Integral Transforms and Special Functions
Vol. 14, No. 4, August 2003, pp. 349–361

SOME FINITE SUMMATION FORMULAS INVOLVING MULTIVARIABLE HYPERGEOMETRIC POLYNOMIALS

LAZAR N. DJORDJEVIĆ[a], DRAGAN M. MILOŠEVIĆ[a], GRADIMIR V. MILOVANOVIĆ[b] and H. M. SRIVASTAVA[c,*]

[a]*Department of Computer Sciences;* [b]*Department of Mathematics, Faculty of Electronic Engineering, University of Niš, YU-18000 Niš, Serbia, Yugoslavia;*
[c]*Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada*

(Received 7 May 2002; In final form 29 May 2002)

The main purpose of this paper is to present a family of finite summation formulas and to apply it in order to derive several functional relationships involving various multivariable hypergeometric polynomials and the Gauss hypergeometric function. A number of special and limit cases of these functional relationships are also considered.

Keywords: Finite summation formulas; Multivariable hypergeometric polynomials; Functional relationships; Gauss hypergeometric function; Orthogonal and biorthogonal polynomials; Multinomial theorem; (Srivastava–Daoust) multivariable hypergeometric functions; Lauricella functions; Carlitz–Srivastava polynomials; Laguerre polynomials; Multivariable Appell polynomials; Jacobi polynomials

MSC 2000: Primary: 33C50, 33C65; Secondary: 33C45

1 INTRODUCTION AND PRELIMINARIES

In terms of a bounded multiple sequence $\{\Omega(k_1, \ldots, k_r)\}$ of essentially arbitrary (real or complex) parameters, let

$$\Theta_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}(x_1, \ldots, x_r) := \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} (-n_1)_{m_1 k_1} \cdots (-n_r)_{m_r k_r}\, \Omega(k_1, \ldots, k_r) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!} \tag{1.1}$$

$$(n_j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ m_j \in \mathbb{N};\ j = 1, \ldots, r),$$

* Corresponding author. E-mail: hmsri@uvvm.uvic.ca

ISSN 1065-2469 print; ISSN 1476-8291 online
DOI: 10.1080/1065246031000081643

where $[\kappa]$ denotes the greatest integer in $\kappa \in \mathbb{R}$ and $(\lambda)_k$ is the Pochhammer symbol (or, more precisely, the *shifted* factorial, since $(1)_k = k!$ $(k \in \mathbb{N}_0)$) defined, in terms of Gamma functions, by

$$(\lambda)_k := \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0;\ \lambda \neq 0) \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{1.2}$$

$\mathbb{N}$ being the set of *positive* integers.

For different choices of the multiple sequence $\{\Omega(k_1, \ldots, k_r)\}$ and with

$$m_j = 1 \quad (j = 1, \ldots, r),$$

the multivariable polynomials [cf. Eq. (1.1)]

$$\Theta^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}(x_1, \ldots, x_r)$$

would readily yield, as special cases, various classes of orthogonal and biorthogonal polynomials associated with hypergeometric functions of two and more variables [see, for details, Refs. 1, 6, 7, 12–18].

Motivated essentially by these and sundry other occurrences of special multivariable hypergeometric polynomials in the mathematical and physical sciences literature, we first propose to derive here a family of finite summation formulas involving the polynomials defined by (1.1) and then show how this general result can be applied in order to deduce several functional relationships between various multivariable hypergeometric polynomials and the Gauss hypergeometric function which corresponds to the familiar special case

$$p - 1 = q = 1$$

of the generalized hypergeometric ${}_pF_q$ function with p numerator and q denominator parameters, defined by

$${}_pF_q[(\alpha_p); (\beta_q); z] = {}_pF_q\left[\begin{matrix} (\alpha_p); \\ \\ (\beta_q); \end{matrix} \quad z\right] := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{z^k}{k!} \tag{1.3}$$

$$(p, q \in \mathbb{N}_0;\ p \leq q+1;\ p \leq q \text{ and } |z| < \infty;\ p = q+1 \text{ and } |z| < 1;$$
$$p = q+1,\ |z| = 1, \text{ and } \Re(\Xi) > 0),$$

where (*and throughout this paper*) we find it to be convenient to abbreviate the p-parameter array:

$$\alpha_1, \ldots, \alpha_p \quad (p \in \mathbb{N})$$

by (α_p), the array being empty when $p = 0$, with similar interpretations for (β_q), etc., and

$$\Xi := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{q} \alpha_j \quad (\beta_j \notin \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}).$$

2 FINITE SUMMATION FORMULAS

We begin by recalling the multinomial theorem in the form [cf., *e.g.*, Ref. 3, p. 13, Eq. 2.3(9)]:

