QUADRATURE FORMULAS OF RADAU TYPE ON $(0, +\infty)$*

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Abstract. In this paper we consider two classes of Radau type quadrature formulas for integrals extended over the positive real axis, assuming given algebraic behavior of the integrand at the origin and at infinity. The parameters of these formulas are expressible in terms of Gauss-Jacobi quadratures. A numerical example is included.

1. Introduction

For integrals on half-infinite intervals W. Gautschi [4] has developed two types of quadrature formulas. One has maximum polynomial degree of exactness, while the other has maximum rational degree of exactness. Here we treat in a similar spirit the Radau type of integrals over the half-infinite interval $[0, +\infty)$ and integrands that have an algebraic singularity at the origin of type x^α, $\alpha > -1$, and behave like $x^{-\beta}$, $\beta > 1$, as $x \to +\infty$. We show that both types of formulas can be reduced to Gaussian quadratures relative to appropriate Jacobi weight functions with different parameters like in [4].

We note that W. M. Harper [6] and S. Haber [5] have developed special symmetric quadrature rules for integrals extended over the whole real line, whose integrands go to zero like $|x|^\beta$ when $|x| \to +\infty$, that integrate exactly $(1+x^2)^{-\beta/2} \times f(x)$, for some types of rational functions $f(x)$. R. Kumar and M. K. Jain [7] have considered quadratures with the maximum "rational" degree of exactness. These formulas have not limitation on the allowed numbers of quadrature points. Similar problems were also considered in other papers (cf. [1], [3] and also [2, pp. 225–226] and [8, p. 52]).

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2. Gauss-Radau Quadrature Rules of Maximum Algebraic Degree of Exactness

Let $\mathcal{P}_m$ be the set of algebraic polynomial of degree at most m. In this section we wish to find a quadrature formula

$$\int_0^{\infty} f(x)\frac{x^{\alpha}}{(1+x)^{\beta}}\,dx = a_0 f(0) + \sum_{k=1}^{n} a_k f(x_k) + r_{n+1}(f) \tag{2.1}$$

of Gauss-Radau type with weight function $x \mapsto x^{\alpha}/(1+x)^{\beta}$.

In order to construct (2.1), we consider the following quadrature problem

$$\int_0^{\infty} F(x)\frac{x^{\alpha}}{(1+x)^{\beta}}\,dx = \sum_{k=1}^{n} a_k F(x_k) + r_{n+1}(F) \tag{2.2,}$$

with $F(x) = xg(x)$, such that $r_{n+1}(F) = 0$ whenever $g \in \mathcal{P}_{2n-1}$.

In formula (2.2) we should find $a_k,\ x_k\ (k = 1, \dots, n)$ from

$$\int_0^{\infty} \frac{x^{\alpha}}{(1+x)^{\beta}} x g(x)\,dx = \sum_{k=1}^{n} a_k x_k g(x_k) + r_n(g), \tag{2.3}$$

such that $r_n(g) = 0$ for $g(x) = x^{\lambda},\ \lambda = 0, 1, \dots, 2n-1$. Thus, for the new weight function $x \mapsto x^{\alpha+1}/(1+x)^{\beta}$, formula (2.3) will be exact for all $g \in \mathcal{P}_{2n-1}$.

To assure integrability (2.3), we assume

$$\alpha > -1, \qquad \beta - \alpha > 2n + 1. \tag{2.4}$$

Changing variables

$$\frac{1-x}{1+x} = t, \quad \text{i.e.,} \quad x = \frac{1-t}{1+t}, \tag{2.5}$$

we obtain

$$\begin{aligned}\int_{-1}^{1} (1-t)^{\lambda}(1+t)^{2n-\lambda-1}(1-t)^{\alpha+1}(1+t)^{\beta-\alpha-2n-2}\,dt \\ = 2^{\beta-1}\sum_{k=1}^{n}\Big\{a_k\frac{1-t_k}{(1+t_k)^{2n}}\Big\}(1-t_k)^{\lambda}(1+t_k)^{2n-\lambda-1}.\end{aligned} \tag{2.6}$$

Here we have set, in conformity with (2.5),

$$\frac{1 - x_k}{1 + x_k} = t_k, \qquad x_k = \frac{1 - t_k}{1 + t_k}, \tag{2.7}$$

