FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. 8 (1993), 35–43

POLYNOMIALS RELATED TO THE GENERALIZED HERMITE POLYNOMIALS

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Abstract. A class of polynomials $H_n^\lambda(z)$ $(\lambda \geq 0)$ which are related to the generalized Hermite polynomials $h_{n,m}^1(z)$ (see [2]) is introduced and considered. Some characteristic properties for the polynomials $H_n^\lambda(z)$ and some special cases of these polynomials are given. Also, some observations about the distribution of zeros of $H_n^1(z)$ are included.

1. Introduction

In [1] K. Dilcher considered the expansion

$$G^{\lambda,\nu}(z,t) = \big(1-(1+z+z^2)t+\lambda z^2t^2\big)^{-\nu} = \sum_{n=0}^{+\infty} f_n^{\lambda,\nu}(z)t^n,$$

where $\nu > 1/2$ and λ is a real parameter. Comparing this with the generating function for the Gegenbauer polynomials $C_n^\nu(z)$, he obtained

$$f_n^{\lambda,\nu}(z) = \lambda^{n/2}z^nC_n^\nu\Big(\frac{1+z+z^2}{2\sqrt{\lambda}z}\Big).$$

In this paper we consider the corresponding generalized Hermite case and study some characteristic properties for polynomials obtained in this way. In Section 2 we introduce the polynomials $H_n^\lambda(z)$ and derive a recurrence relation for their coefficients $C_{n,k}^\lambda$. Some expressions for $C_{n,k}^\lambda$ are given in Section 3. Finally, in Section 4 we deal with some special cases of the polynomials $H_n^\lambda(z)$ and give the distribution of zeros for the polynomial $H_n^1(z)$.

Received November 17, 1992.
1991 *Mathematics Subject Classification*. Primary 33C45.
This work was supported in part by the Science Fund of Serbia under grant 0401F.

2. Polynomials $H_n^\lambda(z)$

At first, we introduce the polynomials $H_n^\lambda(z)$:

Definition 2.1. The polynomials $H_n^\lambda(z)$ $(\lambda \geq 0)$ are given by the following generating function

$$F(z,t) = e^{(1+z+z^2)t-\lambda z^m t^m} = \sum_{n=0}^{\infty} H_n^\lambda(z) t^n. \tag{2.1}$$

Comparing (2.1) with the generating function (see [2])

$$e^{2zt-t^m} = \sum_{n=0}^{\infty} h_{n,m}^1(z) t^n,$$

we get the following representation

$$H_n^\lambda(z) = z^n \lambda^{n/m} h_{n,m}^1\left(\frac{1+z+z^2}{2\lambda^{1/m} z}\right). \tag{2.2}$$

From the recurrence relation (cf. [2])

$$n h_{n,m}^1(x) = 2x h_{n-1,m}^1(x) - m h_{n-m,m}^1(x), \quad n \geq m,$$

with initial values: $h_{n,m}^1(x) = (2x)^n/n!$, $n = 0, 1, \ldots, m-1$, and (2.2), we obtain

$$n H_n^\lambda(z) = (1+z+z^2) H_{n-1}^\lambda(z) - m\lambda z^m H_{n-m}^\lambda(z), \quad n \geq m, \tag{2.3}$$

with starting polynomials: $H_n^\lambda(z) = (1+z+z^2)^n/n!$, $n = 0, 1, \ldots, m-1$.

Now, from (2.2) we find that the polynomials $H_n^\lambda(z)$ are self-inverse, i.e., $H_n^\lambda(z) = z^{2n} H_n^\lambda(1/z)$.

Then, the polynomials $H_n^\lambda(z)$ have the following form

$$H_n^\lambda(z) = C_{n\,n}^\lambda + C_{n\,n-1}^\lambda z + \cdots + C_{n\,0}^\lambda z^n + C_{n\,1}^\lambda z^{n+1} + \cdots + C_{n\,n}^\lambda z^{2n}, \tag{2.4}$$

where $\operatorname{dg} H_n^\lambda = 2n$. From (2.3) and (2.4), we get

$$C_{n\,k}^\lambda = \frac{1}{n}\left[C_{n-1,k-1}^\lambda + C_{n-1,k}^\lambda + C_{n-1,k+1}^\lambda\right] - \frac{m}{n}\lambda C_{n-m,k}^\lambda, \tag{2.5}$$

where $C_{n,k}^\lambda = C_{n,-k}^\lambda$.

