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SOME MODIFICATIONS OF THE CHEBYSHEV MEASURES AND THE CORRESPONDING ORTHOGONAL POLYNOMIALS

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A b s t r a c t. A few rational modifications of the Chebyshev measures of the first and second kind and the corresponding orthogonal polynomials on the finite interval $[-1, 1]$ *are studied, included also a convex combination of these two Chebyshev measures. Also, an non-rational modification of the Chebyshev measure of the second kind, i.e.,* $\mathrm{d}\sigma^{n,s}(x) = |\widehat{U}_n(x)|^{2s}(1-x^2)^{s+1/2}\,\mathrm{d}x$ *on* $[-1, 1]$*, for* $n \in \mathbb{N}$ *and a real number* $s > -1/2$*, is studied, as well as certain properties of the corresponding orthogonal polynomials, including explicit expressions for the coefficients in their three-term recurrence relation.*

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1. *Introduction and preliminaries*

In this paper we consider a few classes of the real polynomials orthogonal to some modifications of the Chebyshev measures of the first and second kind, including symbolic calculations of their three-term recurrence relations, as well as certain differential properties of such classes of polynomials. We start with one of the basic concepts of the constructive theory of orthogonal polynomials on the real line [7] and mention a few recent results in this direction.

1.1. Rational Christoffel modification of the measure

For a given positive measure $\mathrm{d}\mu(x)$ with finite support $\operatorname{supp}(\mathrm{d}\mu) = [a, b]$, we can define a new measure

$$\mathrm{d}\widehat{\mu}(x) = \frac{u(x)}{v(x)}\,\mathrm{d}\mu(x), \quad x \in [a, b],$$

where

$$u(x) = \pm \prod_{k=1}^{\ell}(x - u_k) \quad \text{and} \quad v(x) = \prod_{k=1}^{m}(x - v_k)$$

are two real polynomials, relative prime and not vanishing on $[a, b]$. The sign $+$ or $-$ should be chosen so that $u(x)/v(x) > 0$ on $[a, b]$.

One of main problems in the constructive theory of orthogonal polynomials is to generate the coefficients in three-term recurrence relation for polynomials orthogonal with respect to the modified measure, $\widehat{\alpha}_k(\mathrm{d}\widehat{\mu})$ and $\widehat{\beta}_k(\mathrm{d}\widehat{\mu})$, from those of the original measure, $\alpha_k(\mathrm{d}\mu)$ and $\beta_k(\mathrm{d}\mu)$. Methods for providing this transformation are known as (rational) modification algorithms.

The first results on this subject are due to Christoffel from 1858, who expressed $u(x)\pi_n(\mathrm{d}\widehat{\mu}; x)$ ($v \equiv 0$ and $\mathrm{d}\widehat{\mu}(x) = \mathrm{d}x$) in determinantal form as a linear combination of orthogonal polynomials $\pi_{n+\nu}(\mathrm{d}\mu; x)$, $\nu = 0, 1, \ldots, \ell$ (see [3]). The case $v(x) \not\equiv 0$ was solved one hundred years later by Uvarov [20]. In 1982 Gautschi developed a general method for linear and quadratic factors, as well as linear and quadratic divisors [7]. In [11], Gautschi and Li considered the construction of orthogonal polynomials $\pi_n(\mathrm{d}\widehat{\mu}; x)$ and their recursion coefficients from the coefficients $\alpha_k(\mathrm{d}\mu)$ and $\beta_k(\mathrm{d}\mu)$ of polynomials $\pi_n(\mathrm{d}\mu; x)$, where $\mathrm{d}\widehat{\mu}(x) = [\pi_n(\mathrm{d}\mu; x)]^2\,\mathrm{d}\mu(x)$. For all four Chebyshev measures they obtained the desired coefficients in the closed analytic form.

For the generalized Chebyshev case, i.e., for the measure

$$\mathrm{d}\sigma^{n,s}(x) = \frac{|\widehat{T}_n(x)|^{2s}}{\sqrt{1 - x^2}}\,\mathrm{d}x,$$

where $\widehat{T}_n(x)$ is the n-th degree monic Chebyshev polynomial of the first kind, Milovanović and his colloborators obtained coefficients of three-term recurrence relation $\widehat{\beta}_k^{n,s}$, $k \in \mathbb{N}$, in closed analytic form when $s \in \mathbb{N}$, as well as for each $s > -1/2$ and $n = 2$ (see [5] and [6]). Note that in the last case ($s > -1/2$ and $n = 2$) we do not have a rational modification of the Chebyshev measure!

2. *Rational modification of the Chebyshev measures*

Following some results from [15, p. 374], Milovanović [18] recently obtained explicit expressions for coefficients in three-term recurrence relation for monic orthogonal polynomials with respect to the Szegő-Bernstein measures

$$\mathrm{d}\mu_\nu(x) = \frac{W(x)}{(c-x)^\nu}\,\mathrm{d}x, \quad \nu \geq 1,$$

on $(-1,1)$, when $W(x)$ is

$$(1-x^2)^{\mp 1/2}, \quad \sqrt{(1+x)/(1-x)} \quad \text{and} \quad \sqrt{(1-x)/(1+x)}.$$

