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MOMENT–PRESERVING SPLINE APPROXIMATION ON FINITE INTERVALS AND TURÁN QUADRATURES

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Abstract. We discuss the problem of approximating a function f on the interval $[0, 1]$ by a spline function of degree m and defect d, with n (variable) knots, matching as many of the initial moments of f as possible. Additional constraints on the derivatives of the approximation at one endpoint of $[0, 1]$ may also be imposed. We analyse the case when the defect d is an odd integer ($d = 2s + 1$), and we show that, if the approximation exists, it can be represented in terms of generalized Turán quadrature relative to a measure depending on f. The knots are the zeros of the corresponding s-orthogonal polynomials ($s \geq 1$). A numerical example is included.

1. Introduction

Continuing previous works [4–5], Milovanović and Kovačević [6] have considered the problem of approximating a spherically symmetric function $f(r)$, $r = \|x\|$, $0 \leq r < \infty$, in $\mathbb{R}^d$, $d \geq 1$, by a spline function of degree $m \geq 2$ and defect d $(1 \leq d \leq m)$, with n knots. Under suitable assumptions on f and $d = 2s + 1$, it was shown that the problem as a unique solution if and only if certain generalized Turán quadratures exist corresponding to a measure depending on f. Existence, uniqueness and pointwise convergence of such approximation were analyzed.

In [1] Frontini, Gautschi and Milovanović considered the analogous of the problem treated in [5] on an arbitrary finite interval. If the approximations exist, they can be represented in terms of generalized Gauss-Lobatto and Gauss-Radau quadrature formulas relative to appropriate measures depending on f. In this paper we discuss the case of approximating a function $f = f(t)$ on some given finite interval $[a, b]$, which can be standardized to $[a, b] = [0, 1]$, by a spline function of degree $m \geq 2$ and defect d $(1 \leq d \leq m)$, with n knots. Under suitable assumptions on f and $d = 2s + 1$ we will show that our problem has a unique solution if and only if certain generalized Turán-Radau and Turán-Lobatto quadratures formulas exist corresponding to measures depending on f. Existence, uniqueness and pointwise convergence is assured if f is completely monotonic on $[0, 1]$. One simple numerical example is included.

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2. Spline approximation on $[0,1]$

A spline function of degree $m \geq 2$ and defect d, with n (distinct) knots τ_1, τ_2, ..., τ_n in the interior of $[0,1]$, can be written in terms of truncated powers in the form

$$s_{n,m}(t) = p_m(t) + \sum_{\nu=1}^{n} \sum_{i=m-d+1}^{m} a_{i,\nu}(\tau_\nu - t)_+^i, \tag{2.1}$$

where $a_{i,\nu}$ are real numbers and $p_m(t)$ is a polynomial of degree $\leq m$.

Similarly as in [1] we will consider two related problems:

Problem I. Determine $s_{n,m}$ in (2.1) such that

$$\int_0^1 t^j s_{n,m}(t)\,dt = \int_0^1 t^j f(t)\,dt, \qquad j = 0, 1, \ldots, (d+1)n + m. \tag{2.2}$$

Problem I.* Determine $s_{n,m}$ in (2.1) such that

$$s_{n,m}^{(k)}(1) = p_m^{(k)}(1) = f^{(k)}(1), \qquad k = 0, 1, \ldots, m, \tag{2.3}$$

and such that (2.2) holds for $j = 0, 1, ..., (d+1)n - 1$.

In this paper we will reduce our problems to the power-orthogonality (s-orthogonality) and generalized Gauss-Turán quadratures by restricting the class of functions f (see [6]).

In order to reduce our problems (2.2) and (2.3) to the power-orthogonality, we have to put $d = 2s + 1$, i.e., the defect of the spline function (2.1) should be odd.

