Some Generalized Orthogonal Systems and Their Connections

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Abstract. In this survey paper we consider some important classes of rational functions and generalized polynomial systems, the so-called Müntz systems, which are an important extension of the orthogonal polynomial systems. Rational functions are orthogonal with respect to certain inner products defined on some lines or on the unit circle in the complex plane. In particular, we give a short account of Malmquist-Takenaka systems which are orthogonal on the unit circle. In the second part of the paper we consider several Müntz systems, including some classes of algebraic-logarithmic type. Defining an unusual inner product of the generalized polynomials we introduce the corresponding class of orthogonal Müntz polynomials and connect them with a Malmquist-Takenaka system. Finally, we give a short account on numerical evaluation of orthogonal Müntz polynomials and point out an application of such orthogonal polynomials in numerical integration.

1 Introduction

Polynomial systems are very attractive in many applications in mathematics, physics, and other computational and applied sciences (electronics and communication, signal processing, control system theory, process identification, etc.). In particular, classical orthogonal polynomials (cf. [25,30,26]) play a very important role in many problems in approximation theory and numerical analysis, as well as in applied sciences. Such polynomials are very useful for design and construction of electrical network, transfer functions, orthogonal filters, adaptive control, etc. These applications are mainly based on the least squares polynomial approximations. The orthogonality of these polynomials enables the construction of optimal network and optimal filters. Moreover, the Laplace transforms of the classical polynomials (or their modifications) are rational functions, which can be easily factorized. This property is very convenient in constructing simple procedures for several applications. For instance, for designing orthogonal filters and optimal transfer functions may be used some modifications of the Jacobi polynomials, which are orthogonal on the interval $(-1,1)$. By changing variables $x = 2\,\mathrm{e}^{-at} - 1$ $(a > 0)$, one can find exponential polynomials orthogonal on $(0,+\infty)$. Then, applying the Laplace transform, one can obtain orthogonal rational functions. The following approach shows it (see [12]).

Starting from the orthogonality relation for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $\alpha, \beta > -1$, we obtain, after changing variables $x+1 = 2\,\mathrm{e}^{-2t}$,

$$\int_0^{+\infty} \varphi_n(t)\varphi_m(t)\ \mathrm{d}t = \delta_{nm},$$

where

$$\varphi_n(t) = \frac{2^{(\alpha+\beta+2)/2}}{\|P_n^{(\alpha,\beta)}\|}\ \mathrm{e}^{-(\beta+1)t}(1-\mathrm{e}^{-2t})^{\alpha/2}P_n^{(\alpha,\beta)}(2\,\mathrm{e}^{-2t}-1).$$

Then, the Laplace transform of φ_n can be expressed in the form

$$\begin{aligned} W_n(s) = \mathcal{L}[\varphi_n(t)] &= \int_0^{+\infty} \mathrm{e}^{-st}\,\varphi_n(t)\ \mathrm{d}t \\ &= \frac{2^{-(s+1)/2}}{\|P_n^{(\alpha,\beta)}\|}\int_{-1}^{1}(1-x)^{\alpha/2}(1+x)^{(s+\beta-1)/2}P_n^{(\alpha,\beta)}(x)\ \mathrm{d}x \\ &\qquad \times {}_3F_2\left(-n,\alpha+\beta+n+1,\tfrac{1}{2}\alpha+1;\alpha+1,\tfrac{1}{2}(\alpha+\beta+s+3);1\right), \end{aligned}$$

where the hypergeometric function ${}_3F_2$ is reduced to the following series

$$\sum_{k=0}^{+\infty}(-1)^k\binom{n}{k}\frac{(\alpha+\beta+n+1)_k\left(\frac{1}{2}\alpha+1\right)_k}{(\alpha+1)_k\left(\frac{1}{2}(\alpha+\beta+s+3)\right)_k}.$$

In a simpler case when $\alpha = 0$, the function ${}_3F_2$ can be reduced to ${}_2F_1$, and then we find (see [12])

$$W_n(s) = \sqrt{2(2n+\beta+1)}\ \frac{\prod\limits_{\nu=0}^{n-1}(s-(2\nu+1+\beta))}{\prod\limits_{\nu=0}^{n}(s+(2\nu+1+\beta))}.$$

Such rational functions are orthogonal in certain sense in the complex plane. In this survey paper we consider several classes of orthogonal rational functions and connect them with some generalized polynomial systems, the so-called Müntz systems.

The paper is organized as follows. In Section 2 we study several systems of rational functions which are orhogonal with respect to certain inner products defined on some lines or on the unit circle in the complex plane. Section 3 is devoted to the orthogonal Müntz systems which represent an important extension of the orthogonal polynomial systems. Finally, in Section 5 we give some remarks on numerical evaluation of the Müntz polynomials and mention an application in numerical integration.

2 Orthogonal Systems of Rational Functions

We start this section with a system of generalized exponential polynomials defined on $(0, +\infty)$ and then we connect their orthogonality with some systems of rational functions in the complex domain.

2.1 Generalized exponential polynomials and orthogonal rational functions

Let $A = \{\alpha_0, \alpha_1, \alpha_2, \dots\}$ be a complex sequence such that $\operatorname{Re}\alpha_k > 0$. For each k ($k \ge 0$) one denotes by $m_k \ge 1$ the multiplicity of the appearance of the numbers α_k in the set $A_k = \{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_k\}$. With the sequence A we associate the sequence of functions $\{t^{m_k-1}\,\mathrm{e}^{-\alpha_k t}\}_{k=0}^{+\infty}$, which can be orthogonalized with respect to the inner product

$$(f, g) = \int_0^{+\infty} f(t)\overline{g(t)}\,\mathrm{d}t, \tag{1}$$

for example, using the well-known Gram-Schmidt method. Such an orthonormal system $\{q_k(t)\}_{k=0}^{+\infty}$ is unique up to a multiplicative constant of the form $\mathrm{e}^{\mathrm{i}\gamma_k}$ $(\operatorname{Im}\gamma_k = 0)$.