Repeated modifications for (distinct and the same) linear divisors have been studied by Gautschi [9] using numerical algorithms and applied them to generate special Gaussian rules for dealing with nearby poles. Among interesting examples, Gautschi applied it to evaluate integrals of the form $\int_0^\infty f(t)\varepsilon_r(t)\,\mathrm{d}t$, with Bose-Einstein weight function

$$\varepsilon_r(t) = \left(\frac{t}{\mathrm{e}^t - 1}\right)^r,$$

for a given (well-behaved) function f, as well as to the Szegő-Bernstein measure

$$\mathrm{d}\mu_m(t) = \frac{1}{(c_1^2-t^2)(c_2^2-t^2)\cdots(c_m^2-t^2)}\frac{\mathrm{d}t}{\sqrt{1-t^2}}, \quad -1<t<1,$$

with $c_k = 1 + 1/k > 1$ for each $k = 1, \ldots, m$, and $m \leq 24$ (working in 52-digit arithmetic). It is clear that all α-coefficients are zero, whereas the β_k are all equal to $1/4$ for $k > m+1$. We mention that, for this Szegő-Bernstein measure, the orthogonal polynomials of degree $n > m$ are explicitly known [19, Theorem 2.6]. Some of these algorithms can also be used for neutralizing nearby singularities in numerical quadratures (Gautschi [10]).

Similar problems can be treated in the symbolic form. Here, we mention the cases of the modified Chebyshev measure of the first kind with the identical quadratic divisors $c^2 + t^2$, $c > 0$, i.e., when the measure is given by

$$\mathrm{d}\mu_m(t) = \frac{1}{(c^2+t^2)^m}\frac{\mathrm{d}t}{\sqrt{1-t^2}}, \quad t \in (-1,1). \tag{2.1}$$

In this case the moments $\mu_k^{(m)} = \int_{-1}^1 t^k\,\mathrm{d}\mu_m(t)$, $k \geq 0$, are $\mu_k^{(m)} = 0$ for odd k, and for even k,

$$\mu_k^{(m)} = 2\int_0^\pi \frac{\cos^k\theta}{(c^2+\cos^2\theta)^m}\,\mathrm{d}\theta = 2^{m-k/2}\int_0^{2\pi}\frac{(1+\cos x)^{k/2}}{(a+\cos x)^m}\,\mathrm{d}x,$$

where $a = 2c^2 + 1 > 1$. If we put $c = \sinh\varphi$, then

$$a = \cosh 2\varphi = \frac{1}{2}(\mathrm{e}^{2\varphi} + \mathrm{e}^{-2\varphi}) = \frac{1}{2}\left(\frac{1}{X} + X\right), \quad X = \mathrm{e}^{-2\varphi}, \tag{2.2}$$

where, evidently $0 < X < 1$. After a long computation, we can prove:

Proposition 2.1. *For the measure* (2.1)*, the moments* $\mu_k^{(m)}$ *for even* m *can be expressed in terms of the hypergeometric function*

$$\mu_k^{(m)} = \pi\binom{k}{k/2}2^{2m-k+1}\frac{X^m}{(1+X)^{2m}}\,{}_2F_1\left(\frac{1}{2}, m; \frac{k}{2}+1; \frac{4X}{(1+X)^2}\right),$$

where X *is defined in* (2.2)*. The corresponding recurrence coefficients are* $\alpha_\nu^{(m)} = 0$*,* $\nu \ge 0$*, and* $\beta_\nu^{(m)}$

$$\beta_\nu^{(m)} = \frac{1}{4}, \quad \nu \ge m+2, \qquad \beta_{m+1}^{(m)} = \frac{1}{4}\left(1 + X^m\right),$$

$$\beta_m^{(m)} = \frac{1 - X^{2m} + mX^{m-1}(1-X^2)}{4(1+X^m)}, \quad \textit{etc.}$$

Remark 2.1. For distinct quadratic divisors, $c_\nu^2 + t^2$ ($c_\nu > 0$), $\nu = 1, \dots, m$, the corresponding recurrence coefficients can be expressed in terms of symmetric functions of X_ν $(= \mathrm{e}^{-2\varphi_\nu})$, $\nu = 1, \dots, m$.

2.1. Convex combination of two Chebyshev measures

Polynomials orthogonal with respect to multiple component distributions, e.g., $\mathrm{d}\mu(t) = [(1-t^2)^{-1/2} + a]\,\mathrm{d}t$ on $[-1,1]$, $a > 0$ (adding a constant to the Chebyshev weight function), was considered in [7].

Here we consider a convex combination of two Chebyshev measures of the first and second kind on $[-1,1]$, i.e.,

$$\mathrm{d}\mu_c(t) = \left[\frac{1-c}{\sqrt{1-t^2}} + c\sqrt{1-t^2}\right]\mathrm{d}t, \quad 0 \le c \le 1.$$

Such a measure is, in fact, the modified Chebyshev measure of the first kind by a quadratic factor $1 - ct^2$,

$$\mathrm{d}\mu_c(t) = \frac{1-ct^2}{\sqrt{1-t^2}}\,\mathrm{d}t, \quad 0 \le c \le 1,$$

for which the moments are

$$\mu_k = \begin{cases} \dfrac{[(1-c)k+2-c]\pi}{2^k(k+2)}\dbinom{k}{k/2}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

We can apply the classical Chebyshev algorithm in symbolic form to generate any number N of recurrence coefficients α_k, β_k, $k = 0, 1, \ldots, N-1$, using the first $2N$ moments, μ_k, $k = 0, 1, \ldots, 2N-1$. For example, the first $N = 50$ recurrence coefficients are obtained by the Mathematica package `OrthogonalPolynomials` ([4], [17]), using the commands

```
<<orthogonalPolynomials`
mom=Table[If[OddQ[k],0,((1-c)k+2-c)Pi Binomial[k,k/2]/
      (2^k(k+2))],{k,0,99}];
{alSym,beSym}=aChebyshevAlgorithm[mom,
                       Algorithm->Symbolic]
```