Let

$$\phi_k = \frac{(-1)^k}{m!} f^{(k)}(1), \quad b_k = \frac{(-1)^k}{m!} p_m^{(k)}(1), \quad k = 0, 1, ..., m, \tag{2.4}$$

applying $m+1$ integration by parts to the integrals in the moment equation (2.2) we obtain (see [1])

$$\begin{aligned} &\sum_{k=0}^{m} b_k \left[D^{m-k} t^{m+1+j}\right]_{t=1} + \sum_{\nu=1}^{n} \sum_{i=m-2s}^{m} a_{i,\nu} \tau_\nu^{j+i+1} \frac{i!(m+j+1)!}{m!(j+i+1)!} \\ &= \sum_{k=0}^{m} \phi_k \left[D^{m-k} t^{m+1+j}\right]_{t=1} + \frac{(-1)^{m+1}}{m!} \int_0^1 t^{m+1+j} f^{(m+1)}(t)\,dt, \\ &\qquad\qquad j = 0, 1, \ldots, 2(s+1)n + m, \end{aligned} \tag{2.5}$$

where D is the standard differentiation operator.

For the second sum in (2.5) we may observe that

$$\sum_{\nu=1}^{n}\sum_{i=m-2s}^{m} a_{i,\nu}\tau_\nu^{j+i+1}\frac{i!(m+j+1)!}{m!(j+i+1)!} = \sum_{\nu=1}^{n}\sum_{i=m-2s}^{m}\frac{i!}{m!}a_{i,\nu}\left[D^{m-i}t^{m+j+1}\right]_{t=\tau_\nu},$$

changing indices ($k=m-i$), the second sum on the right becomes

$$\sum_{k=0}^{2s}\frac{(m-k)!}{m!}a_{m-k,\nu}\left[D^k(t^{m+1}t^j)\right]_{t=\tau_\nu}, \tag{2.6}$$

hence defining the measure

$$d\lambda(t) = \frac{(-1)^{m+1}}{m!}f^{(m+1)}(t)\,dt \quad \text{on} \quad [0,1], \tag{2.7}$$

equations (2.5) may be rewrite

$$\begin{aligned}\sum_{k=0}^{m} b_k\left[D^{m-k}t^{m+1+j}\right]_{t=1} + \sum_{\nu=1}^{n}\sum_{k=0}^{2s}\frac{(m-k)!}{m!}a_{m-k,\nu}\left[D^k(t^{m+1+j})\right]_{t=\tau_\nu}\\ = \sum_{k=0}^{m}\phi_k\left[D^{m-k}t^{m+1+j}\right]_{t=1} + \int_0^1 t^{m+1+j}\,d\lambda(t),\\ j=0,1,\dots,2(s+1)n+m,\end{aligned} \tag{2.8}$$

Now we can state the main result for *Problem I*:

Theorem 2.1. *Let $f\in C^{m+1}[0,1]$. There exists a unique spline function* (2.1) *on* $[0,1]$, *with $d=2s+1$, satisfying* (2.2) *if and only if the measure $d\lambda(t)$ in* (2.7) *admits a generalized Gauss-Lobatto-Turán quadrature*

$$\begin{aligned}\int_0^1 g(t)\,d\lambda(t) = \sum_{k=0}^{m}\left[\alpha_k g^{(k)}(0)+\beta_k g^{(k)}(1)\right]\\ +\sum_{\nu=1}^{n}\sum_{i=0}^{2s}A_{i,\nu}^{L}g^{(i)}(\tau_\nu^{(n)}) + R_{n,m}^{L}(g;d\lambda),\end{aligned} \tag{2.9}$$

where

$$R_{n,m}^{L}(g;d\lambda)=0 \quad \textit{for all} \quad g\in\mathcal{P}_{2(s+1)n+2m+1}, \tag{2.10}$$

with distinct real zeros $\tau_\nu^{(n)}$, $\nu=1,2,\dots,n$, all contained in the open interval $(0,1)$. *The spline function in* (2.1) *is given by*

$$\tau_\nu=\tau_\nu^{(n)}, \quad a_{m-k,\nu}=\frac{m!}{(m-k)!}A_{k,\nu}^{L}, \quad \nu=1,2,\dots,n;\ k=0,1,\dots,2s, \tag{2.11}$$