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} x_1^{n_1} \cdots x_r^{n_r} = (x_1 + \cdots + x_r)^n \tag{2.1}$$
$$(n, n_j \in \mathbb{N}_0; j = 1, \ldots, r; r \in \mathbb{N}\backslash\{1\}),$$

where, *and in what follows*,

$$\binom{n}{n_1, \ldots, n_r} := \frac{n!}{n_1! \cdots n_r!}. \tag{2.2}$$

Since

$$(-n_j)_{m_j k_j} = (-1)^{m_j k_j} \frac{n_j!}{(n_j - m_j k_j)!} \quad \left(0 \leq k_j \leq \left[\frac{n_j}{m_j}\right]; j = 1, \ldots, r\right), \tag{2.3}$$

by virtue of the definition (1.2), we can make use of the multinomial theorem (2.1) in conjunction with the definition (1.1) to show that

$$\begin{aligned}
&\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} \Theta_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}(x_1, \ldots, x_r) t_1^{n_1} \cdots t_r^{n_r} \\
&= T^n \sum_{k_1,\ldots,k_r=0}^{m_1k_1+\cdots+m_rk_r \leq n} (-n)_{m_1k_1+\cdots+m_rk_r} \Omega(k_1, \ldots, k_r) \frac{\{x_1(t_1/T)^{m_1}\}^{k_1}}{k_1!} \cdots \frac{\{x_r(t_r/T)^{m_r}\}^{k_r}}{k_r!}
\end{aligned} \tag{2.4}$$

$$(T := t_1 + \cdots + t_r; n, n_j \in \mathbb{N}_0; m_j \in \mathbb{N}; j = 1, \ldots, r).$$

With a view to applying the general finite summation formula (2.4) to the following familiar *special* case of the (Srivastava–Daoust) generalized Lauricella functions, defined by [cf., *e.g.*, Ref. 9, p. 38, Eq. 1.4(24)]

$$\begin{aligned}
&F_{q:q_1;\ldots;q_r}^{p:p_1;\ldots;p_r}\left[\begin{matrix} (\alpha_p)\colon (\gamma'_{p_1}); \ldots; (\gamma^{(r)}_{p_r}); \\ \\ (\beta_q)\colon (\delta'_{q_1}); \ldots; (\delta^{(r)}_{q_r}); \end{matrix} \; z_1, \ldots, z_r\right] \\
&:= \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_{k_1+\cdots+k_r} \prod_{j=1}^{p_1} (\gamma'_j)_{k_1} \cdots \prod_{j=1}^{p_r} (\gamma^{(r)}_j)_{k_r}}{\prod_{j=1}^{q} (\beta_j)_{k_1+\cdots+k_r} \prod_{j=1}^{q_1} (\delta'_j)_{k_1} \cdots \prod_{j=1}^{q_r} (\delta^{(r)}_j)_{k_r}} \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!},
\end{aligned} \tag{2.5}$$

whenever the multiple hypergeometric series in (2.5) converges or terminates, we conveniently set

$$m_1 = \cdots = m_r = 1,$$

and we find from (2.4) that

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1,\ldots,n_r} F^{p:p_1+1;\ldots;p_r+1}_{q:q_1+1;\ldots;q_r+1} \left[\begin{matrix} (\alpha_p){:}-n_1,(\gamma'_{p_1});\ldots;-n_r,(\gamma^{(r)}_{p_r}); \\ \\ (\beta_q){:}\quad (\delta'_{q_1});\ldots;\quad (\delta^{(r)}_{q_r}); \end{matrix} x_1,\ldots,x_r\right] t_1^{n_1}\cdots t_r^{n_r}$$
$$= T^n F^{p+1:p_1;\ldots;p_r}_{q:q_1;\ldots;q_r}\left[\begin{matrix} -n,(\alpha_p){:}\,(\gamma'_{p_1});\ldots;(\gamma^{(r)}_{p_r}); \\ \\ (\beta_q){:}\,(\delta'_{q_1});\ldots;(\delta^{(r)}_{q_r}); \end{matrix} \frac{x_1t_1}{T},\ldots,\frac{x_rt_r}{T}\right] \tag{2.6}$$
$$(T := t_1+\cdots+t_r;\; n, n_j \in \mathbb{N};\; j=1,\ldots,r).$$

Next, by appealing appropriately to the multiple series identity [cf. Ref. 8; see also Ref. 9, p. 39]:

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \omega(n_1+\cdots+n_r)(\lambda_1)_{n_1}\cdots(\lambda_r)_{n_r}\frac{z^{n_1}}{n_1!}\cdots\frac{z^{n_r}}{n_r!} = \sum_{n=0}^{\infty}\omega(n)(\lambda_1+\cdots+\lambda_r)_n\frac{z^n}{n!} \tag{2.7}$$

and its multivariable hypergeometric form:

$$F^{p:1;\ldots;1}_{q:0;\ldots;0}\left[\begin{matrix} (\alpha_p){:}\,\lambda_1;\ldots;\lambda_r; \\ \\ (\beta_q){:}\,—;\ldots;—; \end{matrix} z,\ldots,z\right] = {}_{p+1}F_q\left[\begin{matrix} (\alpha_p),\lambda_1+\cdots+\lambda_r; \\ \\ (\beta_q); \end{matrix} z\right], \tag{2.8}$$

the second members of (2.4) and (2.6) can be simplified considerably in the special cases:

$$\Omega(k_1,\ldots,k_r) = (\lambda_1)_{k_1}\cdots(\lambda_r)_{k_r}\,\omega(k_1+\cdots+k_r), \quad m_j = m, \quad \text{and} \quad x_j = \left(\frac{\tau}{t_j}\right)^m \tag{2.9}$$

$$(m \in \mathbb{N};\; k_j \in \mathbb{N}_0;\; \lambda_j \in \mathbb{C};\; j=1,\ldots,r)$$

and

$$p_j - 1 = q_j = 0 \;\; (\gamma^{(j)}_{p_j} = \lambda_j) \quad \text{and} \quad x_jt_j = \tau \quad (j=1,\ldots,r), \tag{2.10}$$

respectively. We thus find from (2.4) that

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} \Phi_{n_1,\ldots,n_r}^{\lambda_1,\ldots,\lambda_r}\left(m; \frac{\tau}{t_1}, \ldots, \frac{\tau}{t_r}\right) t_1^{n_1} \cdots t_r^{n_r}$$

$$= T^n \sum_{k=0}^{[n/m]} (-n)_{mk} (\lambda_1 + \cdots + \lambda_r)_k \, \omega(k) \frac{(\tau/T)^{mk}}{k!} \tag{2.11}$$

$$(T := t_1 + \cdots + t_r;\ m \in \mathbb{N};\ n, n_j \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C};\ j = 1, \ldots, r),$$

where [cf. Eq. (1.1)]

$$\Phi_{n_1,\ldots,n_r}^{\lambda_1,\ldots,\lambda_r}(m; x_1, \ldots, x_r) := \sum_{k_1=0}^{[n_1/m]} \cdots \sum_{k_r=0}^{[n_r/m]} (-n_1)_{mk_1} (\lambda_1)_{k_1} \cdots (-n_r)_{mk_r} (\lambda_r)_{k_r}$$

$$\cdot\, \omega(k_1 + \cdots + k_r) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!} \tag{2.12}$$

$$(n_j \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C};\ j = 1, \ldots, r;\ m \in \mathbb{N})$$

in terms of a bounded sequence $\{\omega(n)\}_{n=0}^{\infty}$ of essentially arbitrary (real or complex) parameters.

Furthermore, under the constraints given by (2.10), the finite summation formula (2.6) similarly yields

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} F_{q:0;\ldots;0}^{p:2;\ldots;2} \left[\begin{array}{l} (\alpha_p){:}-n_1, \lambda_1; \ldots; -n_r, \lambda_r; \\ \qquad\qquad\qquad\qquad\qquad \dfrac{\tau}{t_1}, \ldots, \dfrac{\tau}{t_r} \\ (\beta_q){:} \qquad —; \ldots; \qquad —; \end{array} \right] t_1^{n_1} \cdots t_r^{n_r}$$

$$= T^n \, {}_{p+2}F_q \left[\begin{array}{l} -n, \lambda_1 + \cdots + \lambda_r, (\alpha_p); \\ \qquad\qquad\qquad\qquad \dfrac{\tau}{T} \\ \qquad\qquad (\beta_q); \end{array} \right] \tag{2.13}$$

$$(T := t_1 + \cdots + t_r;\ n, n_j \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C};\ j = 1, \ldots, r).$$

