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A SEQUENCE OF KUREPA'S FUNCTIONS

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Dedicated to the memory of Professor Đ. Kurepa

Abstract: In this paper we define and study a sequence of functions $\{K_m(z)\}_{m=-1}^{+\infty}$, where $K_{-1}(z) = \Gamma(z)$ is the gamma function and $K_0(z) = K(z)$ is the Kurepa function [5–6]. We give several properties of $K_m(z)$ including a discussion on their zeros and poles.

Keywords: Gamma function, Kurepa function, left factorial, meromorphic function, zeros, poles.

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1. INTRODUCTION

The left factorial function $z \mapsto K(z)$ was defined by Professor Đ. Kurepa (see [5–6]) in the following way

$$K(z) = \int_0^\infty \frac{t^z - 1}{t - 1}\, e^{-t}\, dt \qquad (\operatorname{Re} z > 0). \tag{1.1}$$

Firstly, he introduced so-called left factorial as

$$!0 = 0, \quad !n = 0! + 1! + \cdots + (n-1)! \quad (n \in \mathbb{N})$$

and then extended it to the right side of the complex plane by (1.1). The function $K(z)$ can be extended analytically to the hole complex plane by

$$K(z) = K(z+1) - \Gamma(z+1), \tag{1.2}$$

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where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \quad (\operatorname{Re} z > 0) \quad \text{and} \quad z\Gamma(z) = \Gamma(z+1).$$

Kurepa [6] proved that $K(z)$ is a meromorphic function with simple poles at the points $z_k = -k$ $(k \in \mathbb{N} \setminus \{2\})$. Graphs of the gamma and Kurepa functions for real values of z are displayed in Fig. 1.1.

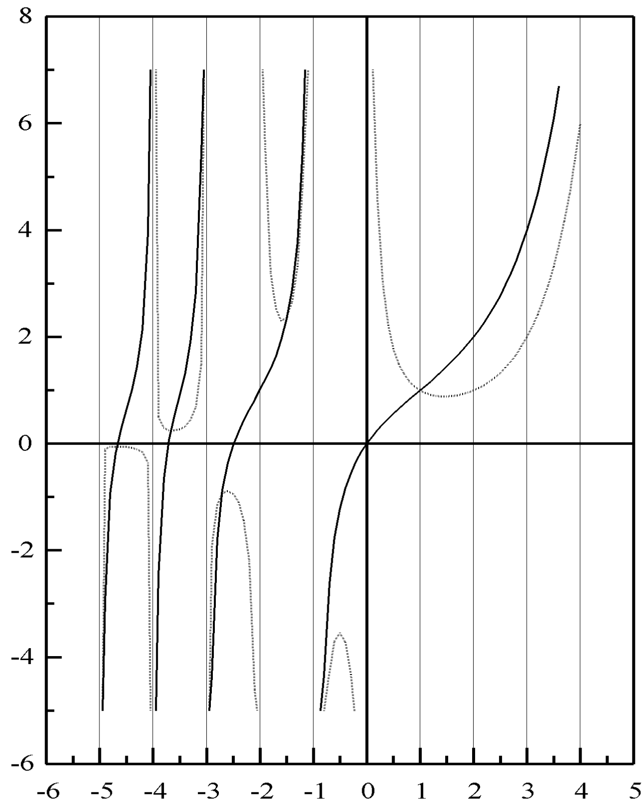

FIG. 1.1: The gamma function $\Gamma(x) = K_{-1}(x)$ (dotted line) and the Kurepa function $K(x) = K_0(x)$ (solid line)

Slavić [10] found the representation

$$K(z) = -\frac{\pi}{e}\cot \pi z + \frac{1}{e}\left(\sum_{n=1}^{\infty} \frac{1}{n!n} + \gamma\right) + \sum_{n=0}^{\infty} \Gamma(z-n),$$

where γ is Euler's constant. These formulas were mentioned also in the book [8]. A number of problems and hypotheses, especially in number theory, were posed by Kurepa and then considered by several mathematicians. For details and a complete list of references see a recent survey written by Ivić and Mijajlović [4].