For example, if we take $A = \{1/2, 1, 1, 2, 5/2, \dots\}$, for which $m_0 = m_1 = 1$, $m_2 = 2$, $m_3 = m_4 = 1$, ..., using MATHEMATICA package:

```
In[1]:= <<LinearAlgebra`Orthogonalization`

In[2]:=L[t_]:={Exp[-t/2],Exp[-t],t Exp[-t],Exp[-2t],Exp[-5t/2]}

In[3]:= q=GramSchmidt[L[t],InnerProduct->
        (Integrate[#1  #2,{t,0,Infinity}]&)] //Simplify
```

we get

$$\begin{aligned}
q_0(t) &= \mathrm{e}^{-t/2},\\
q_1(t) &= -\sqrt{2}\,\mathrm{e}^{-t}\big(-3 + 2\,\mathrm{e}^{t/2}\big),\\
q_2(t) &= -\sqrt{2}\,\mathrm{e}^{-t}\big(-5 + 6\,\mathrm{e}^{t/2} - 6t\big),\\
q_3(t) &= -2\,\mathrm{e}^{-2t}\big(-15 + 8\,\mathrm{e}^{t} + 6\,\mathrm{e}^{3t/2} - 12t\,\mathrm{e}^{t}\big),\\
q_4(t) &= \tfrac{1}{2}\sqrt{5}\,\mathrm{e}^{-5t/2}\big(147 - 240\,\mathrm{e}^{t/2} + 80\,\mathrm{e}^{3t/2} + 15\,\mathrm{e}^{2t} - 48t\,\mathrm{e}^{3t/2}\big).
\end{aligned}$$

If $m_k = 1$ for each $k \in \mathbb{N}_0$, then

$$q_k(t) = \sum_{i=0}^{k} c_{k,i} e^{-\alpha_i t}, \quad c_{k,i} = \sqrt{2\operatorname{Re}\lambda_k}\;\frac{\prod\limits_{\nu=0}^{k-1}(\alpha_i + \bar{\alpha}_\nu)}{\prod\limits_{\substack{\nu=0\\ \nu\ne i}}^{k}(\alpha_i - \alpha_\nu)} \quad (i = 0, 1, \dots, k).$$

This was given firstly by Erdélyi [18]. A study of exponential polynomials was given by Schwartz [38].

A more effective way for finding these orthogonal functions uses their representation in terms of the Fourier integrals. Namely, Djrbashian [16] (see also [17]) proved the following result:

Theorem 1. *Let*

$$\psi_0(z) = \sqrt{\frac{\operatorname{Re}\alpha_0}{\pi}}\,\frac{\mathrm{i}}{z+\mathrm{i}\alpha_0}, \quad \psi_k(z) = \sqrt{\frac{\operatorname{Re}\alpha_k}{\pi}}\,\frac{\mathrm{i}}{z+\mathrm{i}\alpha_k}\prod_{\nu=0}^{k-1}\frac{z-\mathrm{i}\bar{\alpha}_\nu}{z+\mathrm{i}\alpha_\nu} \quad (k\in\mathbb{N}).$$

Then

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\psi_k(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega = \begin{cases} q_k(t), & t\in(0,+\infty),\\ 0, & t\in(-\infty,0).\end{cases}$$

The condition

$$\sum_{k=0}^{+\infty}\frac{\operatorname{Re}\alpha_k}{1+|\alpha_k|^2} = +\infty$$

is necessary and sufficient for the $L^2(0,+\infty)$ completeness of the orthogonal system $\{q_k(t)\}_{k=0}^{+\infty}$.

The proof is based on the fact that the rational functions ψ_k belong to the class $H^2(G^+)$ of analytic functions $f(z)$ in the upper half-plane $G^+ = \{z\in\mathbb{C}\,|\,\operatorname{Im} z>0\}$ for which

$$\|f\| = \sup_{0<y<+\infty}\left(\int_{-\infty}^{+\infty}|f(x+\mathrm{i}y)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} < +\infty.$$

Using the Paley-Wiener theorem and Parseval's equality one can find

$$(q_k,q_m) = \int_{-\infty}^{+\infty}\psi_k(x)\overline{\psi_m(x)}\,\mathrm{d}x = \frac{1}{\pi}\int_{-\infty}^{+\infty}\Psi_k(x)\overline{\Psi_m(x)}\,\mathrm{d}x, \tag{2}$$

where the inner product $(\,\cdot\,,\,\cdot\,)$ is given by (1) and $\psi_k(z) = (\mathrm{i}/\sqrt{\pi})\Psi_n(z)$. Putting $\beta_k = \mathrm{i}\bar{\alpha}_k$, where $\operatorname{Im}\beta_k = \operatorname{Re}\alpha_k > 0$ for each k, the functions $\Psi_k(z)$ can be expressed in the form

$$\Psi_0(z) = \frac{\sqrt{\operatorname{Im}\beta_0}}{z-\bar{\beta}_0}, \quad \Psi_k(z) = \frac{\sqrt{\operatorname{Im}\beta_k}}{z-\bar{\beta}_k}\prod_{\nu=0}^{k-1}\frac{z-\beta_\nu}{z-\bar{\beta}_\nu} \quad (k\in\mathbb{N}). \tag{3}$$

Using Cauchy's residue theorem, it is very easy to prove the following orthogonality result for the rational functions $\Psi_k(z)$ (cf. Djrbashian [16]):

Theorem 2. *The system of functions $\{\Psi_k(z)\}_{k=0}^{+\infty}$ is orthonormal on the real line, i.e.,*

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\Psi_k(x)\overline{\Psi_m(x)}\,\mathrm{d}x = \delta_{k,m}. \tag{4}$$

The corresponding completeness condition for this orthogonal system reduces to

$$\sum_{k=0}^{+\infty}\frac{\operatorname{Im}\beta_k}{1+|\beta_k|^2} = +\infty.$$

Obviously, (2) and (4) give $(q_k,q_m) = \delta_{k,m}$.