Hence, we obtain the following triangle

$$\begin{array}{ccccc} & & C^{\lambda}_{0,0} & & \\ & C^{\lambda}_{1,1} & C^{\lambda}_{1,0} & C^{\lambda}_{1,1} & \\ C^{\lambda}_{2,2} & C^{\lambda}_{2,1} & C^{\lambda}_{2,0} & C^{\lambda}_{2,1} & C^{\lambda}_{2,2} \\ & & \vdots & & \end{array} \tag{2.6}$$

For $m = 2$ the triangle (2.6) becomes

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & 1 & 1 & & \\ & \frac{1}{2} & 1 & \frac{3}{2}-\lambda & 1 & \frac{1}{2} & \\ \frac{1}{6} & \frac{1}{2} & 1-\lambda & \frac{7}{6}-\lambda & 1-\lambda & \frac{1}{2} & \frac{1}{6} \\ & & & \vdots & & & \end{array}$$

3. Coefficients $C^{\lambda}_{n,k}$

The main purpose in this section is to study the coefficients $C^{\lambda}_{n,k}$. First, we derive the following result:

Theorem 3.1. *We have*

$$C^{\lambda}_{n,k} = \sum_{s=0}^{[(n-k)/m]} \frac{(-\lambda)^s}{s!(n-ms)!} \sum_{j=0}^{[(n-k-ms)/2]} \binom{n-ms}{2j+k}\binom{2j+k}{k+j}, \tag{3.1}$$

where $[x]$ *denotes the greatest integer function.*

Proof. From the explicit representation (see [2])

$$h^1_{n,m}(x) = \sum_{s=0}^{[n/m]} (-1)^s \frac{(2x)^{n-ms}}{s!(n-ms)!}$$

and (2.2), we get

$$H^{\lambda}_n(z) = \sum_{s=0}^{[n/m]} (-\lambda)^s \frac{z^{ms}(1+z+z^2)^{n-ms}}{s!(n-ms)!}. \tag{3.2}$$

Using the expansion

$$\begin{aligned}(1+z+z^2)^r &= \sum_{j=0}^{r}\sum_{i=0}^{j}\binom{r}{j}\binom{j}{i} z^{2j-i} \\ &= \sum_{p=0}^{2r} z^p \sum_{j=0}^{[p/2]} \binom{r}{p-j}\binom{p-j}{p-2j},\end{aligned} \tag{3.3}$$

where r is a positive integer, and (3.2) for $r = n - ms$, we find

$$
\begin{aligned}
H_n^\lambda(z) &= \sum_{s=0}^{[n/m]} (-\lambda)^s \frac{z^{ms}}{s!(n-ms)!} \sum_{p=0}^{2(n-ms)} z^p \sum_{j=0}^{[p/2]} \binom{n-ms}{p-j}\binom{p-j}{p-2j} \\
&= \sum_{k=-n}^{n} z^{n-k} \sum_{s=0}^{[(n-k)/m]} \frac{(-\lambda)^s}{s!(n-ms)!} \times \\
&\qquad \times \sum_{j=0}^{[(n-k-ms)/2]} \binom{n-ms}{n-k-j-ms}\binom{n-k-j-ms}{n-k-2j-ms},
\end{aligned}
\tag{3.4}
$$

where $\binom{n}{k} = 0$ for $k < 0$.

Sice

$$
\binom{n-ms}{n-k-j-ms}\binom{n-k-j-ms}{n-k-2j-ms} = \binom{n-ms}{2j+k}\binom{2j+k}{k+j},
$$

the theorem follows from (3.4). □

Now, we prove another representation of $C_{n,k}^\lambda$:

Theorem 3.2. *We have*

$$
C_{n,k}^\lambda = \sum_{s=0}^{[(n-k)/m]} (-\lambda)^s \binom{n-k-(m-1)s}{s} \frac{B_k^{(n-k-ms)}}{k!(n-k-(m-1)s)!},
\tag{3.5}
$$

where

$$
B_k^{(r)} = \sum_{j=0}^{[r/2]} \binom{2j}{j}\binom{r}{2j}\binom{k+j}{j}^{-1}.
\tag{3.6}
$$

Proof. Equalities (3.5) and (3.6) give

$$
\begin{aligned}
&\sum_{s=0}^{[(n-k)/m]} \frac{(-\lambda)^s (n-k-(m-1)s)!}{(n-k-ms)!k!(m-k-(m-1)s)!} \times \\
&\qquad \times \sum_{j=0}^{[(n-k-ms)/2]} \frac{(2j)!(n-k-ms)!j!k!}{(j!)^2(2j)!(n-k-2j-ms)!(k+j)!} \\
&= \sum_{s=0}^{[(n-k)/m]} \frac{(-\lambda)^s}{s!(n-ms)!} \sum_{j=0}^{[(n-k-ms)/2]} \binom{n-ms}{k+2j}\binom{k+2j}{k+j}.
\end{aligned}
$$

Comparing the last equality with (3.1), we conclude that this statement holds. □

Similarly, one can prove the following result:

Theorem 3.3. *We have*

$$C_{n,k}^{\lambda} = \sum_{s=0}^{[(n-k)/m]} \frac{(-\lambda)^s}{s!k!(n-k-ms)!} \cdot \sum_{j=0}^{[r/2]} \frac{2^{2j}}{j!(k+1)_j} \left(-\frac{r}{2}\right)_j \left(\frac{1-r}{2}\right)_j ,$$

where $r = n - k - ms$.