In this paper we define and study a sequence of complex functions $\{K_m(z)\}_{m=-1}^{+\infty}$, such that the first two terms are the gamma function and the Kurepa function, i.e., $K_{-1}(z) = \Gamma(z)$ and $K_0(z) = K(z)$. In Section 2 we give the basic definition of the sequence $\{K_m(z)\}_{m=-1}^{+\infty}$ and main properties of such functions including their graphs for the real values of z. Zeros and poles of $K_m(z)$ are discussed in Section 3. Numerical calculations, series expansions, as well as some applications of such functions will be given elsewhere.

2. BASIC DEFINITIONS AND PROPERTIES

Definition 2.1. The polynomials $t \mapsto Q_m(t;z)$, $m = -1, 0, 1, 2, \ldots$, are defined by

$$Q_{-1}(t;z) = 0, \quad Q_m(t;z) = \sum_{\nu=0}^{m} \binom{m+z}{\nu} (t-1)^{\nu}. \tag{2.1}$$

For example,

$$\begin{aligned}
&Q_0(t;z) = 1,\\
&Q_1(t;z) = 1 + (z+1)(t-1),\\
&Q_2(t;z) = 1 + (z+2)(t-1) + \frac{1}{2}(z^2+3z+2)(t-1)^2,\\
&Q_3(t;z) = 1 + (z+3)(t-1) + \frac{1}{2}(z^2+5z+6)(t-1)^2\\
&\qquad\qquad\qquad\qquad\qquad + \frac{1}{6}(z^3+6z^2+11z+6)(t-1)^3.
\end{aligned}$$

It is easy to see that the following result holds:

Lemma 2.1. *For every $m \in \mathbb{N}_0$ we have*

$$Q_m(t;z) = Q_{m-1}(t;z+1) + \frac{1}{m!}(z+1)(z+2)\cdots(z+m)(t-1)^m.$$

If we define Δ_z as the standard forward difference operator

$$\Delta_z f(z) = f(z+1) - f(z),$$

then equality (1.2) can be expressed in the form

$$\Delta_z K_0(z) = K_{-1}(z+1),$$

where we put $K(z) = K_0(z)$ and $\Gamma(z) = K_{-1}(z)$. Our goal is here to define the functions $K_m(z)$, $m = 1, 2, \ldots$, such that

$$\Delta_z K_m(z) = K_{m-1}(z+1), \qquad m = 0, 1, \ldots .$$

In our considerations we also use the k-th order difference operator Δ_z^k, defined inductively as

$$\Delta_z^0 f(z) \equiv f(z), \qquad \Delta_z^k f(z) = \Delta_z \left(\Delta_z^{k-1} f(z)\right) \quad (k \in \mathbb{N}).$$

Firstly, we prove the following auxiliary result:

Lemma 2.2. *For every $m \in \mathbb{N}_0$ we have*

$$\Delta_z Q_m(t; z) = (t-1) Q_{m-1}(t; z+1).$$

Proof. According to to previous definition we have

$$\begin{aligned}
\Delta_z Q_m(t; z) &= Q_m(t; z+1) - Q_m(t; z) \\
&= \sum_{\nu=0}^{m} \binom{m+z+1}{\nu} (t-1)^\nu - \sum_{\nu=0}^{m} \binom{m+z}{\nu} (t-1)^\nu \\
&= \sum_{\nu=1}^{m} \binom{m+z}{\nu-1} (t-1)^\nu \\
&= (t-1) \sum_{\nu=0}^{m-1} \binom{m-1+z+1}{\nu} (t-1)^\nu \\
&= (t-1) Q_{m-1}(t; z+1). \quad \square
\end{aligned}$$

Definition 2.2. The sequence $\left\{K_m(z)\right\}_{m=-1}^{+\infty}$ is defined by

$$K_m(z) = \int_0^{+\infty} \frac{t^{z+m} - Q_m(t; z)}{(t-1)^{m+1}}\, e^{-t}\, dt \qquad (\operatorname{Re} z > 0), \tag{2.2}$$

where $Q_m(t; z)$ given by (2.1).