2.2 Malmquist-Takenaka systems of orthogonal rational functions

It is easy to see that the simple Moebius transformation $z = \mathrm{i}\,\frac{1+s}{1-s}$ changes the system (3) into the well-known Malmquist–Takenaka system of rational functions $\{\Phi_k(s)\}_{k=0}^{+\infty}$ orthogonal on the unit circle $|s| = 1$ (see Malmquist [22], Takenaka [44], Walsh [47, Sections 9.1 and 10.7], and Djrbashian [17]). This transformation maps the upper-half plane $\mathrm{Im}\, z > 0$ into the unit disk $|s| < 1$, so that the point β_k maps to the point $a_k = (\mathrm{i}\beta_k + 1)/(\mathrm{i}\beta_k - 1)$ inside the unit circle,

$$|a_k|^2 = \frac{1 + |\beta_k|^2 - 2\mathrm{Im}\, \beta_k}{1 + |\beta_k|^2} < 1 \qquad (k = 0, 1, \dots).$$

Usually, such rational functions are represented in the following way (cf. Djrbashian [17])

$$\Phi_0(s) = \frac{(1 - |a_0|^2)^{1/2}}{1 - \bar{a}_0 s},$$

$$\Phi_k(s) = \frac{(1 - |a_k|^2)^{1/2}}{1 - \bar{a}_k s} \prod_{\nu=0}^{k-1} \frac{a_\nu - s}{1 - \bar{a}_\nu s} \cdot \frac{|a_\nu|}{a_\nu} \qquad (k \in \mathbb{N}), \tag{5}$$

where for $a_\nu = 0$ we put $|a_\nu|/a_\nu = \bar{a}_\nu/|a_\nu| = -1$. The following orthogonality result holds:

Theorem 3. *The system of functions $\{\Phi_k(s)\}_{k=0}^{+\infty}$ is orthonormal on the unit circle, i.e.,*

$$(\Phi_k, \Phi_m) = \frac{1}{2\pi \mathrm{i}} \oint_{|s|=1} \Phi_k(s)\overline{\Phi_m(s)}\, \frac{\mathrm{d}s}{s} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_k(\mathrm{e}^{\mathrm{i}\theta})\overline{\Phi_m(\mathrm{e}^{\mathrm{i}\theta})}\, \mathrm{d}\theta = \delta_{k,m}.$$

Practically, this system is the result of an orthogonalization process of the ordered sequence of the rational functions from the system

$$\left\{ \frac{s^{m_k - 1}}{(1 - \bar{a}_k s)^{m_k}} \right\}_{k=0}^{+\infty}$$

on the unit circle $s = \mathrm{e}^{\mathrm{i}\theta}$ $(-\pi \le \theta \le \pi)$ with respect to the Lebesgue measure $\frac{\mathrm{d}\theta}{2\pi}$.

Excluding the normalization constants, the Malquist-Takenaka system (5) can be represented in the form

$$W_n(s) = \frac{\prod_{\nu=0}^{n-1} (s - a_\nu)}{\prod_{\nu=0}^{n} (s - a_\nu^*)}, \tag{6}$$

where $a_\nu^* = 1/\bar{a}_\nu$. For $a_\nu = 0$ we put only s instead of $(s - a_\nu)/(s - a_\nu^*)$.

Supposing that $a_\nu \neq a_\mu$ for $\nu \neq \mu$, we can see that (6) can be written in the form

$$W_n(s) = \sum_{k=0}^{n} \frac{A_{n,k}}{s - a_k^*},$$

where

$$A_{n,k} = \frac{\prod\limits_{\nu=0}^{n-1} (a_k^* - a_\nu)}{\prod\limits_{\substack{\nu=0 \\ \nu\neq k}}^{n} (a_k^* - a_\nu^*)} \qquad (k = 0, 1, \dots, n). \tag{7}$$

The case when $a_\nu = a_\mu$ can be considered as a limiting process $a_\nu \to a_\mu$.

By using Theorem 3 it is easy to see that the system of rational functions $\{W_n(s)\}_{n=0}^{+\infty}$, defined by (6), satisfies the following orthogonality relation

$$(W_n, W_m) = \|W_n\|^2 \delta_{nm},$$

where

$$\|W_n\|^2 = \frac{|a_0 a_1 \cdots a_n|^2}{1 - |a_n|^2}.$$

An important auxiliary result was proved in [29]:

Lemma 1. *Let $-1 \leq t \leq 1$ and let F be defined by*

$$F(t) = \frac{1}{2\pi \mathrm{i}} \oint\limits_{|s|=1} W_n(s)\overline{W_m(ts)}\, \frac{\mathrm{d}s}{s}, \tag{8}$$

where the system functions $\{W_n(s)\}$ are defined by (6) with mutually different numbers a_ν $(\nu = 0, 1, \dots)$ in the unit circle $|s| = 1$. Then

$$F(t) = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{A_{n,i}\bar{A}_{m,j}}{a_i^* \bar{a}_j^* - t},$$

where the numbers $A_{n,k}$ are given in (7).

For $t = 1$, from (8) we obtain that $F(1) = (W_n, W_m)$. Thus,

$$(W_n, W_m) = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{A_{n,i}\bar{A}_{m,j}}{a_i^* \bar{a}_j^* - 1} = \frac{|a_0 a_1 \cdots a_n|^2}{1 - |a_n|^2}\, \delta_{nm}. \tag{9}$$

This equality gives a connection between the Malmquist-Takenaka system of rational functions (6) and a Müntz system, which is orthogonal with respect to an unusual inner product defined in the next section.

In a general case, we can use a positive Borel measure $\mathrm{d}\mu(\theta)$ on $[-\pi,\pi]$ so that the previous inner product reduces to

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\overline{g(s)}\,\mathrm{d}\mu(\theta) \qquad (s = \mathrm{e}^{\mathrm{i}\theta}).$$

The corresponding orthogonal systems of rational functions were intensively investigated in several papers by Djrbashian [13–17], Bultheel, González-Vera, Hendriksen, and Njåstad [9–11], Pan [33–36], etc.

