New Developments on Turán's Extremal Problems for Polynomials

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Abstract

In this paper we give an account of L^r inequalities of Turán type for algebraic polynomials, mainly initiated and studied by the late Professor Arun K. Varma. This paper could be comprehended as a continuation of our previous survey paper [8].

1 Introduction

Let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree at most n and let W_n be some of its subsets. For a given norm $\|\,.\,\|$ we consider extremal problems

$$B_{n,m} = \inf_{P \in W_n} \frac{\|P^{(m)}\|}{\|P\|} \qquad (1 \le m \le n).$$

In comparing with inequalities of Markov's type (cf. Milovanović, Mitrinović, Rassias [9, Chap. 6]), here we have opposite inequalities which are known as *inequalities of Turán type*.

Turán [11] proved the following inequality for polynomials $P \in \mathcal{P}_n$ having all their zeros in $[-1, 1]$,

$$\|P'\|_\infty > \frac{\sqrt{n}}{6}\,\|P\|_\infty\,, \tag{1}$$

taking the uniform norm $\|f\|_\infty = \max_{-1\le t\le 1} |f(t)|$. The constant $\sqrt{n}/6$ is not the best possible.

Turán's inequality (1) has been generalized and extended in several different ways.

Firstly, inequality (1) was sharpened by Erőd [6], who obtained

$$\|P'\|_\infty \geq B_n \|P\|_\infty\,, \tag{2}$$

where $B_2 = 1$, $B_3 = 3/2$, and

$$\begin{aligned} B_{2k} &= \frac{2k}{\sqrt{2k-1}}\left(1-\frac{1}{2k-1}\right)^{k-1}, \\ B_{2k+1} &= \frac{(2k+1)^2}{2k\sqrt{2k+2}}\left(1-\frac{\sqrt{2k+2}}{2k}\right)^{k-1}\left(1+\frac{1}{\sqrt{2k+2}}\right)^{k}, \end{aligned}$$

for $k = 2, 3, \ldots$.

Exactly, equality in (2) is attained for $P(t) = (1-t)^n$, if $n = 1, 2, 3$, and for $P(t) = (1-t)^{n-[n/2]}(1+t)^{[n/2]}$, if $n \geq 4$.

Let W_n be the set of all algebraic polynomials of degree n whose zeros are all real and lie inside $[-1, 1]$. The corresponding inequality for the second derivative of such polynomials was investigated by Babenko and Pichugov [2].

If $P \in W_n$, $n \geq 2$, they proved that

$$\|P''\|_\infty \geq B_{n,2} \|P\|_\infty\,, \tag{3}$$

where $B_{n,2} = \min\bigl\{n, (n-1)n/4\bigr\}$.

If $n = 2, 3, 4, 5$, then $B_{n,2} = (n-1)n/4$, and equality in (3) is attained only for polynomials of the form $P(t) = C(1 \pm t)^n$, where C is an arbitrary real constant different from zero.

In the case $n \geq 6$, they found that $B_{n,2} = n$, and for $n = 2m$ equality in (3) holds only for polynomials of the form $P(t) = C(1-t^2)^m$, where C is an arbitrary real constant different from zero.

An analogue in L^2 norm for algebraic polynomials was considered firstly by Professor A. K. Varma [15]. Taking $\|f\|_2^2 = \int_{-1}^{1} f(t)^2\,dt$ he proved:

Theorem 1 *If $P \in W_n$ we have*

$$\|P'\|_2^2 \geq \frac{n}{2}\,\|P\|_2^2. \tag{4}$$

This result is best posible in the sense that there exists a polynomial P_0 of degree n having all zeros inside $[-1, 1]$ and for which

$$\|P_0'\|_2^2 = \left(\frac{n}{2} + \frac{3}{4} + \frac{3}{4(n-1)}\right)\|P_0\|_2^2, \quad n > 1.$$

The proof of this theorem was based on the following inequality

$$\| \sqrt{1-t^2}P' \|_2^2 \geq \frac{n}{2} \|P\|_2^2 \qquad (P \in W_n),$$

which becomes an equality only for $P(t) = C(1+t)^p(1-t)^q$, $p+q=n$, where C is an arbitrary non-zero constant.

In this survey we give an account of L^r $(r \geq 1)$ inequalities of Turán type.