Theorem 2.3. *For* $\operatorname{Re} z > 0$ *we have*

$$\Delta_z K_m(z) \equiv K_m(z+1) - K_m(z) = K_{m-1}(z+1)$$

and

$$\Delta_z^i K_m(z) = K_{m-i}(z+i), \qquad i = 1, 2, \ldots, m+1.$$

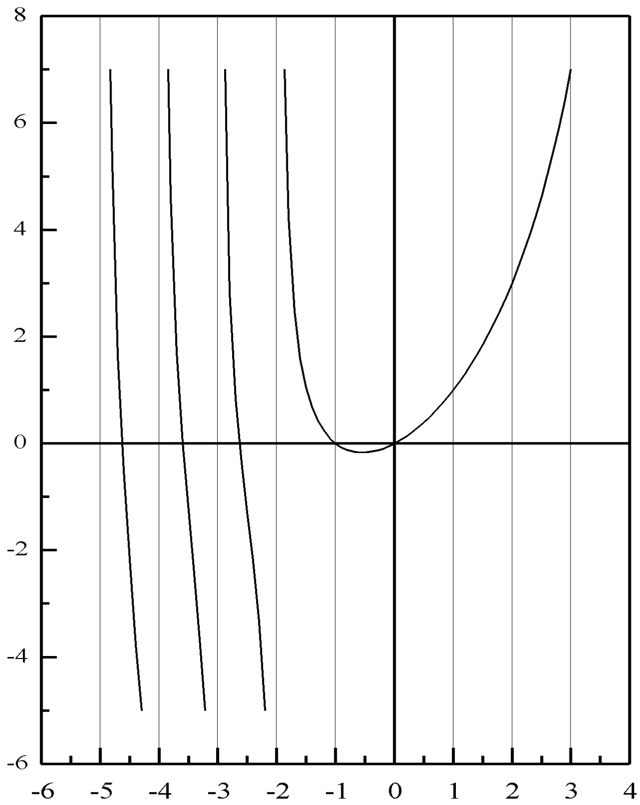

FIG. 2.1: The function $K_1(x)$

Proof. Using Lemma 2.2 we obtain

$$\begin{aligned}\Delta_z\big(t^{z+m} - Q_m(t;z)\big) &= t^{z+1+m} - t^{z+m} - \Delta_z Q_m(t;z)\\ &= (t-1)\big[t^{z+m} - Q_{m-1}(t;z+1)\big].\end{aligned}$$

Then

$$\begin{aligned}\Delta_z K_m(z) &= \int_0^{+\infty} \Delta_z \left[\frac{t^{z+m} - Q_m(t;z)}{(t-1)^{m+1}}\right] e^{-t}\,dt\\ &= \int_0^{+\infty} \frac{t^{z+m} - Q_{m-1}(t;z+1)}{(t-1)^m}\, e^{-t}\,dt\\ &= K_{m-1}(z+1).\end{aligned}$$

Iterating we obtain

$$\Delta_z^i K_m(z) = \Delta_z^{i-1} K_{m-1}(z+1) = \Delta_z^{i-2} K_{m-2}(z+2) = \cdots = K_{m-i}(z+i).$$

For $i = m+1$ we find $\Delta_z^{m+1} K_m(z) = K_{-1}(z+m+1) = \Gamma(z+m+1)$. $\square$