In the extreme case when $\alpha_k = 0$ $(k = 0,1,\dots)$, the corresponding system of functions turns into the system of Szegő polynomials $\{P_k(s)\}_{k=0}^{+\infty}$ (see [43, pp. 287–295] which are orthogonal on the unit circle with the same measure $(2\pi)^{-1}\,\mathrm{d}\mu(\theta)$. This kind of orthogonal polynomials on the unit circle have been introduced and studied by Szegő [41,42] and Smirnov [39,40]. A more general case was considered by Achieser and Kreĭn [1], Geronimus [19,20], Nevai [31,32], Alfaro and Marcellán [2], Marcellán and Sansigre [23], etc. These polynomials are linked with many questions in the theory of time series, digital filters, statistics, image processing, scattering theory, control theory, etc.

In the simplest case, for the Lebesgue measure (when $d\mu(\theta) = \mathrm{d}\theta$), these polynomials reduce to the polynomials $P_k(z) = z^k$ $(k = 0,1,\dots)$.

A survey on orthogonal polynomials, including basic properties of polynomials orthogonal on the unit circle, can be found in [26].

3 Orthogonal Müntz Systems

Let $\Lambda = \{\lambda_0, \lambda_1, \dots\}$ be a given sequence of complex numbers. Taking the following definition for x^λ:

$$x^\lambda = e^{\lambda \log x}, \qquad x \in (0,+\infty),\ \lambda \in \mathbb{C},$$

and the value at $x = 0$ is defined to be the limit of x^λ as $x \to 0$ from $(0,+\infty)$ whenever the limits exists, we will consider orthogonal Müntz polynomials as linear combinations of the Müntz system $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ (see [7,8]). The set of all such polynomials we will denote by $M_n(\Lambda)$, i.e.,

$$M_n(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\},$$

where the linear span is over the complex numbers $\mathbb{C}$ in general. The union of all $M_n(\Lambda)$ is denoted by $M(\Lambda)$.

For real numbers $0 \le \lambda_0 < \lambda_1 < \cdots \to \infty$, it is well-known that the real Müntz polynomials of the form $\sum\limits_{k=0}^{n} a_k x^{\lambda_k}$ are dense in $L^2[0,1]$ if and only if

$$\sum_{k=1}^{+\infty} \lambda_k^{-1} = +\infty.$$

In addition, if $\lambda_0 = 0$ this condition also characterizes the denseness of the Müntz polynomials in $C[0,1]$ in the uniform norm.

The first considerations of orthogonal Müntz systems were made by the Armenian mathematicians Badalyan [4,5] and Taslakyan [45]. Recently, it was rediscovered by McCarthy, Sayre and Shawyer [24]. A complete investigation of such systems, including some inequalities of Markov type, was done by Borwein, Erdélyi, and Zhang [8].

3.1 Müntz-Legendre polynomials

Supposing $\mathrm{Re}\,(\lambda_k) > -1/2$ for each $k \in \mathbb{N}_0$ and $\Lambda_n = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$, we can give the following definition of the Müntz-Legendre polynomials on $(0,1]$ (see [4,45,8]):

Definition 15. Let

$$W_n(s) = \prod_{\nu=0}^{n-1} \frac{s + \bar{\lambda}_\nu + 1}{s - \lambda_\nu} \cdot \frac{1}{s - \lambda_n} \qquad (n \in \mathbb{N}_0), \tag{10}$$

where an empty product for $n = 0$ should be taken to be equal to 1. The nth Müntz-Legendre polynomial on $(0,1]$ is given by

$$P_n(x; \Lambda_n) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma W_n(s) x^s \,\mathrm{d}s \qquad (n = 0, 1, \dots), \tag{11}$$

where the simple contour Γ surrounds all poles of the rational function (10).

For polynomials $P_n(x) \equiv P_n(x; \Lambda_n)$ one can prove an orthogonality relation on $(0,1)$:

Theorem 4. *Let the polynomials $P_n(x)$ be defined by* (11). *Then, for every $n, m = 0, 1, \dots,$ we have*

$$\int_0^1 P_n(x)\overline{P_m(x)}\,\mathrm{d}x = \frac{\delta_{n,m}}{1 + \lambda_n + \bar{\lambda}_n}\,.$$

Evidently, that the polynomials $P_n^*(x) = (1 + \lambda_n + \bar{\lambda}_n)^{1/2} P_n(x)$ are orthonormal. In the simplest case when $\lambda_\nu \neq \lambda_\mu$ $(\nu \neq \mu)$ it is easy to show that polynomials $P_n(x)$ can be expressed in a power form

$$P_n(x) = \sum_{k=0}^{n} c_{n,k} x^{\lambda_k}, \tag{12}$$

where

$$c_{n,k} = \frac{\displaystyle\prod_{\nu=0}^{n-1} (1 + \lambda_k + \bar{\lambda}_\nu)}{\displaystyle\prod_{\substack{\nu=0 \\ \nu \neq k}}^{n} (\lambda_k - \lambda_\nu)}\,.$$

In a limiting case when $\lambda_0 = \lambda_1 = \cdots = \lambda_n = \lambda$, the polynomials (11) are reduced to

$$P_n(x; \Lambda_n) = x^\lambda L_n(-(1 + \lambda + \bar{\lambda}) \log x),$$

where $L_n(x)$ is the Laguerre polynomial orthogonal on $(0, \infty)$ with respect to the exponential weight e^{-x}, and for which $L_n(0) = 1$.