Of course, all α-coefficients in the three-term recurrence relation are zero, wheras the β_k are

$$\beta_0 = \frac{\pi}{2}(2-c), \quad \beta_1 = \frac{3c-4}{4(c-2)}, \quad \beta_2 = \frac{c^2-8c+8}{4(c-2)(3c-4)},$$

$$\beta_3 = \frac{(c-2)\left(5c^2-20c+16\right)}{4(3c-4)\left(c^2-8c+8\right)}, \quad \beta_4 = \frac{(c-2)(3c-4)\left(c^2-16c+16\right)}{4\left(c^2-8c+8\right)\left(5c^2-20c+16\right)},$$

$$\beta_5 = \frac{\left(c^2-8c+8\right)\left(7c^3-56c^2+112c-64\right)}{4(c-2)\left(c^2-16c+16\right)\left(5c^2-20c+16\right)},$$

$$\beta_6 = \frac{\left(5c^2-20c+16\right)\left(c^4-32c^3+160c^2-256c+128\right)}{4(c-2)\left(c^2-16c+16\right)\left(7c^3-56c^2+112c-64\right)},$$

$$\beta_7 = \frac{(c-2)(3c-4)\left(c^2-16c+16\right)\left(3c^3-36c^2+96c-64\right)}{4\left(7c^3-56c^2+112c-64\right)\left(c^4-32c^3+160c^2-256c+128\right)},$$

$$\beta_8 = \frac{(c-2)\left(7c^3-56c^2+112c-64\right)\left(c^4-48c^3+304c^2-512c+256\right)}{4(3c-4)\left(3c^3-36c^2+96c-64\right)\left(c^4-32c^3+160c^2-256c+128\right)},$$

etc. Except β_1 (and β_0), the other coefficients β_k, $k \geq 2$, have the graphics as in Fig. 1, where the β_k, $k = 2, 3, \ldots, 7$, are presented.

The recurrence coefficients β_k, $k \geq 1$, are rational functions in c of the form $\beta_k(c) = P_k(c)/Q_k(c)$, where P_k and Q_k are polynomials of degree k. Evidently, for $c = 0$ and $c = 1$, these coefficients reduce to ones of the Chebyshev polynomials

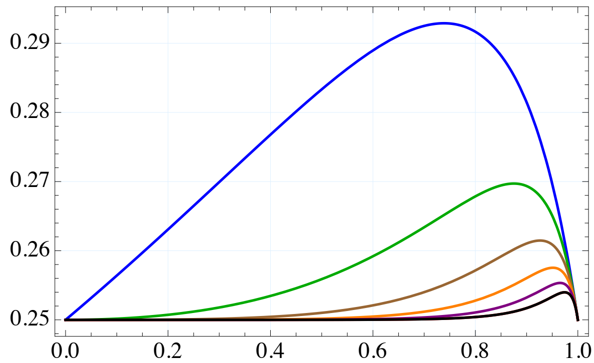

Figure 1: Graphics of the recurrence coefficients $\beta_k(c)$, $k = 2, 3, \ldots, 7$ (from top to bottom) for the polynomials orthogonal to the measure $\mathrm{d}\mu_c(x)$

of the first and second kind, i.e., $\beta_1(0) = 1/2$, $\beta_k(0) = 1/4$ $(k \geq 2)$ and $b_k(1) = 1/4$ $(k \geq 1)$, respectively.

In an analysis of the rational functions $c \mapsto \beta_k(c) = P_k(c)/Q_k(c)$ $(k \in \mathbb{N})$, when c runs over $\mathbb{R}$, after a lot of computation we can state the following conjecture.

Conjecture 2.1. *The rational function* $c \mapsto \beta_k(c) = P_k(c)/Q_k(c)$ *has* k *real zeros and* k *real poles all located in* $(1, \infty)$*. Also,*

$$\lim_{c \to \pm\infty} \beta_k(c) = \begin{cases} \dfrac{k+2}{4k}, & k \ \textit{is odd}, \\[2mm] \dfrac{k-1}{4(k+1)}, & k \textit{ is even}. \end{cases}$$

Remark 2.2. For $c = 1/2$ this β-sequence becomes

$$\Big\{\frac{3\pi}{4}, \frac{5}{12}, \frac{17}{60}, \frac{87}{340}, \frac{495}{1972}, \frac{2873}{11484}, \frac{16733}{66924}, \frac{97515}{390052}, \frac{568347}{2273380}, \frac{3312557}{13250220},$$

$$\frac{19306985}{77227932}, \frac{112529343}{450117364}, \frac{655869063}{2623476244}, \frac{3822685025}{15290740092}, \frac{22280241077}{89120964300},$$

$$\frac{129858761427}{519435045700}, \frac{756872327475}{3027489309892}, \frac{4411375203413}{17645500813644}, \frac{25711378892993}{102845515571964}, \ldots\Big\}.$$