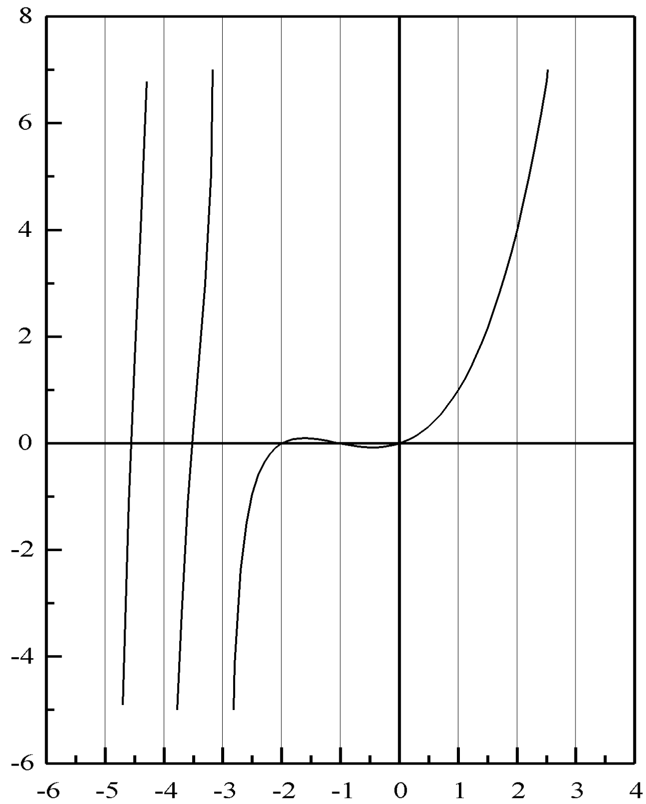

FIG. 2.2: The function $K_2(x)$

It is easy to see that for nonnegative integers the following result holds:

THEOREM 2.4. *For $n, m \in \mathbb{N}_0$ we have*

$$K_m(n) = \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} \sum_{\nu=i}^{n-1} \nu! \binom{m+n}{\nu+m+1}, \qquad K_m(0) = 0.$$

If we put

$$S_\nu = \nu! \sum_{i=0}^{\nu} \frac{(-1)^i}{i!} \qquad (\nu \geq 0),$$

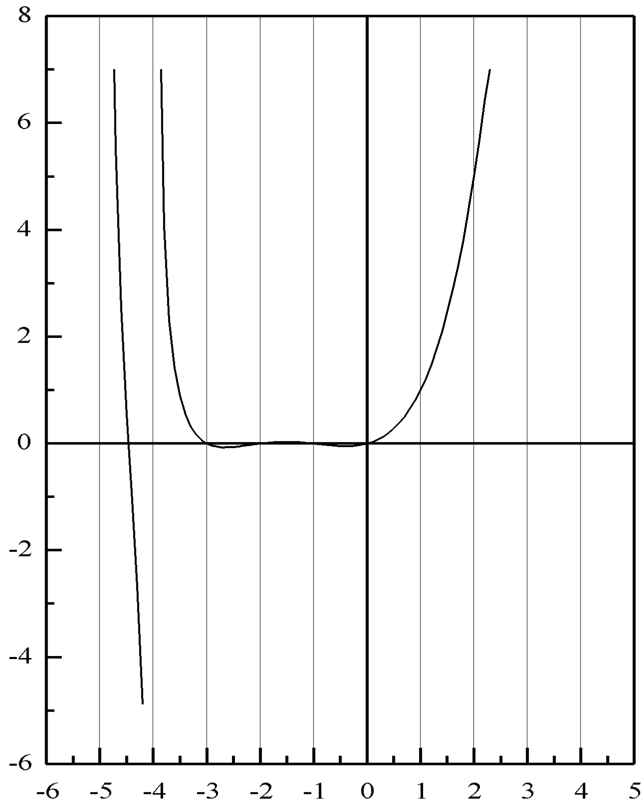

FIG. 2.3: The function $K_3(x)$

i.e., $S_\nu = \nu S_{\nu-1} + (-1)^\nu$ with $S_0 = 1$, then $K_m(n)$ can be expressed in the following form