2 Turán Type Inequalities in L^2 Norm

In [16] Professor Varma gave a more precise form of (4).

Theorem 2 *Let* $\|f\|_2^2 = \int_{-1}^1 f(t)^2\,dt$, $P \in W_n$ *and* $P(1) = P(-1) = 0$. *Then we have*

$$\|P'\|_2^2 \geq \Big(\frac{n}{2} + \frac{3}{4} + \frac{3}{4(n-1)}\Big)\|P\|_2^2, \tag{5}$$

with equality for $P(t) = (1-t^2)^m$, $n = 2m$.

Taking the norm $\|f\|_2^2 = \int_{-1}^1 (1-t^2)f(t)^2\,dt$, in 1979 Varma [17] proved the following result:

Theorem 3 *For* $P \in W_n$ *and* $n \geq 2$ *we have*

$$\|P'\|_2^2 \geq \Big(\frac{n}{2} + \frac{1}{4} - \frac{1}{4(n+1)}\Big)\|P\|_2^2,$$

with equality for $P(t) = (1-t^2)^m$, $n = 2m$.

Later Varma [18] proved an improvement of one of his earlier results.

Theorem 4 *Let* $\|f\|_2^2 = \int_{-1}^1 f(t)^2\,dt$. *If* $P \in W_n$ *and* $n = 2m$; *then*

$$\|P'\|_2^2 \geq \Big(\frac{n}{2} + \frac{3}{4} + \frac{3}{4(n-1)}\Big)\|P\|_2^2, \tag{6}$$

where equality holds if and only if $P(t) = (1-t^2)^m$. *Moreover, if* $n = 2m-1$, *then*

$$\|P'\|_2^2 \geq \Big(\frac{n}{2} + \frac{3}{4} + \frac{5}{4(n-2)}\Big)\|P\|_2^2, \quad n \geq 3, \tag{7}$$

where equality holds if and only if $P(t) = (1-t)^{m-1}(1+t)^m$ *or* $P(t) = (1-t)^m(1+t)^{m-1}$.

This result is an improvement of Theorem 2 in two respects. First, the condition $P(1) = P(-1) = 0$ is not necessary for (5) to hold. Secondly, here there exist precise bounds for n even and also for n odd as mentioned in (6) and (7).

In the same norm, Varma [18] also proved:

Theorem 5 *Let $P \in W_n$, subject to the condition $P(1) = 1$; then*

$$\|P'\|_2^2 \geq \frac{n}{4} + \frac{1}{8} + \frac{1}{8(2n-1)}, \qquad n \geq 1,$$

where equality holds for $P(t) = ((1+t)/2)^n$.

This inequality is an improvement over $\|P'\|_2^2 > n/4$, given by Szabados and Varma [10].

The corresponding inequality for polynomials $P \in W_n$ in L^r norm, defined on $(-1, 1)$ by $\|f\|_r = \left(\int_{-1}^{1} |f(t)|^r \, dt\right)^{1/r}$, was considered by Zhou [20].

Theorem 6 *If $P \in W_n$, then for $1 \leq r \leq +\infty$,*

$$\|P'\|_r \geq C\sqrt{n}\, \|P\|_r,$$

where C is a positive absolute constant.

A similar result for $0 < r < 1$ was obtained also by Zhou [21]. Recently, Zhou [22] proved the following results:

Theorem 7 *If $P \in W_n$, then for $1 \leq r \leq s \leq +\infty$,*

$$\|P'\|_r \geq C n^{1/2 - 1/(2r) + 1/(2s)} \|P\|_s,$$

where C is a positive absolute constant.

The example $P(t) = (1 - t^2)^{[n/2]}$ in the previous theorem shows that the order $n^{1/2-1/(2r)+1/(2s)}$ cannot be improved.

Theorem 8 *Let $1 \leq r \leq s \leq +\infty$ and P be an polynomial of degree n with only real zeros. If at most k zeros of P lie outside the interval $[-1, 1]$, then*

$$\|P'\|_r \geq C_k n^{1/2 - 1/(2r) + 1/(2s)} \|P\|_s,$$

where C_k is a positive constant depending only upon k.