where $\tau_\nu^{(n)}$ *are the interior nodes of the generalized Gauss-Lobatto-Turán quadrature formula and* $A_{k,\nu}^L$ *are the corresponding weights, while the polynomial* $p_m(t)$ *is given by*

$$p_m^{(k)}(1) = f^{(k)}(1) + (-1)^k m!\beta_{m-k}, \quad k = 0, 1, \dots, m, \tag{2.12}$$

where β_{m-k} *is the coefficient of* $g^{(m-k)}(1)$ *in* (2.9).

Proof. Putting $g(t) = t^{m+1}p(t)$, $p \in \mathcal{P}_{2(s+1)n+m}$, in (2.9) and noting (2.10) yields

$$\sum_{k=0}^{m} \beta_k \left[D^k t^{m+1} p(t)\right]_{t=1} + \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu}^L \left[D^k (t^{m+1} p(t)\right]_{t=\tau_\nu}$$
$$= \int_0^1 t^{m+1} p(t)\, d\lambda(t), \quad \forall p \in \mathcal{P}_{2(s+1)n+m},$$

which is identical to (2.8), if we identify

$$b_{m-k} - \phi_{m-k} = \beta_k, \qquad k = 0, 1, \dots, m;$$
$$a_{m-k,\nu} = \frac{m!}{(m-k)!} A_{k,\nu}^L, \quad \nu = 1, 2, \dots, n;\ k = 0, 1, \dots, 2s. \quad \square$$

Remark. The case $s = 0$ of Theorem 2.1 has been obtained in [1].

If f is completely monotonic on $[0, 1]$ then $d\lambda(t)$ in (2.7) is a positive measure for every m, then by virtue of the assumptions in Theorem 2.1 the generalized Gauss-Lobatto-Turán quadrature formula exists uniquely, with n distinct real nodes $\tau_\nu^{(n)}$ in $(0, 1)$.

The solution of *Problem* I^* can be given in a similar way.

Theorem 2.2. *Let* $f \in C^{m+1}[0, 1]$. *There exists a unique spline function on* $[0, 1]$,

$$s_{n,m}^*(t) = p_m^*(t) + \sum_{\nu=1}^{n}\sum_{i=m-2s}^{m} a_{i,\nu}^*(\tau_\nu^* - t)_+^i, \tag{2.13}$$
$$0 < \tau_\nu^* < 1,\ \tau_\nu^* \neq \tau_\mu^* \quad \text{for } \nu \neq \mu,$$

satisfying (2.3) *and* (2.2), *for* $j = 0, 1, \dots, 2(s+1)n - 1$, *if and only if the measure* $d\lambda(t)$ *in* (2.7) *admits a generalized Gauss-Radau-Turán quadrature*

$$\int_0^1 g(t)\, d\lambda(t) = \sum_{k=0}^{m} \alpha_k^* g^{(k)}(0) + \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu}^R g^{(i)}(\tau_\nu^{(n)*}) + R_{n,m}^R(g; d\lambda), \tag{2.14}$$

where

$$R_{n,m}^R(g; d\lambda) = 0 \quad \text{for all} \quad g \in \mathcal{P}_{2(s+1)n+m},$$

with distinct real zeros $\tau_\nu^{(n)*}$, $\nu = 1, 2, \dots, n$, *all contained in the open interval* $(0, 1)$. *The knots* τ_ν^* *in* (2.13) *are then precisely these zeros,*

$$\tau_\nu^* = \tau_\nu^{(n)*}, \quad \nu = 1, \dots, n, \tag{2.15}$$

and

$$a_{m-k,\nu}^* = \frac{m!}{(m-k)!} A_{k,\nu}^R, \quad \nu = 1, 2, \dots, n;\ k = 0, 1, \dots, 2s, \tag{2.16}$$

while the polynomial $p_m^*(t)$ *is given by*

$$p_m^*(t) = \sum_{k=0}^{m} \frac{f^{(k)}(1)}{k!} (t-1)^k. \tag{2.17}$$

3. Error analysis

Similarly as in [1], following [4], we can prove the following statement regarding to the error of spline approximations:

Theorem 3.1. *Define*

$$\rho_x(t) = (t - x)_+^m, \qquad 0 \le t \le 1.$$

Under conditions of Theorem 2.1 *and Theorem* 2.2, *we have*

$$f(x) - s_{n,m}(x) = R_{n,m}^L(\rho_x; d\lambda), \qquad 0 < x < 1, \tag{3.1}$$

and

$$f(x) - s_{n,m}^*(x) = R_{n,m}^R(\rho_x; d\lambda), \qquad 0 < x < 1, \tag{3.2}$$

respectively, where $R_{n,m}^L(g; d\lambda)$ *and* $R_{n,m}^R(g; d\lambda)$ *are the remainder terms in the corresponding Gauss-Turán formulas of Lobatto and Radau type.*

Proof. We will prove (3.1). As in [1] we have

$$f(x) = \sum_{k=0}^{m} \frac{f^{(k)}(1)}{k!} (x-1)^k + \int_0^1 \rho_x(t)\, d\lambda(t). \tag{3.3}$$

By (2.11)

$$s_{n,m}(x) = \sum_{k=0}^{m} \frac{p^{(k)}(1)}{k!} (x-1)^k + \sum_{\nu=1}^{n} \sum_{i=m-2s}^{m} \frac{m!}{i!} A_{m-i,\nu}^L (\tau_\nu - x)_+^i \tag{3.4}$$

and changing indices ($k = m - i$), the third sum on the right becomes

$$\sum_{i=m-2s}^{m} \frac{m!}{i!} A_{m-i,\nu}^{L} (\tau_\nu - x)_+^i = \sum_{k=0}^{m} \frac{m!}{(m-k)!} A_{m-i,\nu}^{L} (\tau_\nu - x)_+^{m-k}$$
$$= \sum_{k=0}^{m} A_{k,\nu}^{L} \left[D^k \rho_x(t) \right]_{t=\tau_\nu} .$$

Equation (3.4) may be rewrite

$$\text{(3.5)} \qquad s_{n,m}(x) = \sum_{k=0}^{m} \frac{p^{(k)}(1)}{k!} (x-1)^k + \sum_{\nu=1}^{n} \sum_{k=0}^{m} A_{k,\nu}^{L} \left[D^k \rho_x(t) \right]_{t=\tau_\nu} .$$

Subtracting (3.5) from (3.3) gives

$$f(x) - s_{n,m}(x) = \int_0^1 \rho_x(t)\, d\lambda(t) + \sum_{k=0}^{m} \frac{1}{k!} \left(f^{(k)}(1) - p^{(k)}(1) \right) (x-1)^k$$
$$- \sum_{\nu=1}^{n} \sum_{k=0}^{m} A_{k,\nu}^{L} \left[D^k \rho_x(t) \right]_{t=\tau_\nu}$$

which, by virtue of (2.12) and (2.4), yields

$$f(x) - s_{n,m}(x) = \int_0^1 \rho_x(t)\, d\lambda(t) - \sum_{k=0}^{m} \frac{m!}{k!} \beta_{m-k} (1-x)^k$$
$$- \sum_{\nu=1}^{n} \sum_{k=0}^{m} A_{k,\nu}^{L} \left[D^k \rho_x(t) \right]_{t=\tau_\nu} .$$

But

$$\rho_x^{(k)}(0) = 0, \quad \rho_x^{(k)}(1) = \frac{m!}{(m-k)!} (1-x)^{m-k}, \quad k = 0, 1, \ldots, m,$$

so that

$$f(x) - s_{n,m}(x) = \int_0^1 \rho_x(t)\, d\lambda(t) - \sum_{k=0}^{m} \beta_{m-k} \rho_x^{(m-k)}(1)$$
$$- \sum_{\nu=1}^{n} \sum_{k=0}^{m} A_{k,\nu}^{L} \left[D^k \rho_x(t) \right]_{t=\tau_\nu}$$

as claimed in (3.1).