3. *One modification of the Chebyshev measure of the second kind*

Let $\mathrm{d}\sigma(x)$ be a positive measure on $\mathbb{R}$, with finite or unbounded support, having finite moments of all orders, and let $\{p_k\}$, be the corresponding (monic)polynomials

$$p_k(x) = p_k(x, \mathrm{d}\sigma), \quad k \in \mathbb{N}_0.$$

They are known to satisfy the three-term recurrence relation (see [16, p. 97])

$$p_{k+1}(x) = (x - \alpha_k)p_k(x) - \beta_k p_{k-1}(x), \quad k \in \mathbb{N}_0, \tag{3.1}$$

with $p_0(x) = 1$ and $p_{-1}(x) = 0$, where $\alpha_k = \alpha_k(\mathrm{d}\sigma) \in \mathbb{R}$, $\beta_k = \beta_k(\mathrm{d}\sigma) > 0$, and by convention, $\beta_0 = \beta_0(\mathrm{d}\sigma) = \mathrm{d}\sigma(\mathbb{R})$. For symmetric measures on $\mathbb{R}$, i.e., when $\mathrm{d}\sigma(-x) = \mathrm{d}\sigma(x)$, the all recurrence coefficients α_k are equal to zero.

For a fixed real number $s > -1/2$, we define the measure

$$\mathrm{d}\sigma^{n,s}(x) = \omega^{n,s}(x)\,\mathrm{d}x = |\widehat{U}_n(x)|^{2s}(1 - x^2)^{1/2+s}\,\mathrm{d}x, \tag{3.2}$$

where $\widehat{U}_n(x)$ is the n-th degree monic Chebyshev polynomial of the second kind. This measure could be recognized as one modification of the Chebyshev measure of the second kind $\mathrm{d}\sigma^{0,0}(x) = \sqrt{1 - x^2}\,\mathrm{d}x$ on $[-1, 1]$. The corresponding (monic) orthogonal polynomials $p_k^{n,s}(x) = p_k^{n,s}(x;\, \mathrm{d}\sigma^{n,s})$, $k \in \mathbb{N}_0$, uniquely exist since $\mathrm{d}\sigma^{n,s}(x)$ is a positive for each $s > -1/2$.

In this section we compute the moments of the measure (3.2) and the recurrence coefficients $\{\beta_k^{n,s}\}$ for the corresponding (monic) orthogonal polynomials $p_k^{n,s}(x)$. In the next section, we derive a differential equality for these orthogonal polynomials and obtain a linear differential equation of the second order, whose one particular solution is just our orthogonal polynomial $p_k^{2,s}$.

3.1. Moments and recurrence coefficients

Case $n = 1$. In this case for $s > -1/2$, i.e., for the measure

$$\mathrm{d}\sigma^{1,s}(x) = |x|^{2s}(1 - x^2)^{1/2+s}\,\mathrm{d}x,$$

the coefficient $\beta_0^{1,s}$ in the corresponding three-term recurrence relation (3.1) is

$$\beta_0^{1,s} = \mu_0^{1,s} = \int_{-1}^{1} |x|^{2s}(1 - x^2)^{1/2+s}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2^{2s+1}}\frac{\Gamma\big(s + \frac{1}{2}\big)}{\Gamma(s + 1)}.$$

Using formulas for β_{2k} and β_{2k-1} (see [16, p. 147]) for $\alpha = s + 1/2$, $\gamma = 2s$ and $\beta = -1/2$, we get directly

$$\beta_k^{1,s} = \begin{cases} \dfrac{k}{4(k + 2s)}, & k \text{ is even } (\geq 2), \\[2mm] \dfrac{k + 4s + 1}{4(k + 2s + 1)}, & k \text{ is odd } (\geq 1), \end{cases}$$

or alternatively

$$\beta_k^{1,s} = \frac{(k+s)(4s+1)+k^2-s(-1)^k(4s+2k+1)}{4(2s+k+1)(2s+k)}, \quad k \geq 1.$$

This result, as well as one more general for the generalized Gegenbauer weight, was proved in 1953 by Laščenov [13] (see also [2, p. 156]).

The graphics of the weight function $x \mapsto \omega^{1,s}(x)$ for some selected values of the parameter s, are presented in Figure 2 (left).

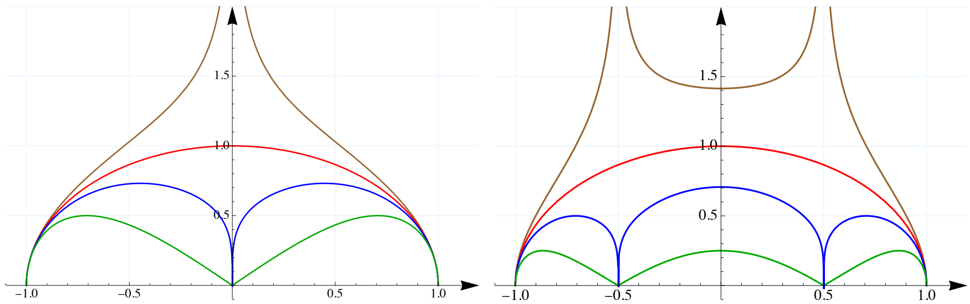

Figure 2: Graphics of the weight functions $x \mapsto \omega^{1,s}(x)$ (left) and $x \mapsto \omega^{2s}(x)$ on $(-1,1)$, for $s = -1/8$ (brown line), $s = 0$ (red line), $s = 1/8$ (blue line) and $s = 1/2$ (green line)

Case $n = 2$. The moments of the weight function

$$x \mapsto \omega^{2,s}(x) = |\widehat{U}_2(x)|^{2s}(1-x^2)^{1/2+s}, \tag{3.3}$$

on $(-1,1)$, for $s > -1/2$ are equal to zero for odd k, while the moments of even order, due to parity of the weight function $x \mapsto \omega^{2,s}(x)$, are given by

$$\mu_k^{2,s} = 2\int_0^1 x^k |\widehat{U}_2(x)|^{2s}(1-x^2)^{s+1/2}\,\mathrm{d}x, \quad k = 0, 1, \ldots .$$

The graphics of the weight function (3.3) for some selected values of the parameter s, are presented in Figure 2 (right).