After Professor Varma's death, the following result [14] has appeared:

Theorem 9 *Let* $\|f\|^2 = \int_{-1}^{1}(1-t^2)^\alpha f(t)^2\,dt$*,* $P \in W_n$ $(n \geq 2)$ *and* $\alpha > 1$ *real. Then we have* $(n = 2m)$

$$\|P'\|^2 \geq \frac{n^2(2n+2\alpha+1)}{4(n+\alpha-1)(n+\alpha)}\|P\|^2,$$

with equality if and only if $P(t) = c(1-t^2)^m$*. If* $P(\pm 1) = 0$*, then the previous inequality remains valid for* $\alpha > -1$*.*

This result was proved earlier by Varma for the cases $\alpha = 0$ and $\alpha = 1$. In the same paper [14], Underhill and Varma invesigated the corresponding inequality in L^4 norm for $\alpha = 3$:

Theorem 10 *Let* $\|f\|_4^4 = \int_{-1}^{1}(1-t^2)^3 f(t)^4\,dt$ *and* $P \in W_n$*. Then we have* $(n = 2m)$

$$\|P'\|_4^4 \geq \frac{3n^3(4n+7)(4n+5)}{4(4n+6)(4n+4)(4n+2)}\|P\|_4^4,$$

with equality if and only if $P(t) = c(1-t^2)^m$*.*

Also, they considered the cases when $\alpha = 1$ and $\alpha = 2$, as well as an inequality in L^r norm, when $r \geq 2$ is even. In [19] Varma proved:

Theorem 11 *Let* $P \in W_n$*, subject to the condition* $P(1) = 0$*. Then, for* $r \geq 1$*, we have*

$$\int_{-1}^{1}|P'(t)|^r\,dt \geq \frac{n^r}{2^{r-1}((n-1)r+1)},$$

wuth equality if and only if $P(t) = ((1+t)/2)^n$*.*

3 Bojanov's Solution

More general results on Turán type inequalities were obtained by Bojanov [3]. Introducing the notations

$$p_{n,k}(t) = (-1)^{n-k}\frac{n^n}{2^n k^k (n-k)^{n-k}}\,(t+1)^k(t-1)^{n-k},$$

for $k = 0, 1, \ldots, n$ $(n \in \mathbb{N})$, Bojanov [3] proved the following results:

Theorem 12 *Let* $x \mapsto \varphi(x)$ *be any continuously differentiable, strictly increasing convex function in* $[0, +\infty)$*. Then for every* $n \in \mathbb{N}$ *and* $m \in \{1, \ldots, n\}$*, there exists a constant* $A_{n,m} > 0$ *such that*

$$\int_{-1}^{1}\varphi(|P^{(m)}(t)|)\,dt \geq A_{n,m}\|P\|_\infty \qquad (P \in W_n).$$

Moreover,

$$A_{n,m} = \min_{0\le k\le n} \left\{ \int_{-1}^{1} \varphi(|p_{n,k}^{(m)}(t)|)\, dt \right\}$$

and this is the exact constant.

Theorem 13 *For any given n and m, there exists a constant $B_{n,m}$ such that*

$$\|P^{(m)}\|_\infty \ge B_{n,m}\|P\|_\infty \qquad (P \in W_n).$$

Moreover,

$$B_{n,m} = \min_{0\le k\le n} \{\|p_{n,k}^{(m)}\|_\infty\}.$$

Using this theorem one could get the exact previous result of Erőd [6] and Babenko and Pichugov [2], treating the case $m = 1$ and $m = 2$, respectively. In the first case we have that

$$B_{n,1} = \|p'_{n,k}\|_\infty \quad \text{for} \quad k = \left[\frac{n}{2}\right].$$

Combining an idea of Babenko and Pichugov [2] with Theorem 13, Bojanov [3] obtained an explicit value of $B_{n,2}$.