Since $\{(1-t)^\lambda(1+t)^{2n-\lambda-1} \mid \lambda = 0, 1, \dots, 2n-1\}$ forms a basis in $\mathcal{P}_{2n-1}$, it follows from (2.6) that

$$t_k = \tau_k^J, \quad 2^{\beta-1}(1 - t_k)(1 + t_k)^{-2n} a_k = \omega_k^J, \quad k = 1, \dots, n, \tag{2.8}$$

where

$$\tau_k^J = \tau_k^{(n)}(\alpha + 1, \beta - \alpha - 2n - 2), \quad \omega_k^J = \omega_k^{(n)}(\alpha + 1, \beta - \alpha - 2n - 2) \tag{2.9}$$

are the n-point Gaussian nodes and weights relative to the Jacobi weight function with parameters $(\alpha + 1, \beta - \alpha - 2n - 2)$. Note, by assumption (2.4) that both parameters are larger than -1, as required by the theory of Gauss-Jacobi quadratures. In this way, (2.7) and (2.8) gives

$$x_k = \frac{1 - \tau_k^J}{1 + \tau_k^J}, \qquad a_k = \frac{(1 + \tau_k^J)^{2n}\omega_k^J}{2^{\beta-1}(1 - \tau_k^J)}, \quad k = 1, \dots, n, \tag{2.10}$$

for the desired abscissas and weights in the quadrature formula (2.2).

Supposing that $F(0) = 0$ and let $f(x)$ be given by $F(x) = f(x) - f(0)$. Then comparing (2.1) and (2.2) we obtain

$$a_0 = \int_0^\infty \frac{x^\alpha}{(1 + x)^\beta}\, dx - \sum_{k=1}^n a_k, \tag{2.11}$$

where a_k is defined by (2.10).

Using the transformation (2.5) we find

$$\begin{aligned} \int_0^\infty \frac{x^\alpha}{(1 + x)^\beta}\, dx &= \frac{1}{2^{\beta-1}} \int_{-1}^1 (1 - t)^\alpha (1 + t)^{\beta-\alpha-2}\, dt \\ &= \frac{1}{2^{\beta-1}} \|P_0^{(\alpha,\beta-\alpha-2)}\|^2 \\ &= \frac{\Gamma(\alpha + 1)\Gamma(\beta - \alpha - 1)}{\Gamma(\beta)}. \end{aligned} \tag{2.12}$$

Finally, using (2.9), (2.11), and (2.12), we find that

$$a_0 = \frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha-1)}{(\beta-1)\Gamma(\beta-1)} - \sum_{k=1}^{n} \frac{(1+\tau_k^J)^{2n}\omega_k^J}{2^{\beta-1}(1-\tau_k^J)}. \tag{2.13}$$

We consider now a generalized Gauss-Radau quadrature rule with the weight function $x \mapsto x^\alpha/(1+x)^\beta$ and point 0 of multiplicity m,

$$\begin{aligned} \int_0^\infty \frac{x^\alpha}{(1+x)^\beta} f(x)\,dx &= K_0 f(0) + K_1 f'(0) + \cdots \\ &+ K_{m-1} f^{(m-1)}(0) + \sum_{k=1}^{n} A_k f(X_k) + R_{n+m}(F), \end{aligned} \tag{2.14}$$

where the remainder $R_{n+m}(F)$ is zero whenever F is a polynomial of degree $\le 2n+m-1$.

In this case, we assume that the function F is given by

$$F(x) = f(x) - \sum_{i=0}^{m-1} \frac{f^{(i)}(0)}{i!} x^k.$$

In order to construct (2.14), we consider the corresponding quadrature problem

$$\int_0^\infty \frac{x^\alpha}{(1+x)^\beta} F(x)\,dx = \sum_{k=1}^{n} A_k F(X_k) + R_{n+m}(F),$$

for which $R_{n+m}(F) = 0$ whenever $g \in \mathcal{P}_{2n-1}$ and $F(x) = x^m g(x)$. Using the same technique as before we obtain that

$$X_k = \frac{1-T_k^J}{1+T_k^J}, \quad A_k = \frac{(1+T_k^J)^{2n+m-1}\Omega_k^J}{2^{\beta-1}(1-T_k^J)^m} \quad (k=1,\ldots,n), \tag{2.15}$$

where

$T_k^J = \tau_k^{(n)}(\alpha+m, \beta-\alpha-2n-m-1), \quad \Omega_k^J = \omega_k^{(n)}(\alpha+m, \beta-\alpha-2n-m-1)$

are the parameters of n-point Gaussian quadrature formula with the Jacobi weight function with parameters $(\alpha+m, \beta-\alpha-2n-m-1)$.