If we put $x = \mathrm{e}^{-t}$, the Müntz-Legendre polynomials are reduced to the generalized exponential polynomials. For example, (12) becomes

$$P_n(\mathrm{e}^{-t}) = \sum_{k=0}^{n} c_{n,k}\, \mathrm{e}^{-\lambda_k t}.$$

In a general case, such polynomials can be expressed in terms of a Laplace transform (see [27]):

Theorem 5. *If* $W_n(s)$ *given by* (10) *and*

$$G_n(s) = -W_n(-s) = \prod_{k=0}^{n-1} \frac{s - (\bar{\lambda}_k + 1)}{s + \lambda_k} \cdot \frac{1}{s + \lambda_n},$$

then $P_n(\mathrm{e}^{-t})$ *is the inverse Laplace transform of* $G_n(s)$, *i.e.,*

$$P_n(\mathrm{e}^{-t}) = \mathcal{L}^{-1}[G_n(s)].$$

Taking, for example, $\alpha > 1/2$ we can prove that

$$P_n(\mathrm{e}^{-t}) = \frac{1}{2\pi \mathrm{i}} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} G_n(s)\, \mathrm{e}^{-st}\, \mathrm{d}s.$$

There is also a kind of the generalized Rodrigues formula for the Müntz-Legendre polynomials (see [24])

$$P_n(x) = \mathrm{D}_{\lambda_0} \mathrm{D}_{\lambda_1} \cdots \mathrm{D}_{\lambda_{n-1}} \ell_n(x),$$

where

$$\ell_n(x) = \sum_{k=0}^{n} \frac{x^{\lambda_k}}{\prod\limits_{j=0, j \neq k}^{n} (\lambda_k - \lambda_j)}$$

and D_λ is the differential operator defined by

$$\mathrm{D}_\lambda f(x) = x^{-\bar{\lambda}} \frac{\mathrm{d}}{\mathrm{d}x} \left(x^{1 + \bar{\lambda}} f(x) \right).$$

It is easy to prove that $\ell_n(x)$ and its first $n - 1$ derivatives vanish at the point $x = 1$.

The polynomials (11) satisfy some recurrence relations, e.g.,

$$xP_n'(x) - xP_{n-1}'(x) = \lambda_n P_n(x) + (1 + \bar{\lambda}_{n-1})P_{n-1}(x) \tag{13}$$

and

$$P_n(x) = P_{n-1}(x) - (\lambda_n + \bar{\lambda}_{n-1} + 1)x^{\lambda_n} \int_x^1 t^{-\lambda_n - 1} P_{n-1}(t)\, dt \quad (x \in (0,1]).$$

Also, it is easy to prove that

$$P_n(1) = 1 \qquad \text{and} \qquad P_n'(1) = \lambda_n + \sum_{k=0}^{n-1} (\lambda_k + \bar{\lambda}_k + 1).$$

An interesting question is connected by the zero distribution of the Müntz-Legendre polynomials. Badalyan [6] proved the following result (see also [8]):

Theorem 6. *For real numbers $\lambda_\nu > -1/2$ ($\nu = 0, 1, \dots$) the Müntz-Legendre polynomial $P_n(x; \Lambda_n)$ has exactly n distinct zeros in $(0,1)$, and it changes sign at each of these zeros. Furthermore, the zeros of the polynomials*

$$P_{n-1}(x; \Lambda_{n-1}) \qquad \textit{and} \qquad P_n(x; \Lambda_n)$$

in $(0,1)$ strictly interlace.

3.2 Some algebraic-logarithmic polynomials

An important special case of the Müntz-Legendre polynomials when

$$\lambda_{2k} = \lambda_{2k+1} = k \qquad (k = 0, 1, \dots)$$

was considered in [27]. Namely, we put $\lambda_{2k} = k$ and $\lambda_{2k+1} = k + \varepsilon$ ($k = 0, 1, \dots$), where ε decreases to zero. The corresponding limiting process leads to orthogonal Müntz polynomials with logarithmic terms,

$$P_n(x) = R_n(x) + S_n(x) \log x \qquad (n = 0, 1, \dots), \tag{14}$$

where $R_n(x)$ and $S_n(x)$ are algebraic polynomials of degree $\left[\frac{n}{2}\right]$ and $\left[\frac{n-1}{2}\right]$, respectively, i.e.,

$$R_n(x) = \sum_{\nu=0}^{[n/2]} a_\nu^{(n)} x^\nu, \qquad S_n(x) = \sum_{\nu=0}^{[(n-1)/2]} b_\nu^{(n)} x^\nu. \tag{15}$$

Notice that $P_n(1) = R_n(1) = 1$. The first few Müntz polynomials (14) are:

$$\begin{aligned}
P_0(x) &= 1,\\
P_1(x) &= 1 + \log x,\\
P_2(x) &= -3 + 4x - \log x,\\
P_3(x) &= 9 - 8x + 2(1 + 6x)\log x,\\
P_4(x) &= -11 - 24x + 36x^2 - 2(1 + 18x)\log x,\\
P_5(x) &= 19 + 276x - 294x^2 + 3(1 + 48x + 60x^2)\log x,\\
P_6(x) &= -21 - 768x + 390x^2 + 400x^3 - 3(1 + 96x + 300x^2)\log x.
\end{aligned}$$

The explicit expressions for the coefficients of the polynomials (15) for arbitrary n are given in [27]. These Müntz polynomials can be used in the proof of the irrationality of $\zeta(3)$ and of other familiar numbers (see [7, pp. 372–381] and [46]).

Similarly, if we take

$$\lambda_{3k} = \lambda_{3k+1} = \lambda_{3k+2} = k \qquad (k = 0, 1, \ldots),$$

i.e., $\lambda_{3k} = k - \varepsilon$, $\lambda_{3k+1} = k$, $\lambda_{3k+2} = k + \varepsilon$ $(k = 0, 1, \ldots)$, where ε tends to zero, we get the corresponding orthogonal Müntz polynomials:

$$\begin{aligned}
P_0(x) &= 1,\\
P_1(x) &= 1 + \log x,\\
P_2(x) &= 1 + 2\log x + \frac{1}{2}\log^2 x,\\
P_3(x) &= -7 + 8x - 4\log x - \frac{1}{2}\log^2 x,\\
P_4(x) &= 29 - 28x + (11 + 24x)\log x + \log^2 x,\\
P_5(x) &= -97 + 98x - 4(7 + 15x)\log x + (36x - 2)\log^2 x,\\
P_6(x) &= 127 - 342x + 216x^2 + (32 - 108x)\log x + (2 - 108x)\log^2 x.
\end{aligned}$$

These polynomials have the form

$$P_n(x) = R_n(x) + S_n(x)\log x + T_n(x)\log^2 x,$$

where $R_n(x)$, $S_n(x)$, and $T_n(x)$ are algebraic polynomials of degree $\left[\frac{n}{3}\right]$, $\left[\frac{n-1}{3}\right]$, and $\left[\frac{n-2}{3}\right]$, respectively. Notice that $P_n(1) = R_n(1) = 1$.