For calculating recurrence coefficients in the three-term recurrence relation (3.1) we use the Chebyshev method of modified moments developed by Gautschi [7] (see also [8]), because, in this case, we get simpler formulas. This modified Chebyshev algorithm can be expressed as a mapping of the sequence of the modified moments, given by

$$m_k = \int_{-1}^{1} q_k(x)\,\mathrm{d}\mu(x), \quad k = 0, 1, \ldots, , \tag{3.4}$$

into the coefficients of the three term recurrence relation α_k and β_k, where q_k are certain monic polynomials ($\deg q_k = k$) choosen to be "close" in some sense to the desired polynomials p_k. Usually, we supposed that q_k satisfy a three-term recurrence relation of the form (3.1), with recursion coefficients a_k $(\in \mathbb{R})$ and b_k (≥ 0) (instead of α_k and β_k). Then, for a fixed N $(\in \mathbb{N})$ there is a unique map $\rho : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ that maps the first $2N$ modified moments (3.4) into the desired N recurrence coefficients α_k and β_k, i.e., $\{m_k\}_{k=0}^{2N-1} \mapsto \{\alpha_k, \beta_k\}_{k=0}^{N-1}$.

In our construction of the recurrence coefficients α_k and β_k, for the polynomials q_k we take the monic Chebyshev polynomials of the second type, $q_k(x) = \widehat{U}_k(x)$, which satisfy the recurrence relation

$$\widehat{U}_{k+1}(x) = x\widehat{U}_k(x) - \frac{1}{4}\widehat{U}_{k-1}(x), \quad k = 0, 1, \ldots,$$

where $\widehat{U}_0(x) = 1$ and $\widehat{U}_1(x) = x$. Then, the modified moments are given with

$$m_k^{2,s} = \int_{-1}^{1} |\widehat{U}_2(x)|^{2s}(1-x^2)^{s+1/2}\widehat{U}_k(x)\,\mathrm{d}x, \quad k = 0, 1, \ldots.$$

Obviosly, $m_k^{2,s} = 0$ for odd k and

$$m_0^{2,s} = \mu_0^{2,s} = 2\int_0^1 |\widehat{U}_2(x)|^{2s}(1-x^2)^{s+1/2}\,\mathrm{d}x.$$

while for the moments of even order $(k = 2j)$ we have

$$m_{2j}^{2,s} = 2\int_0^1 |\widehat{U}_2(x)|^{2s}(1-x^2)^{s+1/2}\widehat{U}_{2j}(x)\,\mathrm{d}x, \quad j = 0, 1, \ldots.$$

Since, $\widehat{U}_2(x) = x^2 - \frac{1}{4}$ and $\widehat{U}_2(x)^2 = x^4 - \frac{1}{2}x^2 + \frac{1}{16}$, after substitution $x = \cos\theta$, we get

$$\begin{aligned}
\widehat{U}_2(\cos\theta)^2 &= \cos^4\theta - \frac{1}{2}\cos^2\theta + \frac{1}{16} \\
&= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3) - \frac{1}{4}(1 + \cos 2\theta) + \frac{1}{16} \\
&= \frac{1}{16}(3 + 4\cos 2\theta + 2\cos 4\theta).
\end{aligned}$$

Then

$$m_{2j}^{2,s} = \frac{2}{2^{4s}}\int_0^{\pi/2} (3 + 4\cos 2\theta + 2\cos 4\theta)^s \sin^{2s+1}\theta\,\widehat{U}_{2j}(\cos\theta)\sin\theta\,\mathrm{d}\theta$$

$$= \frac{1}{2^{4s+2k-1}} \int_0^{\pi/2} (3 + 4\cos 2\theta + 2\cos 4\theta)^s \sin^{2s+1}\theta \sin(2j+1)\theta \, \mathrm{d}\theta,$$

because $\widehat{U}_{2j}(\cos\theta) = 2^{-2j}\sin(2j+1)\theta/\sin\theta$. For $k = 0$ and $s > -1/2$ it gives

$$m_0^{2,s} = \frac{1}{2^{4s-1}} \int_0^{\pi/2} (3 + 4\cos 2\theta + 2\cos 4\theta)^s \sin^{2s+2}\theta \, \mathrm{d}\theta$$

$$= \frac{\pi}{2^{4s}} \frac{\left(\frac{1}{2}\right)_s}{(1)_s} = \frac{\sqrt{\pi}\,\Gamma\left(s+\frac{1}{2}\right)}{2^{4s+1}\Gamma(s+1)}.$$

To calculate the modified moments $m_{2j}^{2,s}$ for an arbitrary j, we divide our calculations in three parts, for $j = 3\nu$, $j = 3\nu + 1$ and $j = 3\nu + 2$, where $\nu \in \mathbb{N}_0$.