Since

$$F(0) = F'(0) = \cdots = F^{(m-1)}(0) = 0,$$

in a similar way as before we find the coefficients K_i $(i = 0, 1, \ldots, m-1)$ in the following form

$$K_i = \frac{1}{i!}\left(\frac{\Gamma(\alpha+i+1)\Gamma(\beta-\alpha-i-1)}{(\beta-1)\Gamma(\beta-1)} - \sum_{k=1}^{n} A_k X_k^i \right),$$

where A_k and X_k are given in (2.15).

3. Gauss-Radau Quadrature Rules of Maximum Rational Degree of Exactness

In [4] Gautschi considered the quadrature rule

$$\int_0^{+\infty} f(x)x^\alpha \, dx = \sum_{k=1}^{n} A_k f(x_k) + R_n(f), \tag{3.1}$$

which is exact (i.e., $R_n(f) = 0$) whenever

$$f(x) = \frac{1}{(1+x)^{\beta+\nu}}, \qquad \nu = 0, 1, \dots, 2n-1.$$

In the case $\alpha = 0$, $\beta = 2$, such a quadrature rule has already been suggested in Stroud and Secrest [8].

In this section we the corresponding Radau quadrature rule

$$\int_0^{+\infty} f(x)x^\alpha \, dx = A_0 f(0) + \sum_{k=1}^{n} A_k f(x_k) + R_{n+1}(f), \tag{3.2}$$

which is exact whenever

$$f(x) = \frac{1}{(1+x)^{\beta+\nu}}, \qquad \nu = 0, 1, \dots, 2n. \tag{3.3}$$

Here the assumptions needed for integrability are

$$\alpha > -1, \quad \beta - \alpha > 1.$$

It is easy to see that the exactness of (3.2) for the rational functions (3.3) is equivalent to

$$\int_0^{+\infty} \frac{x^\alpha}{(1+x)^{\beta+2n}} F(x) \, dx = A_0 F(0) + \sum_{k=1}^{n} A_k \frac{F(x_k)}{(1+x_k)^{\beta+2n-1}}$$

for $F(x) = x^\nu$ $(\nu = 0, 1, \dots, 2n)$.

Using the transformation of variables (2.5) and (2.7), the previous formula becomes

$$\begin{aligned} &\int_{-1}^{1} (1-t)^\alpha (1+t)^{\beta-\alpha-2} (1-t)^\nu (1+t)^{2n-\nu} \, dt \\ &\qquad = 2^{\beta+2n-1} A_0 \delta_{\nu 0} + \frac{1}{2} \sum_{k=1}^{n} (1+t_k)^\beta A_k (1-t_k)^\nu (1+t_k)^{2n-\nu}, \end{aligned}$$

for $\nu = 0, 1, \dots, 2n$. (Here, δ_{ij} is the Kronecker's delta.) In fact, these equalities are conditions for the Radau quadrature formula with Jacobi weight $w^{(\alpha,\beta-\alpha-2)}(t) = (1-t)^\alpha (1+t)^{\beta-\alpha-2}$ on $(-1, 1)$ and fixed node at $t = 1$, i.e.,

$$\int_{-1}^{1} w^{(\alpha,\beta-\alpha-2)}(t) g(t)\, dt = B_0 g(1) + \sum_{k=1}^{n} B_k g(t_k) \quad (g \in \mathcal{P}_{2n}), \tag{3.4}$$

where

$$2^{\beta-1} A_0 = B_0, \quad \frac{1}{2} A_k (1 + t_k)^\beta = B_k.$$

Taking $g(t) = (1 - t)h(t)$, (3.4) reduces to the Gaussian formula

$$\int_{-1}^{1} w^{(\alpha+1,\beta-\alpha-2)}(t) h(t)\, dt = \sum_{k=1}^{n} B_k (1 - t_k) h(t_k),$$

where $h \in \mathcal{P}_{2n-1}$. Thus,

$$B_k(1 - t_k) = \lambda_k^{(n)} = \lambda_k^{(n)}(\alpha + 1, \beta - \alpha - 2),$$
$$t_k = \tau_k^{(n)} = \tau_k^{(n)}(\alpha + 1, \beta - \alpha - 2),$$

where $\tau_k^{(n)}$ and $\lambda_k^{(n)}$ are the n-point Jacobi nodes and weights corresponding to parameters $\alpha + 1$ and $\beta - \alpha - 2$.