$$K_m(n) = \sum_{\nu=0}^{n-1} \binom{m+n}{\nu+m+1} S_\nu.$$

Since

$$S_0 = 1, \quad S_1 = 0, \quad S_2 = 1, \quad S_3 = 2, \quad S_4 = 9, \quad S_5 = 44, \quad \text{etc.},$$

we have

$$K_m(0) = 0, \quad K_m(1) = 1, \quad K_m(2) = m+2,$$

$$K_m(3) = \frac{1}{2}(m^2 + 5m + 8),$$

$$K_m(4) = \frac{1}{6}(m^3 + 9m^2 + 32m + 60),$$

etc.

The function $K_m(z)$, $m \in \mathbb{N}$, can be extended analytically to the hole complex plane by

$$K_m(z) = K_m(z+1) - K_{m-1}(z+1). \tag{2.3}$$

Suppose that we have analytic extensions for all functions $K_\nu(z)$, $\nu < m$. Using (2.2) and (2.3) we define $K_m(z)$ at first for z satisfying $\operatorname{Re} z > -1$, then for $\operatorname{Re} z$ such that $\operatorname{Re} z > -2$, etc. In this way we obtain the function $K_m(z)$ in the hole complex plane.

Evaluation of the Kurepa function $K_0(z)$ for some specific z in $(0,1)$, using quadrature formulas with relatively small accuracy, was done by Slavić and the author of this paper (see [6]). Recently, we [9] gave power series expansions of the Kurepa function $K_0(a+z)$, $a \geq 0$, and determined numerical values of their coefficients $b_\nu(a)$ for $a = 0$ and $a = 1$, in high precision (Q-arithmetic with machine precision $\approx 1.93 \times 10^{-34}$). Using an asymptotic behaviour of $b_\nu(a)$, when $\nu \to \infty$, we gave a transformation of series with much faster convergence. Also, we obtained the Chebyshev expansions for $K_0(1+z)$ and $1/K_0(1+z)$. For similar expansions of the gamma function see e.g. Davis [2], Luke [7], Fransén and Wrigge [3], and Bohman and Fröberg [1].

Graphs of functions $K_m(x)$, $m = 1, 2, 3$, for real values of x are displayed in figures 2.1, 2.2, and 2.3, respectively.

3. ZEROS AND POLES

Poles of $K_m(z)$ are in the points $z_n^{(m)} = -n$, $n = m+1, m+2, \ldots$, except the point $z_2^{(0)}$ when $K_0(z_2^{(0)}) = K_0(-2) = 1$.

The poles of gamma function $\Gamma(z) = K_{-1}(z)$ are $z_n^{(-1)} = -n$, $n = 0, 1, \ldots$, with the corresponding residues

$$\operatorname*{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{n!} \qquad (n = 0, 1, \ldots).$$

Putting

$$R_n^{(m)} = \operatorname*{Res}_{z=-n} K_m(z) \qquad (n \geq m+1),$$

we can prove the following result:

THEOREM 3.2. *For every $n \geq m+3$ we have that*

$$R_n^{(m)} = R_{m+2}^{(m)} - \sum_{\nu=m+2}^{n-1} R_\nu^{(m-1)},$$

where

$$R_{m+1}^{(m)} = (-1)^{m+1}, \qquad R_{m+2}^{(m)} = m(-1)^{m+1}.$$

For $m = 0$ Theorem 4 reduces to Kurepa's result [6, §6]:

$$R_1^{(0)} = \operatorname*{Res}_{z=-1} K_0(z) == 1,$$

$$R_n^{(0)} = \operatorname*{Res}_{z=-n} K_0(z) = -\sum_{\nu=2}^{n-1} \frac{(-1)^\nu}{\nu!} .$$

We note that $z = -2$ is not a pole of $K_0(z)$ ($R_2^{(0)} = 0$).

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