The result of these calculations is summarized in the following theorem:

Theorem 3.1. *For the modified moments of even order $m_{2j}^{2,s}$ for all $s > -1/2$, we have*

$$m_{2j}^{2,s} = \begin{cases} (-1)^\nu m_0^{2,s} \dfrac{(s-\nu+1)_\nu}{2^{6\nu}(s+1)_\nu}, & j = 3\nu, \\ 0, & j = 3\nu+1, \\ (-1)^\nu m_0^{2,s} \dfrac{(s-\nu)_{\nu+1}}{2^{6\nu+4}(s+1)_{\nu+1}}, & j = 3\nu+2, \end{cases}$$

where

$$m_0^{2,s} = \frac{\sqrt{\pi}}{2^{4s+1}} \frac{\Gamma\left(s+\dfrac{1}{2}\right)}{\Gamma(s+1)}.$$

In the previous theorem the Pochhammer symbol is defined by

$$(c)_0 = 1, \quad (c)_\nu = c(c+1)\cdots(c+\nu-1) = \frac{\Gamma(c+\nu)}{\Gamma(c)}.$$

Using the modified moments given in the previous theorems, we are able to calculate coefficients in the corresponding recurrence relations for monic polynomials orthogonal on $(-1,1)$, with respect to the weight functions $x \mapsto \omega^{2,s}(x)$. Here we use the MATHEMATICA package `OrthogonalPolynomials` (see [4], [17]), with the implemented command `aChebyshevAlgorithmModified` in the symbolic mode. This function `aChebyshevAlgorithmModified` returns the sequences of three-term recurrence coefficients of the length N in symbolic form, for a given sequence of the modified moments.

In the following theorem we summarize the obtained results for recurrence coefficients for the polynomials $p_k^{2,s}(x)$ orthogonal with respect to the weight function $x \mapsto \omega^{2,s}(x)$ on $(-1,1)$ and $s > -1/2$.

Theorem 3.2. *For any real parameter* $s > -1/2$, *the polynomials* $p_k^{2,s}(x) = p_k^{2,s}(x;\, \mathrm{d}\sigma^{2,s})$, $k \in \mathbb{N}_0$, *where* $\mathrm{d}\sigma^{2,s}(x) = |\widehat{U}_2(x)|^{2s}(1-x^2)^{1/2+s}\,\mathrm{d}x$, *satisfy three-term recurrence relation*

$$p_{k+1}^{2,s}(x) = xp_k^{2,s}(x) - \beta_k^{2,s} p_{k-1}^{2,s}(x), \quad k \in \mathbb{N}_0,$$

$$p_0^{2,s}(x) = 1, \quad p_{-1}^{2,s}(x) = 0,$$

where

$$\beta_k^{2,s} = \begin{cases} \dfrac{k}{4(k+3s)}, & k = 3\nu, \\[2ex] \dfrac{1}{4}, & k = 3\nu+1, \\[2ex] \dfrac{k+6s+1}{4(k+3s+1)}, & k = 3\nu+2, \end{cases} \tag{3.5}$$

and

$$\beta_0^{2,s} = \frac{\sqrt{\pi}}{2^{4s+1}} \frac{\Gamma\big(s+\frac{1}{2}\big)}{\Gamma(s+1)}.$$

The formal proof of this theorem can be performed in a similar way as the proof of Theorem 3.1 in [6].

Remark 3.1. An alternative analytic expression for the recurrence coefficients (3.5) could be written in the following form

$$\beta_k^{2,s} = \frac{s\,\xi^{2k}\big[(\overline{\gamma+3\delta k})\xi^{2k} - (\gamma+3\delta k)\big] + 2\mathrm{i}\sqrt{3}\big[2s+(6s+1)k+k^2\big]}{4\delta(1+\xi)(3s+k)(3s+1+k)},$$

where $\xi = \mathrm{e}^{\mathrm{i}\pi/3}$, and γ and δ are the complex numbers given by $\gamma = 9s + (9s + 2)\mathrm{i}\sqrt{3}$ and $\delta = 1 + \mathrm{i}\sqrt{3}$.

Remark 3.2. We list now the first nine orthogonal polynomials $p_k^{2,s}(x)$, when $k = 0, 1, \ldots, 8$:

$$\begin{aligned}
&p_0^{2,s}(x) = 1,\ \ p_1^{2,s}(x) = x,\ \ p_2^{2,s}(x) = x^2 - \frac{1}{4},\ \ p_3^{2,s}(x) = x^3 - \frac{(2+3s)x}{4(s+1)},\\
&p_4^{2,s}(x) = x^4 - \frac{3x^2}{4} + \frac{1}{16(s+1)},\ \ p_5^{2,s}(x) = x^5 - x^3 + \frac{3x}{16},\\
&p_6^{2,s}(x) = x^6 - \frac{(5+3s)x^4}{(4+2s)} + \frac{3(4+2s)x^2}{16(s+2)} - \frac{1}{32(s+2)},\\
&p_7^{2,s}(x) = x^7 - \frac{3x^5}{2} + \frac{(9s+20)x^3}{32+16s} - \frac{x}{8(s+2)},\\
&p_8^{2,s}(x) = x^8 - \frac{7x^6}{4} + \frac{15x^4}{16} - \frac{(20+9s)x^2}{64(s+2)} + \frac{1}{128(s+2)}.
\end{aligned}$$

Evidently, for $s = 0$ these polynomials reduce to the monic Chebyshev polynomials of the second kind $\widehat{U}_k(x)$.