The proof of (3.2) is entirely analogous to the proof of (3.1) and it will be omitted. □

4. Construction of spline approximation

In [7] one of us considered the generalized Gauss-Turán quadrature formula

$$\int_{\mathbb{R}} g(t)\,d\sigma(t) = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu}^{G} g^{(i)}(\tau_\nu^{(n)}) + R_n^G(g), \tag{4.1}$$

where $d\sigma(t)$ is a nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_k = \int_{\mathbb{R}} t^k\,d\sigma(t)$, $k = 0, 1, \dots$, exist and are finite, and $\mu_0 > 0$. The formula (4.1) is exact for all polynomials of degree at most $2(s+1)n - 1$, i.e.,

$$R_n^G(g) = 0 \qquad \text{for} \quad g \in \mathcal{P}_{2(s+1)n-1}.$$

The knots $\tau_\nu^{(n)}$ ($\nu = 1, \dots, n$) in (4.1) are zeros of a (monic) polynomial $\pi_n(t)$, which minimizes the following integral

$$\int_{\mathbb{R}} \pi_n(t)^{2s+2}\,d\sigma(t),$$

where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. In the other words, the polynomial π_n satisfies the following generalized orthogonality conditions

$$\int_{\mathbb{R}} \pi_n(t)^{2s+1} t^k\,d\sigma(t), \qquad k = 0, 1, \dots, n-1. \tag{4.2}$$

This polynomial π_n is known as s-orthogonal (or s-self associated) polynomial with respect to the measure $d\sigma(t)$. For $s = 0$, we have the standard case of orthogonal polynomials, and (4.1) then becomes well-known Gauss-Christoffel formula.

The "orthogonality condition" (4.1) can be interpreted as (see [7])

$$\int_{\mathbb{R}} \pi_\nu^{s,n}(t) t^k\,d\mu(t) = 0, \qquad k = 0, 1, \dots, \nu - 1,$$

where $\{\pi_\nu^{s,n}\}$ is a sequence of standard monic polynomials orthogonal on $\mathbb{R}$ with respect to the new measure $d\mu(t) = d\mu^{s,n}(t) = (\pi_n^{s,n}(t))^{2s} d\sigma(t)$. The polynomials $\{\pi_\nu^{s,n}\}$, $\nu = 0, 1, \dots$, are implicitly defined because the measure $d\mu(t)$ depends on $\pi_n^{s,n}(t)$ $\big(= \pi_n(t)\big)$. Of course, we are interested only in $\pi_n^{s,n}(t)$. A stable algorithm for constructing such (s-orthogonal) polynomials is given in [7].

In order to use this algorithm in construction of spline functions (2.1) and (2.13) we need two auxiliary results. These results give a conection between the generalized Gauss-Turán quadrature (4.1) and the corresponding formulas of Lobatto and Radau type.

Lemma 4.1. *If the measure $d\lambda(t)$ in (2.7) admits the generalized Gauss-Lobatto-Turán quadrature (2.9), with distinct real zeros $\tau_\nu = \tau_\nu^{(n)}$, $\nu = 1, \ldots, n$, all contained in the open interval $(0,1)$, there exists then a generalized Gauss-Turán formula*

$$\int_0^1 g(t)\, d\sigma(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu}^G g^{(i)}(\tau_\nu^{(n)}) + R_n^G(g), \tag{4.3}$$

where $d\sigma(t) = [t(1-t)]^{m+1} d\lambda(t)$, the nodes $\tau_\nu^{(n)}$ are the zeros of s-orthogonal polynomial $\pi_n(\cdot; d\sigma))$, while the weights $A_{i,\nu}^G$ are expressible in terms of those in (2.9) by

$$A_{i,\nu}^G = \sum_{k=i}^{2s} \binom{k}{i} \Big[D^{k-i} \left(t(1-t)\right)^{m+1}\Big]_{t=\tau_\nu} A_{k,\nu}^L, \qquad i = 0, 1, \ldots, 2s. \tag{4.4}$$