3.3 One-parametric Müntz-Jacobi polynomials

A little generalization of the Müntz-Legendre polynomials can be done in the following way. Namely, putting $\lambda_k + \beta/2$ instead of λ_k, $k = 0, 1, \ldots$, in the sequence Λ, we can define a kind of the Müntz-Jacobi polynomials $P_n^{(\beta)}(x)$ by

$$P_n^{(\beta)}(x) = \frac{x^{-\beta/2}}{2\pi \mathrm{i}} \oint_{\Gamma} W_n^{(\beta)}(s) x^s \, \mathrm{d}s,$$

where

$$W_n^{(\beta)}(s) = \prod_{k=0}^{n-1} \frac{s + \bar{\lambda}_k + \beta/2 + 1}{s - \lambda_k - \beta/2} \cdot \frac{1}{s - \lambda_n - \beta/2}.$$

Then, the following result holds:

Theorem 7. *Let $\beta \in \mathbb{R}$ and $\operatorname{Re}\lambda_k > -(\beta+1)/2$ for each $k \in \mathbb{N}_0$. Then*

$$(P_n^{(\beta)}, P_m^{(\beta)}) = \int_0^1 P_n^{(\beta)}(x)\overline{P_m^{(\beta)}(x)}\, x^\beta \, \mathrm{d}x = \frac{\delta_{nm}}{\lambda_n + \bar{\lambda}_n + \beta + 1}.$$

In the special case $\lambda_k = k$ $(k = 0, 1, \dots)$, these generalized polynomials reduce to the classical Jacobi polynomials $\tilde{P}_n^{(0,\beta)}$ $(\beta > -1)$ shifted to $[0,1]$ (see [27]). It would be very interesting to construct the Müntz-Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ orthogonal with respect to the inner product

$$(f,g) = \int_0^1 f(x)\overline{g(x)}\,(1-x)^\alpha x^\beta\,\mathrm{d}x.$$

3.4 Another type of the orthogonal Müntz polynomials

In [12] and [29] we defined an external operation for the Müntz polynomials from $M(\Lambda)$ and the corresponding inner product.

At first, we introduced an operation $\odot$ for monomials in the following way:

$$x^\alpha \odot x^\beta = x^{\alpha\beta} \qquad (x \in (0,\infty),\ \alpha,\beta \in \mathbb{C}).$$

An extension of this operation to the Müntz polynomials $P \in M_n(\Lambda)$ and $Q \in M_m(\Lambda)$, i.e.,

$$P(x) = \sum_{i=0}^{n} p_i x^{\lambda_i} \qquad \text{and} \qquad Q(x) = \sum_{j=0}^{m} q_j x^{\lambda_j}, \tag{16}$$

can be done as

$$(P \odot Q)(x) = \sum_{i=0}^{n}\sum_{j=0}^{m} p_i q_j x^{\lambda_i\lambda_j}. \tag{17}$$

Under the restrictions that for each i and j we have

$$|\lambda_i| > 1, \qquad \mathrm{Re}\,(\lambda_i\bar{\lambda}_j - 1) > 0, \tag{18}$$

we introduce an unusual inner product for Müntz polynomials (16)

$$[P,Q] = \int_0^1 (P \odot \overline{Q})(x)\,\frac{\mathrm{d}x}{x^2}, \tag{19}$$

where $(P \odot Q)(x)$ is defined by (17).

It is not clear immediately that (19) represents an inner product. A proof of this fact was given in [29], where Sylvester's necessary and sufficient conditions [25, p. 214] were used in order to prove a positive definiteness of matrix $H_n = \left[1/(\lambda_i\bar{\lambda}_j - 1)\right]_{i,j=0}^{n}$. Also, it can be done by taking the functions $f_k(\theta) = 1/\left(\lambda_k - \mathrm{e}^{\mathrm{i}\theta}\right)$ $(k = 0, 1, \dots)$ and interpreted H_n as the corresponding Gram's matrix. Indeed,

$$\frac{1}{\lambda_i\bar{\lambda}_j - 1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_i(\theta)\overline{f_j(\theta)}\,\mathrm{d}\theta = (f_i, f_j).$$

Professor Dušan R. Georgijević (Belgrade) pointed out this fact.

Under the conditions (18), we introduced and studied (see [29]) the Müntz polynomials $Q_n(x) \equiv Q_n(x;\Lambda_n)$, $n = 0,1,\ldots$, orthogonal with respect to the inner product (19). These polynomials are associated with the rational functions

$$W_n(s) = \frac{\prod\limits_{\nu=0}^{n-1}(s - 1/\bar{\lambda}_\nu)}{\prod\limits_{\nu=0}^{n}(s-\lambda_\nu)} \qquad (n = 0,1,\ldots) \tag{20}$$

in the sense that

$$Q_n(x) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma W_n(s)x^s \, \mathrm{d}s, \tag{21}$$

where the simple contour Γ surrounds all the points λ_ν $(\nu = 0,1,\ldots,n)$.

Assuming that $\lambda_i \neq \lambda_j$ $(i \neq j)$, we get a representation of (21) in the form

$$Q_n(x) = \sum_{k=0}^{n} A_{n,k}x^{\lambda_k}, \qquad A_{n,k} = \frac{\prod\limits_{\nu=0}^{n-1}(\lambda_k - 1/\bar{\lambda}_\nu)}{\prod\limits_{\substack{\nu=0\\ \nu\neq k}}^{n}(\lambda_k-\lambda_\nu)} \quad (k = 0,1,\ldots,n).$$

The case when $\lambda_i = \lambda_j$ can be considered as a limiting process $\lambda_i \to \lambda_j$.