Now, we have

$$x_k = \frac{1 - t_k}{1 + t_k}, \quad A_k = \frac{2\lambda_k^{(n)}}{(1 - t_k)(1 + t_k)^\beta} \quad (k = 1, \dots, n).$$

Finally, putting $g(t) = 1$ in (3.4) we find

$$B_0 = \int_{-1}^{1} w^{(\alpha,\beta-\alpha-2)}(t)\, dt - \sum_{k=1}^{n} B_k,$$

i.e.,

$$A_0 = \frac{1}{2^{\beta-1}} \left\{ 2^{\beta-1} \frac{\Gamma(\alpha + 1)\Gamma(\beta - \alpha - 1)}{\Gamma(\beta)} - \sum_{k=1}^{n} \frac{\lambda_k^{(n)}}{1 - t_k} \right\}.$$

In applications we use formula (3.2) in the following form

$$\int_{0}^{+\infty} \frac{x^\alpha}{(1 + x)^\beta} \varphi(x)\, dx = C_0 \varphi(0) + \sum_{k=1}^{n} C_k \varphi(x_k) + R_{n+1}(\varphi), \tag{3.5}$$

where $\varphi(x) = (1+x)^\beta f(x)$ and

$$C_k = \frac{A_k}{(1+x_k)^\beta} = 2^{1-\beta} \frac{\lambda_k^{(n)}}{1-t_k} \qquad (k = 1, \ldots, n)$$

and

$$C_0 = \frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha-1)}{\Gamma(\beta)} - 2^{1-\beta} \sum_{k=1}^{n} \frac{\lambda_k^{(n)}}{1-t_k}.$$

The remainder $R_{n+1}(\varphi)$ can be obtained in terms of the corresponding remainder in Gauss-Jacobi formula.

4. Numerical Example

In this section we test formulas (3.1) and (3.2). Formulas with algebraic degree of precision exist only for small values of n (see conditions (2.4)). All computations were done on the MICROVAX 3400 computer using VAX FORTRAN Ver. 5.3 in D- and Q-arithmetic, with machine precision $\approx 2.76 \times 10^{-17}$ and $\approx 1.93 \times 10^{-34}$, respectively.

Consider integral from [4]

$$\int_0^{+\infty} \frac{x^{1/2} \tanh x}{(1+x)^{12.5}}\, dx = 0.34038896750456956178704228900101 9 \times 10^{-2}.$$

Here $\alpha = 1/2$ and $\beta = 12.5$. Using Gaussian quadrature (3.1) and Radau quadrature (3.2), i.e., (3.5) with $\varphi(x) = \tanh x$, we obtain results with the associated relative errors which are presented in Table 4.1. (Numbers in parentheses indicate decimal exponents.)

Relative errors in quadrature rules (3.1) and (3.2)

n	(3.1)	(3.2)
5	1.38(−6)	8.14(−7)
10	5.08(−11)	2.38(−11)
15	2.63(−15)	7.88(−15)
20	7.98(−18)	1.06(−17)
25	1.94(−19)	8.27(−20)
30	1.06(−21)	1.11(−21)
35	2.10(−23)	6.88(−24)
40	3.27(−25)	3.15(−25)
45	9.93(−27)	4.50(−27)
50	7.46(−29)	1.42(−28)
55	1.08(−29)	7.69(−30)
60	2.71(−31)	8.61(−32)

The rational degree of exactness of the Gaussian formula (3.1) is $2n-1$. Since $\varphi(0)=0$ the quadrature sum of (3.2) has also n nodes as the formula (3.1), but its rational degree of exactness is $2n$, which explains its slightly better behavior in Table 4.1.

REFERENCES

1. and *Quadrature formulas based on rational interpolation.* Math. Comp. **61** (1993), 765–783.
2. and *Methods of Numerical Integration.* Academic Press, New York, 1984.
3. *Gauss-type quadrature rules for rational functions.* ISNM Vol. 112, Birkhäuser Verlag, Basel, 1993.
4. *Quadrature formulae on half-infinite intervals.* BIT **31** (1991), 438–446.
5. *A note on some quadrature formulas for the interval* $(-\infty,\infty)$. Math. Comp. **18** (1964), 313–314.
6. *Quadrature formulas for infinite integrals.* Math. Comp. **16** (1962), 170–175.
7. and *Quadrature formulas for semi-infinite integrals.* Math. Comp. **28** (1974), 499–503.
8. and *Gaussian Quadrature Formulas.* Prentice-Hall, Englewood Cliffs, N.J., 1966.

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