Proof. Let $g(t) = \left(t(1-t)\right)^{m+1} p(t),\ p \in \mathcal{P}_{2(s+1)n-1}$ and $\tau_\nu = \tau_\nu^{(n)}$. We have by (2.9)

$$\int_0^1 g(t)\, d\lambda(t) = \sum_{\nu=1}^{n} \sum_{k=0}^{2s} A_{k,\nu}^L \Big[D^k \left(t(1-t)\right)^{m+1} p(t)\Big]_{t=\tau_\nu},$$

and by (4.3)

$$\int_0^1 p(t)\, d\sigma(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu}^G \left[D^i p(t)\right]_{t=\tau_\nu}.$$

Thus, we obtain that

$$\sum_{\nu=1}^{n} \sum_{k=0}^{2s} A_{k,\nu}^L \Big[D^k \left(t(1-t)\right)^{m+1} p(t)\Big]_{t=\tau_\nu} = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu}^G \left[D^i p(t)\right]_{t=\tau_\nu}.$$

Appling the Leibnitz formula to k-th derivative in the second sum, we find

$$\begin{aligned}
&\sum_{k=0}^{2s} A_{k,\nu}^L \Big[D^k \left(t(1-t)\right)^{m+1} p(t)\Big]_{t=\tau_\nu} \\
&\qquad = \sum_{k=0}^{2s} A_{k,\nu}^L \left[\sum_{i=0}^{k} \binom{k}{i} D^{k-i} \left(t(1-t)\right)^{m+1} D^i p(t)\right]_{t=\tau_\nu} \\
&\qquad = \sum_{i=0}^{2s} \left(\sum_{k=i}^{2s} \binom{k}{i} \Big[D^{k-i} \left(t(1-t)\right)^{m+1}\Big]_{t=\tau_\nu} A_{k,\nu}^L \left[D^i p(t)\right]_{t=\tau_\nu} \right) \\
&\qquad = \sum_{i=0}^{2s} A_{i,\nu}^G \left[D^i p(t)\right]_{t=\tau_\nu},
\end{aligned}$$

where

$$A_{i,\nu}^G = \sum_{k=i}^{2s} \binom{k}{i} \left[D^{k-i} \left(t(1-t) \right)^{m+1} \right]_{t=\tau_\nu} A_{k,\nu}^L, \quad i = 0, 1, \ldots, 2s. \quad \square$$

Similarly we can prove:

Lemma 4.2. *If the measure* $d\lambda(t)$ *in* (2.7) *admits the generalized Gauss-Radau-Turán quadrature* (2.14), *with distinct real zeros* $\tau_\nu = \tau_\nu^{(n)*}$, $\nu = 1, \ldots, n$, *all contained in the open interval* $(0,1)$, *there exists then a generalized Gauss-Turán formula* (4.3), *where* $d\sigma(t) = d\sigma^*(t) = t^{m+1} d\lambda(t)$, *the nodes* $\tau_\nu^{(n)*}$ *are the zeros of s-orthogonal polynomial* $\pi_n(\cdot;\, d\sigma^*)$, *while the weights* $A_{i,\nu}^G$ *are expressible in terms of those in* (2.14) *by*

$$A_{i,\nu}^G = \sum_{k=i}^{2s} \binom{k}{i} \left[D^{k-i} t^{m+1} \right]_{t=\tau_\nu} A_{k,\nu}^R, \qquad i = 0, 1, \ldots, 2s. \tag{4.5}$$

Now, we can state a construction procedure of our spline approximations:

1° For a given $t \mapsto f(t)$ and (n, m, s), we find the measure $d\lambda(t)$ and the corresponding Jacobi matrix $J_N(d\lambda)$, where $N = (s+1)n + 2m + 2$ in the Lobatto case, and $N = (s+1)n + m + 1$ in the Radau case. The latter can be computed by the discretized Stieltjes procedure (see [2, § 2.2]).