In the sequel we state the results for the modified moments $m_{2k}^{n,s}$, obtained for the weight function $x \mapsto \omega^{n,s}(x)$, when $n \in \mathbb{N}$ and $s > -1/2$, as well as the recurrence coefficients $\beta_k^{n,s}$ of the corresponding orthogonal polynomials $p_k^{n,s}(x)$, $k \in \mathbb{N}_0$.

Theorem 3.3. *For the modified moments of even order $m_{2k}^{n,s}$ for all $s > -1/2$, we have*

$$m_{2k}^{n,s} = \begin{cases} (-1)^\nu m_0^{n,s} \dfrac{(s-\nu+1)_\nu}{2^{2(n+1)\nu}(s+1)_\nu}, & k \equiv 0 \pmod{n+1},\\ 0, & k \equiv 1 \pmod{n+1},\\ (-1)^\nu m_0^{n,s} \dfrac{(s-\nu)_{\nu+1}}{2^{2(n+1)\nu+4}(s+1)_{\nu+1}}, & k \equiv n \pmod{n+1}, \end{cases}$$

where

$$m_0^{n,s} = \frac{\sqrt{\pi}}{2^{2ns+1}} \frac{\Gamma\big(s+\frac{1}{2}\big)}{\Gamma(s+1)}.$$

Theorem 3.4. *For any real parameter $s > -1/2$, the polynomials $p_k^{n,s}(x) = p_k^{n,s}(x; \mathrm{d}\sigma^{n,s})$, $k \in \mathbb{N}_0$, orthogonal with respect to the measure*

$$\mathrm{d}\sigma^{n,s}(x) = |\widehat{U}_n(x)|^{2s}(1-x^2)^{1/2+s}\,\mathrm{d}x,$$

satisfy the three-term recurrence relation

$$\begin{aligned}
&p_{k+1}^{n,s}(x) = x p_k^{n,s}(x) - \beta_k^{n,s} p_{k-1}^{n,s}(x), \quad k = 0, 1, \ldots,\\
&p_0^{n,s}(x) = 1, \quad p_{-1}^{n,s}(x) = 0,
\end{aligned}$$

where

$$\beta_k^{2,s} = \begin{cases} \dfrac{k}{4(k+(n+1)s)}, & k \equiv 0 \ (\mathrm{mod}\ n+1), \\ \dfrac{1}{4}, & k \equiv 1 \ (\mathrm{mod}\ n+1), \\ \dfrac{k+2(n+1)s+1}{4(k+(n+1)s+1)}, & k \equiv n \ (\mathrm{mod}\ n+1). \end{cases}$$

and

$$\beta_0^{n,s} = \frac{\sqrt{\pi}}{2^{2ns+1}} \frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(s+1)}.$$

4. *Differential equality and differential equation*

Since the considered families of semiclassical polynomials are characterized by a structure relation and a linear homogeneous second order differential equation (see [1], [12], [14]), in this section we derive a differential equality for representing the expression $-(1-x^2)\widehat{U}_2(x)(p_k^{2,s}(x))'$ in terms of two orthogonal polynomials, $p_k^{2,s}(x)$ and $p_{k-1}^{2,s}(x)$, as well as a linear differential equation of the second order for orthogonal polynomials $p_k^{2,s}(x)$.

Theorem 4.1. *For every polynomial $p_k^{2,s}(x)$ there exist polynomials $P_k^{2,s}(x)$ and $Q_k^{2,s}(x)$ of degrees 3 and 2, respectively, such that*

$$-(1-x^2)\widehat{U}_2(x)(p_k^{2,s}(x))' = P_k^{2,s}(x)p_k^{2,s}(x) + Q_k^{2,s}(x)p_{k-1}^{2,s}(x). \tag{4.1}$$

PROOF. Denote $\phi = 1 - x^2$. Following [6, Theorem 4.1] we have

$$\left(\phi(x)\widehat{U}_2(x)p_k^{2,s}(x)w^{2,s}(x)\right)' = w^{2,s}(x)R_{k,2}(x),$$

where $\omega^{2,s}(x)$ is given by (3.3) and $R_{k,2}(x)$ is a polynomial of degree $k+3$ defined by

$$R_{k,2}(x) = \Big[-x(3+2s)\widehat{U}_2(x)+\phi(x)(2s+1)\widehat{U}_2'(x)\Big]p_k^{2,s}(x)+\phi(x)\widehat{U}_2(x)(p_k^{2,s}(x))'. \tag{4.2}$$

Using an integration by parts and the orthogonality of the polynomials $p_k^{2,s}(x)$ with respect to the weight function (3.3), after some computation, we conclude that

$$R_{k,2}(x) = r_k^{2,s}(x)p_k^{2,s}(x) + s_k^{2,s}(x)p_{k-1}^{2,s}(x), \tag{4.3}$$

where $r_k^{2,s}(x)$ and $s_k^{2,s}(x)$ are some polynomials of the third and the second degree, respectively. Finally, from (4.2) and (4.3) we get (4.1), where

$$P_k^{2,s}(x) = (2s+1)\phi(x)\widehat{U}_2'(x) - x(3+2s)\widehat{U}_2(x) - r_k^{2,s}(x)$$

and $Q_k^{2,s}(x) = -s_k^{2,s}(x)$.

As we can see $\deg(P_k^{2,s}(x)) = 3$ and $\deg(Q_k^{2,s}(x)) = 2$.

Remark 4.1. Similarly to [6, Remark 4.2] we can show that

$$P_k^{2,s}(x) = kx^3 + a_k x \quad \text{and} \quad Q_k^{2,s}(x) = b_k x^2 + c_k, \tag{4.4}$$

where a_k, b_k, c_k are constants depending on k and s.