Following Bojanov [3], let $P \in W_n$ and $t_1 \le t_2 \le \cdots \le t_n$ be the zeros of $t \mapsto P(t)$. Then we have

$$P'(t) = P(t)\sigma(t), \quad P''(t) = P'(t)\sigma(t) + P(t)\sigma'(t) \quad (P \in W_n),$$

where

$$\sigma(t) = \sum_{\nu=1}^{n} \frac{1}{t - t_k}\,.$$

Suppose that $\|P\|_\infty = |P(\tau)|$ and $\tau \in (-1, 1)$. Then $P'(\tau) = 0$ and therefore $\sigma(\tau) = 0$. Thus, $|P''(\tau)| = |\sigma'(\tau)|$. Choose $P = p_{n,k}$, where $k = 1, \ldots, n-1$. Then $\tau = b_{n,k} = (2k - n)/n$ and

$$\sigma(t) = \frac{k}{t+1} + \frac{n-k}{t-1}\,.$$

Therefore,

$$\|p''_{n,k}\|_\infty \ge |p''_{n,k}(\tau)| = |\sigma'(\tau)| = \frac{n^2}{4}\Big(\frac{1}{k} + \frac{1}{n-k}\Big).$$

But the last expression attains its minimal value for $k = [n/2]$ and this minimal value is n for even n, respectively $n\big(1 + 1/(n^2 - 1)\big)$, for odd n. Adding the obvious fact that

$$\|p''_{n,1}\|_\infty = \|p''_{n,n}\|_\infty = \frac{1}{4}\, n(n-1) \ge n \qquad (\text{for } n > 4),$$

we get $B_{n,2} = n$ for even $n \geq 6$, and

$$B_{n,2} \geq n\Big(1 + \frac{1}{n^2 - 1}\Big) \quad \text{for odd } n \geq 5.$$

Bojanov [3] also proved:

Theorem 14 *Let $x \mapsto \varphi(x)$ be any continuously differentiable, strictly increasing convex function in $[0, +\infty)$. Then for every $n \in \mathbb{N}$ and $m \in \{1, \ldots, n\}$,*

$$\int_{-1}^{1} \varphi(|P^{(m)}(t)|)\, dt \geq \int_{-1}^{1} \varphi(|p_{n,n}^{(m)}(t)|)\, dt$$

for every polynomial $P \in W_n$ such that $P(1) = 1$.

If $\varphi(x) = x^r$ $(1 \leq r < +\infty)$ this theorem reduces to the following result:

Corollary 15 *Let $P \in \mathcal{P}_n$, $P(1) = 1$, and $1 \leq r < +\infty$. Then*

$$\|P^{(m)}\|_r \geq \frac{n!}{2^m (n-m)!}\Big(\frac{2}{(n-m)r + 1}\Big)^{1/r}.$$

Notice that for $m = 1$ this corollary gives Theorem 11.

Inequalities of Turán type for trigonometric polynomials were investigated by Babenko and Pichugov [1]–[2], Zhou [20], Tyrygin [12]–[13], and Bojanov [3]–[4].

4 A Result of Chen

In this section we mention a recent result of Chen [5], which can be expressed in the same way as the Markov's inequality in [7] (see also [9]).

We consider a general case with a given non-negative measure $d\sigma(t)$ on the real line $\mathbb{R}$, with compact or infinite support, for which all moments

$$\mu_\nu = \int_{\mathbb{R}} t^\nu \, d\sigma(t), \qquad \nu = 0, 1, \ldots,$$

exist and are finite, and $\mu_0 > 0$. Then there exists a unique set of orthonormal polynomials $\pi_\nu(\cdot) = \pi_\nu(\cdot\,; d\sigma)$, $\nu = 0, 1, \ldots$, defined by

$$\pi_\nu(t) = a_\nu t^\nu + \text{lower degree terms}, \quad a_\nu > 0,$$

and

$$\int_{\mathbb{R}} \pi_\nu(t)\pi_\mu(t)\, d\sigma(t) = \delta_{\nu\mu}, \qquad \nu, \mu \geq 0. \tag{8}$$

For each polynomial $P \in \mathcal{P}_n$, with complex coefficients, we take

$$\|P\| = \Big(\int_{\mathbb{R}} |P(t)|^2\, d\sigma(t)\Big)^{1/2}.$$

As a restricted subset of $\mathcal{P}_n$ Chen [5] took

$$W_n = \mathcal{P}_{n,m}(d\sigma) = \big\{P \in \mathcal{P}_n \mid P \perp \mathcal{P}_{m-1}\big\},$$

i.e., $P \in W_n$ if $P \in \mathcal{P}_n$ and $(P, \pi_\nu) = 0$ for each $\nu = 0, 1, \ldots, m-1$.