2° By repeated application of the algorithms in [3, § 4.1] corresponding to multiplication of a measure by $t(1-t)$ and t, from the above Jacobi matrices, we generate the Jacobi matrices $J_{(s+1)n}(d\sigma)$ and $J_{(s+1)n}(d\sigma^*)$, respectively. Here, $d\sigma(t) = (t(1-t))^{m+1} d\lambda(t)$ and $d\sigma^*(t) = t^{m+1} d\lambda(t)$.

3° Using the algorithm for the construction of s-orthogonal polynomials, given in [7], we obtain the Jacobi matrix $J_n(d\mu)$, where $d\mu(t) = (\pi_n(t))^{2s} d\sigma(t)$, or $d\mu(t) = (\pi_n(t))^{2s} d\sigma^*(t)$.

4° From $J_n(d\mu)$ we determine the Gaussian nodes $\tau_\nu^{(n)}$ (resp. $\tau_\nu^{(n)*}$ in the Radau case) and the corresponging weights $A_{i,\nu}^G$ ($\nu = 1, \ldots, n$; $i = 0, 1, \ldots, 2s$).

5° From the triangular systems of linear equations (4.4) and (4.5), we find the coefficients $A_{k,\nu}^L$ and $A_{k,\nu}^R$, respectively.

6° Using (2.11) and (2.12), or (2.15), (2.16) and (2.17), we determine the spline approximation $s_{n,m}(t)$, or $s_{n,m}^*(t)$, respectively.

5. Numerical example

We consider the spline approximations of the exponential function $f(t) = e^{-ct}$, $0 \le t \le 1$, where $c > 0$. All computations were done on the PC/AT in double precision (machine precision $\approx\ 8.88 \times 10^{-16}$).

In this example the function f is completely monotonic and the associated measure (2.7) is positive. Thus

$$d\lambda(t) = \frac{c^{m+1}}{m!}\, e^{-ct}\, dt \quad \text{on} \;\; [0,1].$$

In the discretized Stieltjes algorithm (Step 1° in the procedure given in the previous section), we use Fejér quadrature rule as the modus of discretization.

We analyzed the cases when $n \le 10$, $2 \le m \le 5$, $s \le 2$, $c = 1, 2, 4$. For example, for $n = m = 3$, $s = 1$, $c = 1$, the parameters of the spline function in the Lobatto case,

$$s_{n,m}(t) = \sum_{k=0}^{m} \gamma_k (1-t)^k + \sum_{\nu=1}^{n} \sum_{i=m-2s}^{2s} a_{i,\nu} (\tau_\nu - t)_+^i,$$

are given in Table 5.1 (to 10 decimals only, to save space). Numbers in parenthesis idicate decimal exponents. The last row of this table contains the coefficients $\gamma_0, \gamma_1, \dots, \gamma_m$.

TABLE 5.1
The coefficients of spline function $s_{n,m}(t)$, for $n = m = 3,\ s = 1,\ c = 1$

ν	τ_ν	$a_{1,\nu}$	$a_{2,\nu}$	$a_{3,\nu}$
1	1.939368619(−1)	3.448547172(−2)	5.226278048(−4)	3.456311754(−4)
2	4.880999986(−1)	3.217255538(−2)	−4.235816948(−4)	4.941612712(−4)
3	7.907411411(−1)	2.039617915(−2)	−6.985434349(−4)	2.256692493(−4)
γ_k	3.678793085(−1)	3.678989078(−1)	1.833595896(−1)	6.681611249(−2)

Table 5.2 shows the accuracy of the spline approximation $s_{n,m}$ (Lobatto case), i.e.,

$$e_{n,m} = \max_{0 \le t \le 1} |s_{n,m}(t) - e^{-ct}|,$$

for $n = 1, 3, 5, 10$, $m = 2, 3, 4, 5$, $s = 1$, and $c = 1, 2, 4$.