4. Special Cases and Distribution of Zeros

For $m = 2$ the polynomials $H_n^{\lambda}(z)$ can be expressed by the classical Hermite polynomials $H_n(z)$ (see [3], [5]), i.e.,

$$H_n^{\lambda}(z) = \frac{1}{n!} z^n \lambda^{n/2} H_n\left(\frac{1+z+z^2}{2\lambda^{1/2} z}\right). \tag{4.1}$$

In that case, (3.1) reduces to

$$C_{n,k}^{\lambda} = \sum_{s=0}^{[(n-k)/2]} \frac{(-\lambda)^s}{s!(n-2s)!} \sum_{j=0}^{[(n-k-2s)/2]} \binom{n-2s}{2j+k}\binom{2j+k}{k+j}.$$

On the other hand, (2.4) for $z = 1$ gives

$$\sum_{k=-n}^{n} C_{n,k}^{\lambda} = \lambda^{n/m} h_{n,m}^{1}\left(\frac{3}{2\lambda^{1/m}}\right).$$

Also, for $m = 2$, (4.1) reduces to

$$\sum_{k=-n}^{n} C_{n,k}^{\lambda} = \frac{1}{n!} \lambda^{n/2} H_n\left(\frac{3}{2\lambda^{1/2}}\right).$$

Using relation (2.2) and the expression (see [2])

$$\frac{(2x)^n}{n!} = \sum_{k=0}^{[n/m]} \frac{1}{k!} h_{n-mk,m}^{1}(x), \qquad m \geq 2,$$

we can find

$$\frac{(1+z+z^2)^n}{n!} = \sum_{k=0}^{[n/m]} \frac{\lambda^k}{k!} z^{mk} H_{n-mk}^{\lambda}(z). \tag{4.2}$$

Similarly, from the relation

$$u^n h_{n,m}^1\left(\frac{x}{n}\right)=\sum_{k=0}^{[n/m]}\frac{(1-n^m)^k}{k!}h_{n-mk,m}^1(x),$$

derived in [2], and equality (2.2), we get the following relation

$$(uz)^n\lambda^{n/m}h_{n,m}^1\left(\frac{1+z+z^2}{2\lambda^{1/m}uz}\right)=\sum_{k=0}^{[n/m]}\lambda^k\frac{1-u^m)^k}{k!}z^{mk}H_{n-mk}^{\lambda}(z).$$

At the end of this section, we consider the monic polynomials $\hat{H}_n^1(z)$, obtained for $m=2$ and $\lambda=1$. For $n=1(1)5$ we have the following explicit expressions

$$\begin{aligned}
&\hat{H}_1^1(z)=1+z+z^2,\\
&\hat{H}_2^1(z)=1+2z+z^2+2z^3+z^4,\\
&\hat{H}_3^1(z)=1+3z+z^3+3z^5+z^6,\\
&\hat{H}_4^1(z)=1+4z-2z^2-8z^3-5z^4-8z^5-2z^6+4z^7+z^8,\\
&\hat{H}_5^1(z)=1+5z-5z^2-30z^3-15z^4-29z^5-15z^6-30z^7-5z^8+5z^9+z^{10}.
\end{aligned}$$

We note that $\hat{H}_n^1(0)=1$ and $\operatorname{dg}\hat{H}_n^1=2n$.

Theorem 4.1. *All zeros of $H_n^1(z)$, $n\geq 2$, defined by (4.1), are simple and located on the unit circle $|z|=1$ and the real line. For $n=1$ the zeros are given by $z_1^{\pm}=\big(-1\pm\sqrt{3}\big)/2$.*

Proof. Let $H=\{x_\nu \mid H_n(x_\nu),\ \nu=1,\ldots,n\}$ be the set of all zeros of the Hermite polynomial $H_n(x)$. It is known that these zeros are simple and that non-zero zeros are irrational (cf. Subramanian [4]). Divide H into two sets

$$H_C=\left\{x_\nu \;\middle|\; -\frac{1}{2}<x_\nu<\frac{3}{2}\right\}\quad\text{and}\quad H_R=H\setminus H_C.$$

Let z_ν, $\nu=1,\ldots,2n$, be the zeros of $\hat{H}_n^1(z)$. For them we can introduce the notation $z_\nu^{\pm}$, $\nu=1,\ldots,n$. According to (4.1) these zeros can be expressed in the form

$$z_\nu^{\pm}=\frac{1}{2}\left[2x_\nu-1\pm\sqrt{4x_\nu^2-4x_\nu-3}\right],\quad \nu=1,\ldots,n.$$

We note that $z_\nu^+z_\nu^-=1$. If $4x_\nu^2-4x_\nu-3<0$, i.e., $-1/2<x_\nu<3/2$, the zeros $z_\nu^{\pm}$ are complex and lie on the unit circle. Otherwise, they are real and have the same sign.