We note that the rational functions (20) form a Malmquist-Takenaka system. Indeed, putting $a_\nu = 1/\bar{\lambda}_\nu$ $(\nu = 0,1,\ldots)$, these functions are reduced to (6) and the previous coefficients $A_{n,k}$ become

$$A_{n,k} = \frac{\prod\limits_{\nu=0}^{n-1}(1/\bar{a}_\nu - a_\nu)}{\prod\limits_{\substack{\nu=0\\ \nu\neq k}}^{n}(1/\bar{a}_k - 1/\bar{a}_\nu)} \qquad (k = 0,1,\ldots,n),$$

i.e., (7). Then,

$$[Q_n, Q_m] = \int_0^1 (Q_n \odot \overline{Q}_m)(x)\frac{\mathrm{d}x}{x^2} = \sum_{i=0}^{n}\sum_{j=0}^{m} A_{n,i}\overline{A}_{n,j}\int_0^1 x^{\lambda_i\bar{\lambda}_j-2}\,\mathrm{d}x,$$

i.e.,

$$[Q_n, Q_m] = \sum_{i=0}^{n}\sum_{j=0}^{m}\frac{A_{n,i}\overline{A}_{n,j}}{\lambda_i\bar{\lambda}_j - 1}.$$

According to Lemma 1 and equality (9), we see that $[Q_n, Q_m] = (W_n, W_m)$ and prove the following orthogonality relation for the polynomials $Q_n(x)$:

Theorem 8. *Under conditions* (18) *on the sequence* Λ, *the Müntz polynomials* $Q_n(x)$, $n = 0, 1, \ldots$, *defined by* (21), *are orthogonal with respect to the inner product* (19), *i.e.,*

$$[Q_n, Q_m] = \frac{\delta_{n,m}}{(|\lambda_n|^2 - 1)|\lambda_0\lambda_1 \cdots \lambda_{n-1}|^2}.$$

We mention now some recurrence relations for the polynomials $Q_n(x)$:

Theorem 9. *Suppose that* Λ *is a complex sequence satisfying* (18). *Then the polynomials* $Q_n(x)$, *defined by* (21), *satisfy*

$$xQ_n'(x) = xQ_{n-1}'(x) + \lambda_n Q_n(x) - (1/\bar{\lambda}_{n-1})Q_{n-1}(x),$$

$$xQ_n'(x) = \lambda_n Q_n(x) + \sum_{k=0}^{n-1}(\lambda_k - 1/\bar{\lambda}_k)Q_k(x),$$

$$xQ_n''(x) = (\lambda_n - 1)Q_n'(x) + \sum_{k=0}^{n-1}(\lambda_k - 1/\bar{\lambda}_k)Q_k'(x),$$

$$Q_n(1) = 1, \quad Q_n'(1) = \lambda_n + \sum_{k=0}^{n-1}(\lambda_k - 1/\bar{\lambda}_k),$$

$$Q_n(x) = Q_{n-1}(x) - (\lambda_n - 1/\bar{\lambda}_{n-1})x^{\lambda_n}\int_x^1 t^{-\lambda_n - 1}Q_{n-1}(t)\,dt \quad (x \in (0,1]).$$

When $\lambda_\nu \to \lambda$ for each ν, we obtain the following particular result of Müntz polynomials (21):

Corollary 1. *Let* $Q_n(x)$ *be defined by* (21) *and let* $\lambda_0 = \lambda_1 = \cdots = \lambda_n = \lambda$. *Then*

$$Q_n(x) = x^\lambda L_n\bigl(-(\lambda - 1/\bar{\lambda})\log x\bigr),$$

where $L_n(x)$ *is a Laguerre polynomial.*

For real sequences Λ we can prove ([29,37]):

Theorem 10. *Let* Λ *be a real sequence such that* $1 < \lambda_0 < \lambda_1 < \cdots$. *Then the polynomial* $Q_n(x)$, *defined by* (21), *has exactly* n *simple zeros in* $(0,1)$ *and no other zeros in* $[1, \infty)$.

The graphics of polynomials $Q_4(x)$ and $Q_5(x)$ are displayed in Fig. 1. Their zeros are more densely distributed around 0 than in other parts of the interval $[0,1]$. From Fig. 1 we can see that only two zeros of Q_5 are in the interval $[0.1, 1]$. In Fig. 2 we display the graphic of $Q_5(x)$ on the intervals $[0, 10^{-7}]$, $[10^{-7}, 10^{-3}]$, and $[10^{-3}, 0.1]$. Notice that $Q_5(x)$ has one zero in each of these intervals.

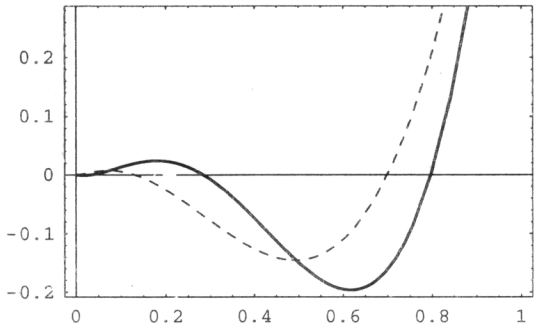

Fig. 1. Graphics $x \mapsto Q_4(x)$ (solid line) and $x \mapsto Q_5(x)$ (broken line)

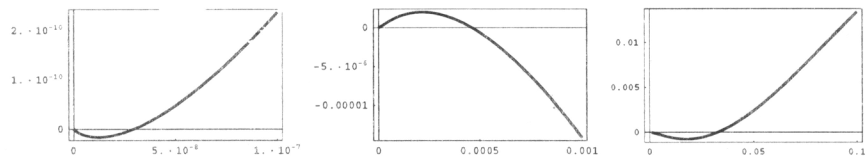

Fig. 2. The Müntz polynomial $Q_5(x)$ on $[0, 10^{-7}]$, $[10^{-7}, 10^{-3}]$, and $[10^{-3}, 0.1]$

4 Numerical Evaluation of the Orthogonal Müntz Polynomials and an Application

In this section we give a short account on numerical evaluation of the orthogonal Müntz polynomials and point out an application of such orthogonal polynomials in numerical integration.