For $k = 1$ and $k = 2$, the polynomials (4.4) are not unique. Namely, $a_1 = \alpha$, $b_1 = -5/4 - \alpha$, $c_1 = 1/4$ and $a_2 = \alpha$, $b_2 = -2 - \alpha$, $c_2 = 1/2 + \alpha/4$, where α is a free parameter.

In the case $k \geq 3$ the polynomials (4.4) are unique, e.g., for $k = 3$ we have

$$a_3 = -\frac{3}{4}, \quad b_3 = -\frac{3s+4}{2(s+1)}, \quad c_3 = \frac{3s+2}{4(s+1)}.$$

The following auxiliary result, which enables us to get explicit expressions of the polynomials in (4.4), was proved in [6, Lemma 4.3].

Lemma 4.1. *Let $F_k(x) = (d_k x^2 + e_k)p_k^{2,s}(x) + (f_k x^3 + g_k x)p_{k-1}^{2,s}(x)$, where $d_k,\ e_k,\ f_k,\ g_k$ are constants and $p_{k-1}^{2,s}(x)$ and $p_k^{2,s}(x)$ are two successive orthogonal polynomials. Then for $k \geq 4$ we have the following equivalence*

$$F_k(x) \equiv 0 \iff d_k = e_k = f_k = g_k = 0.$$

Theorem 4.2. *The coefficients a_k, b_k, c_k of the unique polynomials $P_k^{2,s}(x)$ and $Q_k^{2,s}(x)$, $k \geq 4$, in (4.4) are given by*

$$a_k = \begin{cases} -\dfrac{k}{4}, & k = 3\nu, \\[2mm] -\dfrac{k-6s}{4}, & k = 3\nu+1, \\[2mm] -\dfrac{k}{4} + 3s, & k = 3\nu+2; \end{cases}$$

$$b_k = \begin{cases} -\dfrac{k(k+1+3s)}{2(k+3s)}, & k = 3\nu, \\[2mm] -\dfrac{k+3s+1}{2}, & k = 3\nu+1, \\[2mm] -\dfrac{k+6s+1}{2}, & k = 3\nu+2; \end{cases}$$

$$c_k = \begin{cases} \dfrac{k(k+6s+1)}{8(k+3s)}, & k = 3\nu, \\[2mm] \dfrac{k+1}{8}, & k = 3\nu+1, \\[2mm] \dfrac{k+6s+1}{8}, & k = 3\nu+2. \end{cases}$$

The proof of this result is the same with one in [6, Theorem 4.4].

Theorem 4.3. *The polynomials* $P_k^{2,s}(x)$ *and* $Q_k^{2,s}(x)$ *in* (4.1) *can be expressed in terms of* $\widehat{U}_2(x)$ *as*

$$P_k^{2,s}(x) = \begin{cases} kx\widehat{U}_2(x), & k = 3\nu, \\ x(k\widehat{U}_2(x) + 3s/2), & k = 3\nu + 1, \\ x(k\widehat{U}_2(x) + 3s), & k = 3\nu + 2, \end{cases}$$

and

$$Q_k^{2,s}(x) = \begin{cases} -\dfrac{k}{2(k+3s)}\Big((k+3s+1)\widehat{U}_2(x) - \frac{3s}{4}\Big), & k = 3\nu, \\ -\frac{1}{2}\Big((3s+k+1)\widehat{U}_2(x) + \frac{3s}{4}\Big), & k = 3\nu + 1, \\ -\dfrac{6s+k+1}{2}\widehat{U}_2(x), & k = 3\nu + 2. \end{cases}$$

Finally, as in [6], we obtain the second order differential equation for the polynomials $p_k^{2,s}(x)$.

Theorem 4.4. *The polynomial* $y = p_k^{2,s}(x)$*,* $k \in \mathbb{N}$*, satisfies the differential equation*

$$A_k(x)y'' + B_k(x)y' + C_k(x)y = 0,$$

where

$$\begin{aligned} A_k(x) &= \frac{\phi_2(x)^2}{Q_k^{2,s}(x)} = \frac{(1-x^2)^2\widehat{U}_2(x)^2}{Q_k^{2,s}(x)}, \\ B_k(x) &= \phi_2(x)\left[\frac{\phi_2(x)}{Q_k^{2,s}(x)}\right]' + \frac{\phi_2(x)}{Q_k^{2,s}(x)}\left[P_k^{2,s}(x) + P_{k-1}^{2,s}(x) + x\frac{Q_{k-1}^{2,s}(x)}{\beta_{k-1}^{2,s}}\right], \\ C_k(x) &= \phi_2(x)\left[\frac{P_k^{2,s}(x)}{Q_k^{2,s}(x)}\right]' + \frac{P_k^{2,s}(x)}{Q_k^{2,s}(x)}\left[P_{k-1}^{2,s}(x) + x\frac{Q_{k-1}^{2,s}(x)}{\beta_{k-1}^{2,s}}\right] + \frac{Q_{k-1}^{2,s}(x)}{\beta_{k-1}^{2,s}}, \end{aligned}$$

and $\phi_2(x) = (1-x^2)\widehat{U}_2(x)$.

Using expressions for polynomials $P_k^{2,s}(x)$ and $Q_k^{2,s}(x)$ given in Theorem 4.3, we can obtain concrete differential equations for $k = 3\nu + \ell$, when $\ell = 0, 1, 2$.

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