -2 \end{pmatrix}$ we obtain

$$A_{11}^{-1} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & -2 \\ -1 & -3 \end{pmatrix} \text{[5]}.$$

Using (G_{10a}) we obtain

$$A^{\dagger} = \begin{pmatrix} -\frac{5}{34} & -\frac{3}{17} & \frac{1}{34} & -\frac{1}{34} & \frac{3}{17} & \frac{5}{34} \\ \frac{4}{51} & \frac{13}{102} & -\frac{5}{102} & \frac{5}{102} & -\frac{13}{102} & -\frac{4}{51} \\ \frac{7}{102} & \frac{5}{102} & \frac{1}{51} & -\frac{1}{51} & -\frac{5}{102} & -\frac{7}{102} \\ \frac{1}{17} & -\frac{1}{34} & \frac{3}{34} & -\frac{3}{34} & \frac{1}{34} & -\frac{1}{17} \end{pmatrix}.$$

Example 4.3. For the matrix $A = \begin{pmatrix} 4 & -1 & 1 & 2 \\ -2 & 2 & 0 & -1 \\ 6 & -3 & 1 & 3 \\ -10 & 4 & -2 & -5 \end{pmatrix}$ we obtain

$$R = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}, \quad T_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad T_2 = \left(\begin{smallmatrix} -1 & -2 \\ -1 & -3 \end{smallmatrix}\right), \quad F = I_4.$$

Then, the following results can be obtained:

$$\left(R^{-1|r} T_1\right)^* = \begin{pmatrix} -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}, \quad R^* \begin{bmatrix} I_r \\ (T_1^{-1}T_2)^* \end{bmatrix} = \begin{pmatrix} -4 & -6 \\ 1 & 3 \\ -1 & -1 \\ -2 & -3 \end{pmatrix}.$$

Finally, using (G_{11}), we get

$$A^\dagger = \begin{pmatrix} \frac{8}{81} & \frac{10}{81} & -\frac{2}{81} & -\frac{2}{27} \\ \frac{47}{162} & \frac{79}{162} & -\frac{16}{81} & -\frac{5}{54} \\ \frac{7}{54} & \frac{11}{54} & -\frac{2}{27} & -\frac{1}{18} \\ \frac{4}{81} & \frac{5}{81} & -\frac{1}{81} & -\frac{1}{27} \end{pmatrix}.$$

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