Consider now the extremal problem

$$A_{n,m} = A_{n,m}(d\sigma) = \inf_{P \in W_n} \frac{\|P^{(m)}\|}{\|P\|} \qquad (1 \le m \le n). \tag{9}$$

Theorem 16 *The best constant $A_{n,m}$ defined in (9) is given by*

$$A_{n,m} = \big(\lambda_{\min}(B_{n,m})\big)^{1/2}, \tag{10}$$

where $\lambda_{\min}(B_{n,m})$ is the minimal eigenvalue of the matrix

$$B_{n,m} = \big[b_{i,j}^{(m)}\big]_{m \le i,j \le n},$$

whose elements are given by

$$b_{i,j}^{(m)} = \int_{\mathbb{R}} \pi_i^{(m)}(t)\pi_j^{(m)}(t)\, d\sigma(t), \qquad m \le i, j \le n. \tag{11}$$

An extremal polynomial is

$$P^*(t) = \sum_{\nu=m}^{n} c_\nu \pi_\nu(t),$$

where $\big[c_k, c_{k+1}, \ldots, c_n\big]^T$ is an eigenvector of the matrix $B_{n,m}$ corresponding to the eigenvalue $\lambda_{\min}(B_{n,m})$.

Proof. Let $P \in W_n$. Then we can write $P(t) = \sum_{\nu=m}^{n} c_\nu \pi_\nu(t)$ and

$$P^{(m)}(t) = \sum_{\nu=m}^{n} c_\nu \pi_\nu^{(m)}(t), \qquad m \le n,$$

where the coefficients c_ν are uniquely determined. Hence, by (8) and (11), we have

$$\|P\|^2 = \sum_{\nu=m}^{n} |c_\nu|^2 \qquad \text{and} \qquad \|P^{(m)}\|^2 = \sum_{i,j=m}^{n} c_i \bar{c}_j b_{i,j}^{(m)}.$$

Then

$$\frac{\|P^{(m)}\|^2}{\|P\|^2} = \frac{\displaystyle\sum_{i,j=m}^{n} c_i \bar{c}_j b_{i,j}^{(m)}}{\displaystyle\sum_{i=m}^{n} |c_i|^2} = \frac{\langle B_{n,m}\boldsymbol{c}, \boldsymbol{c}\rangle}{\langle \boldsymbol{c}, \boldsymbol{c}\rangle}, \tag{12}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in an $(n-m+1)$-dimensional space.

The matrix $B_{n,m}$ is evidently positive definite. Since the right side in (12) is not smaller than the minimal eigenvalue of this matrix, we obtain

$$\|P^{(m)}\|^2 \geq \lambda_{\min}(B_{n,m})\|P\|^2. \tag{13}$$

In order to show that $A_{n,m}$, given by (10), is the best possible, we note that (13) reduces to an equality if we put $P(t) = P^*(t) = \sum_{\nu=m}^{n} c_\nu^* \pi_\nu(t)$, where $\left[c_m^*, c_{m+1}^*, \ldots, c_n^*\right]^T$ is an eigenvector of the matrix $B_{n,m}$ corresponding to $\lambda_{\min}(B_{n,m})$. Q.E.D.

An alternative result like Theorem 16 is the following theorem:

Theorem 17 *Let* $Q_{n,m} = \left[q_{ij}^{(m)}\right]_{m\leq i,j\leq n}$ *be an upper triangular matrix of the order* $n-m+1$*, whose elements* $q_{ij}^{(m)}$ *are given by the following inner product*

$$q_{ij}^{(m)} = (\pi_j^{(m)}, \pi_{i-m}) \qquad (m \leq i, j \leq n).$$

Then the best constant $A_{n,m}$ *defined in* (9) *is given by*

$$A_{n,m} = \left(\lambda_{\min}(Q_{n,m}Q_{n,m}^T)\right)^{1/2}. \tag{14}$$

Alternatively, (14) *can be expressed in the form*

$$A_{n,m} = \left(\lambda_{\max}(C_{n,m})\right)^{-1/2}, \tag{15}$$

where $C_{n,m} = \left(Q_{n,m}Q_{n,m}^T\right)^{-1}$.