4.1 Numerical evaluation of orthogonal Müntz polynomials

A direct evaluation of the Müntz polynomials $P_n(x)$ (or $Q_n(x)$) in the power form can be problematic in finite arithmetic, especially when n is a large number and x is close to 1. The polynomial coefficients become very large numbers when n increases, but their sums are always equal to 1, i.e., $P_n(1) = 1$ and $Q_n(1) = 1$. An illustrative numerical example was considered in [27].

Using (11) and an integration along a contour in the complex plane, we can prove the following result ([27]):

Theorem 11. *Let* $\sigma < -1/2$, $\omega = \log(1/x)$, *and*

$$\varphi_n(t;\omega) = \frac{1}{2\mathrm{i}}\Big(f_n(t;\omega)\,\mathrm{e}^{\mathrm{i}t} + f_n(-t;\omega)\,\mathrm{e}^{-\mathrm{i}t}\Big),$$

where

$$f_n(t;\omega) = \prod_{\nu=0}^{n-1} \frac{t + \mathrm{i}(\sigma + \bar{\lambda}_\nu + 1)\omega}{t + \mathrm{i}(\sigma - \lambda_\nu)\omega} \cdot \frac{1}{t + \mathrm{i}(\sigma - \lambda_n)\omega}. \tag{22}$$

Then the Müntz-Legendre polynomials can be represented in the integral form

$$P_n(x) = \frac{x^\sigma}{\pi} \int_0^{+\infty} \varphi_n(t;\omega)\, \mathrm{d}t.$$

For real sequences Λ we get a useful result:

Theorem 12. *Let $\Lambda = \{\lambda_0, \lambda_1, \dots\}$ be a real sequence such that $\lambda_k > -1/2$ for every $k \ge 0$, $f_n(t;\omega)$ be defined by (22), and $\sigma < -1/2$. Then, the Müntz-Legendre polynomials have a computable representation*

$$P_n(x) = \frac{x^\sigma}{\pi} \operatorname{Im}\Big\{L_1(f_n(\,\cdot\,;\omega)) + L_2(f_n(\,\cdot\,;\omega))\Big\},$$

where

$$L_1(f_n(\,\cdot\,;\omega)) = \pi \int_0^1 \left[\sum_{k=1}^m f_n\big(\pi(y+k-1);\omega\big) e^{i\pi(y+k-1)}\right] \mathrm{d}y,$$

$$L_2(f_n(\,\cdot\,;\omega)) = (-1)^m \int_0^{+\infty} \psi_n(y;\omega)\, \mathrm{e}^{-y}\, \mathrm{d}y,$$

and $\psi_n(y;\omega) = \mathrm{i} f_n(a + \mathrm{i}y;\omega)$, $m \ge 1$.

In the numerical implementation we use the Gauss-Legendre rule on $(0,1)$ and the Gauss-Laguerre rule for calculating $L_1(f_n(\,\cdot\,;\omega))$ and $L_2(f_n(\,\cdot\,;\omega))$, respectively. Numerical experiments show that a convenient choice for the parameter σ is $\lambda_{\min} - \pi/\omega$, where $\lambda_{\min} = \min\{\lambda_0, \lambda_1, \dots\}$.

For evaluating the Müntz polynomials $Q_n(x)$, defined by (21), we can use the same procedure with rational function (20).

4.2 Some remarks on an application in numerical integration

Let $d\sigma(x)$ be a given nonnegative measure on $[0,1]$ and

$$\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\} \qquad (x \in [0,1]), \tag{23}$$

be a system of linearly independent functions chosen to be complete in some suitable space of functions. If the quadrature rule

$$\int_0^1 f(x) d\sigma(x) = \sum_{k=1}^n A_k f(x_k) + R_n(f) \tag{24}$$

is such that it integrates exactly the first $2n$ functions in (23), we call the rule (24) *Gaussian* with respect to the system (23). The existence and uniqueness of a Gaussian quadrature rule (24) with respect to the system (23), or shorter a *generalized Gaussian formula*, is always guaranteed if the first $2n$ functions of this system constitute a Chebyshev system on $[0,1]$. Then, all the weights $A_1, \ldots, A_n$ in (24) are positive.

A numerical algorithm for the construction of generalized Gaussian quadratures was investigated by Ma, Rokhlin and Wandzura [21]. They take a Chebyshev system of functions $\{\phi_0, \phi_1, \ldots, \phi_{2n-1}\}$ with certain conditions on $[0,1]$ and call it *extended Hermite* (EH) *system*. Their algorithm is ill conditioned.

Namely, in order to obtain the double precision results, the authors ([21]) performed the computations in an extended precision (Q-arithmetic – REAL* 16) for generating Gaussian quadratures up to order 20, and in MATEMATICA (120 digits operations) for generating Gaussain quadratures of higher orders ($n \leq 40$). In particular, the following important cases of EH systems:

$$\{1, \log x, x, x \log x, \ldots, x^{n-1}, x^{n-1} \log x\} \tag{25}$$

and

$$\{1, x^s, x, x^{s+1}, \ldots, x^{n-1}, x^{s+n-1}\} \tag{26}$$

for $s = 1/3$, $s = -1/3$, $s = -2/3$, were considered in [21]. The case (25) was also considered by Andronov [3].

Recently, we presented a stable numerical method for constructing the generalized Gaussian quadratures for the Müntz polynomials $\{P_0, P_1, \ldots, P_{2n-1}\}$. Our constructive method [28] is based on an application of the orthogonal Müntz polynomials $P_n(x)$, as well as on a numerical procedure for evaluation of such polynomials with a high-precision [27].

Notice that for

$$\lambda_{2k} = k, \qquad \lambda_{2k+1} = k + s \qquad (k = 0, 1, \ldots),$$

the Müntz polynomials $\{P_0, P_1, \ldots, P_{2n-1}\}$ reduces to (26). The case $s = 0$ corresponds to (25). Such algebraic-logarithmic polynomials are considered in Subsection 3.2.

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