For $n=1$ the result is clear ($x_1=0$).

Let $n \geq 2$. With C and R we denote the sets of those zeros of $\hat{H}_n^1(z)$ which lie on the unit circle and on the real line $\mathbb{R}$, respectively. Evidently, if $x_\nu \in H_C$ then $z_\nu^\pm \in C$, and $x_\nu \in H_R$ then $z_\nu^\pm \in R$.

To finish the proof it is enough to prove that the sets H_C and H_R (or equivalently, C and R) are not empty.

Since for $n \geq 2$ (see Szegő [5, §6.31]),

$$\min_\nu |x_\nu| \leq \begin{cases} \left(\dfrac{5/2}{2n+1}\right)^{1/2}, & n \ \text{ even}, \\ \left(\dfrac{21/2}{2n+1}\right)^{1/2}, & n \ \text{ odd}, \end{cases}$$

we conclude that $\min\limits_\nu |x_\nu| < 3/2$ for any $n \geq 2$, i.e., at least one of zeros x_ν belongs to C.

Similarly, using the following very rough estimate for the largest zero (cf. [5, §6.2])

$$\max_\nu |x_\nu| > \left(\frac{n-1}{2}\right)^{1/2}, \qquad n \geq 2,$$

we conclude that there is one zero, say x_μ, such that

$$x_\mu < -\sqrt{\frac{n-1}{2}} \leq -\sqrt{\frac{1}{2}} < -\frac{1}{2}\,.$$

This means that $x_\mu \in R$. $\square$

Remark. It would be interested to determine numbers of complex zeros, and positive and negative real zeros (resp. $N_C(n)$, $N_{R_-}(n)$, and $N_{R_+}(n)$). Numerical experiments for $2 \leq n \leq 50$ show that

$N_C(n) = 2$, for $n = 1, 2, 4$;
$N_C(n) = 4$, for $n = 3, 5, 7$;
$N_C(n) = 6$, for $n = 6$, 8–13, 15, 17, 19;
$N_C(n) = 8$, for $n = 14, 16, 18, 20, 22, 24, 26$;
$N_C(n) = 10$, for $n = 21, 23, 25$, 27–34, 36, 38, 40, 42;
$N_C(n) = 12$, for $n = 35, 37, 39, 41$, 43–45, 47, 49, ...;
$N_C(n) = 14$, for $n = 46, 48, 50, \ldots$

Also, for the number of negative zeros we obtained:

$N_{R_-}(n) = N_{R_-}(n+1) = n$, for $n = 2$;
$N_{R_-}(n) = N_{R_-}(n+1) = N_{R_-}(n+2) = n$, for $n = 4$;
$N_{R_-}(n) = N_{R_-}(n+1) = n-1$, for $n = 7(2)17$;
$N_{R_-}(n) = N_{R_-}(n+1) = N_{R_-}(n+2) = n-1$, for $n = 19$;
$N_{R_-}(n) = N_{R_-}(n+1) = n-2$, for $n = 22(2)40$;
$N_{R_-}(n) = N_{R_-}(n+1) = N_{R_-}(n+2) = n-2$, for $n = 42$;
$N_{R_-}(n) = N_{R_-}(n+1) = n-3$, for $n = 45, 47, 49$, etc.

Notice that $N_C(n) + N_{R_-}(n) + N_{R_+}(n) = 2n$. All numbers $N_C(n)$, $N_{R_-}(n)$, and $N_{R_+}(n)$ are even. Also, $N_{R_-}(n) > N_{R_+}(n)$, for $n \geq 2$.

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POLINOMI POVEZANI SA GENERALISANIM HERMITEOVIM POLINOMIMA

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U radu se uvodi i proučava klasa polinoma $H_n^{\lambda}(z)$ $(\lambda \geq 0)$, koja je u vezi sa generalisanim Hermiteovim polinomima $h_{n,m}^{1}(z)$ (videti [2]). Daju se neke karakteristične osobine za polinome $H_n^{\lambda}(z)$ i razmatraju neki specijalni slučajevi ovih polinoma. Takodje, data je i distribucija nula polinoma $H_n^{1}(z)$.