Proof. It is enough to consider only a real polynomial set $\mathcal{P}_n$. Let $P \in W_n$ and $\pi_j^{(m)}(t) = \sum_{i=m}^{n} q_{ij}^{(m)}\pi_{i-m}(t)$, where $q_{ij}^{(m)} = (\pi_j^{(m)}, \pi_{i-m})$. Then

$$P^{(m)}(t) = \sum_{j=m}^{n} c_j \sum_{i=m}^{j} q_{ij}^{(m)}\pi_{i-m}(t) = \sum_{i=m}^{n}\Bigl(\sum_{j=m}^{n} c_j q_{ij}^{(m)}\Bigr)\pi_{i-m}(t)$$

and

$$\|P^{(m)}\|^2 = \sum_{i=m}^{n} \Bigl(\sum_{j=i}^{n} c_j q_{ij}^{(m)} \Bigr)^2 = \sum_{i=m}^{n} y_i^2,$$

where

$$y_i = \sum_{j=i}^{n} c_j q_{ij}^{(m)}, \qquad i = m, \ldots, n. \tag{16}$$

Let $\boldsymbol{c} = [c_m, \ldots, c_n]^T$, $\boldsymbol{y} = [y_m, \ldots, y_n]^T$, and $Q_{n,m} = \bigl[q_{ij}^{(m)}\bigr]_{m \le i,j \le n}$. Since $\boldsymbol{y} = Q_{n,m}\boldsymbol{c}$, it follows that

$$\frac{\|P^{(m)}\|^2}{\|P\|^2} = \frac{\langle \boldsymbol{y}, \boldsymbol{y}\rangle}{\langle \boldsymbol{c}, \boldsymbol{c}\rangle} = \frac{\langle \boldsymbol{y}, \boldsymbol{y}\rangle}{\langle (Q_{n,m}Q_{n,m}^T)^{-1}\boldsymbol{y}, \boldsymbol{y}\rangle}.$$

Thus (14) and (15) hold. Q.E.D.

Now, we will consider a few special measures.

1° $d\sigma(t) = e^{-t^2}dt$, $-\infty < t < +\infty$. Here we have

$$\pi_\nu(t) = \hat{H}_\nu(t) = (\sqrt{\pi}\, 2^\nu \nu!)^{-1/2} H_\nu(t),$$

where H_ν is a Hermite polynomial of degree ν. Since

$$H_\nu'(t) = 2\nu H_{\nu-1}(t) \qquad \text{and} \qquad \hat{H}_\nu'(t) = \sqrt{2\nu}\,\hat{H}_{\nu-1}(t),$$

we have

$$\hat{H}_\nu^{(m)}(t) = \sqrt{2\nu}\sqrt{2(\nu-1)} \cdots \sqrt{2(\nu-m+1)}\,\hat{H}_{\nu-m}(t),$$

i.e.,

$$\hat{H}_\nu^{(m)}(t) = \sqrt{2^m m! \binom{\nu}{m}}\,\hat{H}_{\nu-m}(t),$$

and

$$b_{ij}^{(m)} = 2^m m! \binom{i}{m} \delta_{ij}, \qquad m \le i, j \le n.$$

Thus, we find $\lambda_{\min}(B_{n,m}) = 2^m m!$ and $A_{n,m} = 2^{m/2}\sqrt{m!}$.

2° $d\sigma(t) = t^s e^{-t}dt$, $0 < t < +\infty$. Here we have the generalized Laguerre case with

$$\pi_\nu(t) = \hat{L}_\nu^s(t) = \sqrt{\nu!/\Gamma(\nu+s+1)} \sum_{i=0}^{\nu} (-1)^{\nu-i} \binom{\nu+s}{\nu-i} \frac{t^i}{i!},$$

where Γ is the gamma function.