TABLE 5.2
Accuracy of the spline approximation $s_{n,m}$

c	n	$m = 2$	$m = 3$	$m = 4$	$m = 5$
1	1	2.1(−3)	4.8(−5)	6.9(−7)	1.9(−8)
	3	3.2(−4)	4.5(−6)	3.9(−8)	6.7(−10)
	5	1.0(−4)	1.1(−6)	7.1(−9)	9.2(−11)
	10	1.9(−5)	1.2(−7)	4.9(−10)	4.2(−12)
2	1	1.1(−2)	4.7(−4)	1.4(−5)	7.5(−7)
	3	1.7(−3)	4.5(−5)	8.1(−7)	2.6(−8)
	5	5.3(−4)	1.1(−5)	1.5(−7)	3.6(−9)
	10	1.0(−4)	1.2(−6)	1.9(−8)	1.7(−10)
4	1	4.3(−2)	3.0(−3)	2.1(−4)	2.5(−5)
	3	6.0(−3)	2.8(−4)	1.1(−5)	6.5(−7)
	5	2.4(−3)	8.7(−5)	2.0(−6)	9.8(−8)
	10	4.5(−4)	9.9(−6)	1.5(−7)	4.8(−9)

The corresponding errors in Radau case,

$$e_{n,m}^* = \max_{0 \le t \le 1} |s_{n,m}^*(t) - e^{-ct}|$$

are given in Table 5.3.

TABLE 5.3
Accuracy of the spline approximation $s_{n,m}^*$

c	n	$m = 2$	$m = 3$	$m = 4$	$m = 5$
1	1	3.8(−3)	1.7(−4)	1.7(−5)	1.5(−6)
	3	4.5(−4)	8.5(−6)	1.2(−7)	3.4(−9)
	5	1.4(−4)	1.8(−6)	1.8(−8)	2.9(−10)
	10	2.3(−5)	1.7(−7)	8.2(−10)	7.9(−12)
2	1	1.8(−2)	1.8(−3)	3.3(−4)	5.8(−5)
	3	2.4(−3)	9.4(−5)	2.5(−6)	1.4(−7)
	5	7.6(−4)	2.0(−5)	3.3(−7)	1.2(−8)
	10	1.2(−4)	1.8(−6)	1.6(−8)	3.3(−10)
4	1	6.2(−2)	1.3(−2)	4.4(−3)	1.4(−3)
	3	1.1(−2)	7.7(−4)	3.8(−5)	3.9(−6)
	5	3.4(−3)	1.5(−4)	5.1(−6)	3.3(−7)
	10	5.0(−4)	1.4(−5)	2.6(−7)	9.7(−9)

We can see that the approximation error is more easily reduced by increasing m rather than n. Also, the spline $s_{n,m}$ is only slightly more accurate than the spline $s_{n,m}^*$.

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SPLAJN APROKSIMACIJE NA KONAČNIM INTERVALIMA KOJE OČUVAVAJU MOMENTE I TURÁNOVE KVADRATURE

Marco Frontini and Gradimir V. Milovanović

Razmatra se problem aproksimacije funkcije f na konačnom intervalu $[0,1]$ pomoću splajn funkcije reda m i defekta d, sa n (promenljivih) čvorova, zadržavajući pritom što je mogućno više početnih momenata funkcije f. Dodatna ograničenja na izvode u jednoj od krajnjih tačaka intervala $[0,1]$ takođe se mogu nametnuti. U radu se analizira slučaj kada je defekt d neparan broj ($d = 2s + 1$), i pokazuje se da u slučaju kada aproksimacija egzistira, tada se ona može reprezentovati pomoću parametara generalisane Turánove kvadrature u odnosu na meru koja zavisi od f. Čvorovi splajna su nule odgovarajućih s-ortogonalnih polinoma ($s \geq 1$). Kao ilustracija aproksimacionog postupka uključen je jedan numerički primer.