First, we consider the simplest case where $m = 1$. Since

$$\frac{d}{dt}\hat{L}_j^s(t) = \sum_{i=1}^{j} q_{ij}^{(1)}\hat{L}_{i-1}^s(t), \qquad q_{ij}^{(1)} = -\sqrt{\frac{j!}{\Gamma(j+s+1)}}\cdot\sqrt{\frac{\Gamma(i+s)}{(i-1)!}},$$

from the equalities (16), it follows that

$$c_i = y_{i+1} - \sqrt{\frac{i+s}{i}}\,y_i, \qquad i = 1,\ldots,n,$$

where we put $y_{n+1} = 0$. The elements $p_{ij}^{(1)}$ of the matrix $P_{n,1} = Q_{n,1}^{-1}$ are

$$p_{ij}^{(1)} = -\sqrt{1+\frac{s}{i}}, \quad i = 1,\ldots,n; \quad p_{i,i+1}^{(1)} = 1, \quad i = 1,\ldots,n-1;$$

$$p_{ij}^{(1)} = 0, \quad \text{otherwise},$$

so that $C_{n,1} = P_{n,1}^T P_{n,1} = -J_n$, where

$$J_n = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathbf{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}$$

and

$$\alpha_0 = -(1+s), \quad \alpha_\nu = -\Big(2+\frac{s}{\nu+1}\Big), \quad \beta_\nu = 1+\frac{s}{\nu}, \quad \nu = 1,\ldots,n-1.$$

We see that J_n is the Jacobi matrix for monic orthogonal polynomials $\{Q_\nu\}$, which satisfy the following three-term recurrence relation

$$Q_{\nu+1}(t) = (t-\alpha_\nu)Q_\nu(t) - \beta_\nu Q_{\nu-1}(t), \quad \nu = 0,1,\ldots,$$

with $Q_{-1}(t) = 0$ and $Q_0(t) = 1$. The eigenvalues of $C_{n,1}$ are $\lambda_\nu = -t_\nu$, where $Q_n(t_\nu) = 0$ for $\nu = 1,\ldots,n$.

The standard Laguerre case ($s = 0$) can be exactly solved. In fact, for $t = 2(z-1)$ with $-1 \le z \le 1$, we have

$$Q_\nu(t) = \cos(2\nu+1)\frac{\theta}{2} \Big/ \cos\frac{\theta}{2}, \qquad z = \cos\theta.$$

The eigenvalues of the matrix $C_{n,1}$ are

$$\lambda_\nu = -t_\nu = 4\sin^2\frac{(2\nu-1)\pi}{2(2n+1)}, \qquad \nu = 1,\ldots,n.$$

Since $\lambda_{\max}(C_{n,1}) = \lambda_n$, we obtain

$$A_{n,1} = \left(2\cos\frac{\pi}{2n+1}\right)^{-1}.$$

Now, we consider the case when $m=2$ and $s=0$. First, we note that

$$\frac{d^m}{dt^m}\hat{L}_j(t) = (-1)^m\sum_{i=m}^{j}\binom{j-i+m-1}{m-1}\hat{L}_{i-m}(t).$$

The formulae (16), for $m=2$, become

$$y_i = \sum_{j=i}^{n}(j-i+1)c_j, \qquad i = 2,\ldots,n.$$

Since $\Delta^2 y_i = c_i$ $(y_{n+1} = y_{n+2} = 0)$, we find a five-diagonal symmetric matrix of order $n-1$

$$C_{n,2} = \begin{bmatrix} 1 & -2 & 1 & & & & & \mathbf{O} \\ -2 & 5 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & 1 & -4 & 6 & -4 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & 6 & -4 \\ \mathbf{O} & & & & & 1 & -4 & 6 \end{bmatrix}.$$

Thus, using the maximal eigenvalue of this matrix, we obtain the best constant $A_{n,2} = \big(\lambda_{\max}(C_{n,2})\big)^{-1/2}$. In the simplest case when $n=2$ and $n=3$ we have $A_{2,2}=1$ and $A_{3,2} = (3-2\sqrt{2})^{1/2}$, respectively.

We conclude this paper with a remark that Varma [17] also studied an extremal problem on $(0,+\infty)$ with respect to the Laguerre measure, i.e., when $\|f\|_2^2 = \int_0^\infty e^{-t} f(t)^2\,dt$.

Theorem 18 *Let P be an algebraic polynomial of degree n whose zeros τ_ν $(\nu = 1, \ldots, n)$ all lie in the interval $[0, \infty)$. If $P(0) = 0$ or*

$$\sum_{\nu=1}^{n} \tau_\nu^{-1} \geq \frac{1}{2};$$

then

$$\|P'\|_2^2 \geq \frac{n}{2(2n-1)} \|P\|_2^2.$$

The equality holds for $P(t) = t^n$.

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