3 APPLICATIONS TO LAURICELLA POLYNOMIALS IN SEVERAL VARIABLES

The various finite summation formulas, which we presented in the preceding section, can indeed be applied in order to derive the corresponding results for many of the orthogonal and biorthogonal multivariable hypergeometric polynomials which were referred to in Section 1. We leave the details involved in the derivations of these consequences of our results as an exercise for the interested reader. We turn instead to some multivariable hypergeometric polynomials which are associated, in particular, with the Lauricella functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$, and $F_D^{(r)}$ of r variables, where [cf. Ref. 1, p. 114; see also Ref. 9, p. 33]

$$F_A^{(r)}[a, b_1, \ldots, b_r; c_1, \ldots, c_r; z_1, \ldots, z_r] = F_{0:1;\ldots;1}^{1:1;\ldots;1}\left[\begin{array}{l} a: b_1; \ldots; b_r; \\ \qquad\qquad\qquad z_1, \ldots, z_r \\ —: c_1; \ldots; c_r; \end{array}\right]$$

$$:= \sum_{k_1, \ldots, k_r=0}^{\infty} \frac{(a)_{k_1+\cdots+k_r}(b_1)_{k_1}\cdots(b_r)_{k_r}}{(c_1)_{k_1}\cdots(c_r)_{k_r}} \frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!} \tag{3.1}$$

$$(|z_1| + \cdots + |z_r| < 1;\ c_j \notin \mathbb{Z}_0^-;\ j = 1, \ldots, r),$$

$$F_B^{(r)}[a_1, \ldots, a_r, b_1, \ldots, b_r; c; z_1, \ldots, z_r] = F_{1:0;\ldots;0}^{0:2;\ldots;2}\left[\begin{array}{l} —: a_1, b_1; \ldots; a_r, b_r; \\ \qquad\qquad\qquad z_1, \ldots, z_r \\ c: \ ——; \ldots; ——; \end{array}\right]$$

$$:= \sum_{k_1, \ldots, k_r=0}^{\infty} \frac{(a_1)_{k_1}\cdots(a_r)_{k_r}(b_1)_{k_1}\cdots(b_r)_{k_r}}{(c)_{k_1+\cdots+k_r}} \frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!} \tag{3.2}$$

$$(\max\{|z_1|, \ldots, |z_r|\} < 1;\ c \notin \mathbb{Z}_0^-),$$

$$F_C^{(r)}[a, b; c_1, \ldots, c_r; z_1, \ldots, z_r] = F_{0:1;\ldots;1}^{2:0;\ldots;0}\left[\begin{array}{l} a, b: —; \ldots; —; \\ \qquad\qquad\qquad z_1, \ldots, z_r \\ ——: c_1; \ldots; c_r; \end{array}\right]$$

$$:= \sum_{k_1, \ldots, k_r=0}^{\infty} \frac{(a)_{k_1+\cdots+k_r}(b)_{k_1+\cdots+k_r}}{(c_1)_{k_1}\cdots(c_r)_{k_r}} \frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!} \tag{3.3}$$

$$(|z_1|^{1/2} + \cdots + |z_r|^{1/2} < 1;\ c_j \notin \mathbb{Z}_0^-;\ j = 1, \ldots, r),$$

and

$$F_D^{(r)}[a, b_1, \ldots, b_r; c; z_1, \ldots, z_r] = F_{1:0;\ldots;0}^{1:1;\ldots;1}\left[\begin{array}{l} a: b_1; \ldots; b_r; \\ \qquad\qquad\qquad z_1, \ldots, z_r \\ c: —; \ldots; —; \end{array}\right]$$

$$:= \sum_{k_1, \ldots, k_r=0}^{\infty} \frac{(a)_{k_1+\cdots+k_r}(b_1)_{k_1}\cdots(b_r)_{k_r}}{(c)_{k_1+\cdots+k_r}} \frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!} \tag{3.4}$$

$$(\max\{|z_1|, \ldots, |z_r|\} < 1;\ c \notin \mathbb{Z}_0^-).$$

First of all, in its special case when

$$p - 1 = q = 0 \ \ (\alpha_1 = a) \quad \text{and} \quad p_j = q_j - 1 = 0 \ \left(\delta_j^{(j)} = c_j;\ j = 1, \ldots, r\right),$$

our summation formula (2.6) yields

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} F_A^{(r)}[a, -n_1, \ldots, -n_r; c_1, \ldots, c_r; x_1, \ldots, x_r] t_1^{n_1} \cdots t_r^{n_r}$$
$$= T^n \, F_C^{(r)}\left[-n, a; c_1, \ldots, c_r; \frac{x_1 t_1}{T}, \ldots, \frac{x_r t_r}{T}\right] \tag{3.5}$$
$$(T := t_1 + \cdots + t_r;\ n, n_j \in \mathbb{N}_0;\ c_j \notin \mathbb{Z}_0^-;\ j = 1, \ldots, r).$$

In the case of the Lauricella hypergeometric polynomials associated with $F_B^{(r)}$, (2.13) with

$$p = q - 1 = 0 \ \ (\beta_1 = c) \quad \text{and} \quad \lambda_j \mapsto b_j \quad (j = 1, \ldots, r)$$

leads us immediately to the functional relationship:

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} F_B^{(r)}\left[-n_1, \ldots, -n_r, b_1, \ldots, b_r; c; \frac{\tau}{t_1}, \ldots, \frac{\tau}{t_r}\right] t_1^{n_1} \cdots t_r^{n_r}$$
$$= T^n \, {}_2F_1\left(-n, b_1 + \cdots + b_r; c; \frac{\tau}{T}\right) \tag{3.6}$$
$$(T := t_1 + \cdots + t_r;\ n, n_j \in \mathbb{N}_0;\ j = 1, \ldots, r;\ c \notin \mathbb{Z}_0^-).$$

For the Lauricella hypergeometric polynomials associated with $F_D^{(r)}$, we similarly find from (2.6) with

$$p = q = 1 \ \ (\alpha_1 = a;\ \beta_1 = c) \quad \text{and} \quad p_j = q_j = 0 \quad (j = 1, \ldots, r)$$

that

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} F_D^{(r)}[a, -n_1, \ldots, -n_r; c; x_1, \ldots, x_r] t_1^{n_1} \cdots t_r^{n_r}$$
$$= T^n \, {}_2F_1\left(-n, a; c; \frac{x_1 t_1 + \cdots + x_r t_r}{T}\right) \tag{3.7}$$
$$(T := t_1 + \cdots + t_r;\ n, n_j \in \mathbb{N}_0;\ j = 1, \ldots, r;\ c \notin \mathbb{Z}_0^-).$$

Since

$$F_A^{(2)} = F_2, \quad F_B^{(2)} = F_3, \quad F_C^{(2)} = F_4, \quad \text{and} \quad F_D^{(2)} = F_1, \tag{3.8}$$

each of the functional relationships (3.5), (3.6), and (3.7) (with $r = 2$) can immediately be rewritten in terms of one or the other of Appell's hypergeometric functions F_1, F_2, F_3, and F_4 of two variables [cf. Ref. 1, p. 14; see also Ref. 9, pp. 22–23].

Next, for the second set of the Carlitz–Srivastava polynomials defined by [2, Part II, p. 143, Eq. (27)]

$$\mathcal{F}_D^{(r)}[(\alpha\colon \vartheta_j)\colon (-n_j, m_j); (\gamma\colon \varphi_j); x_1, \ldots, x_r] \\ := \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} (-n_1)_{m_1 k_1} \cdots (-n_r)_{m_r k_r} \frac{(\alpha)_{k_1\vartheta_1+\cdots+k_r\vartheta_r}}{(\gamma)_{k_1\varphi_1+\cdots+k_r\varphi_r}} \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!} \tag{3.9}$$

$$(n_j \in \mathbb{N}_0;\, m_j \in \mathbb{N};\, \vartheta_j, \varphi_j \in \mathbb{R}^+;\, j = 1, \ldots, r;\, \gamma \notin \mathbb{Z}_0^-),$$

the general result (2.4) readily yields

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} \mathcal{F}_D^{(r)}[(\alpha\colon \vartheta_j)\colon (-n_j, m_j); (\gamma\colon \varphi_j); x_1, \ldots, x_r] t_1^{n_1} \cdots t_r^{n_r} \\ = T^n \sum_{k_1,\ldots,k_r=0}^{m_1k_1+\cdots+m_rk_r \le n} (-n)_{m_1k_1+\cdots+m_rk_r} \frac{(\alpha)_{k_1\vartheta_1+\cdots+k_r\vartheta_r}}{(\gamma)_{k_1\varphi_1+\cdots+k_r\varphi_r}} \\ \cdot \frac{\{x_1(t_1/T)^{m_1}\}^{k_1}}{k_1!} \cdots \frac{\{x_r(t_r/T)^{m_r}\}^{k_r}}{k_r!} \tag{3.10}$$

$$(T := t_1 + \cdots + t_r;\, n, n_j \in \mathbb{N}_0;\, m_j \in \mathbb{N};\, j = 1, \ldots, r;\, \gamma \notin \mathbb{Z}_0^-).$$

In the particular case when

$$m_j = m, \quad \vartheta_j = \rho, \quad \text{and} \quad \varphi_j = \sigma \quad (m, \rho, \sigma \in \mathbb{N}), \tag{3.11}$$

(3.10) would reduce to the form:

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} \mathcal{F}_D^{(r)}[(\alpha\colon \rho)\colon (-n_j, m); (\gamma\colon \sigma); x_1, \ldots, x_r] t_1^{n_1} \cdots t_r^{n_r} \\ = T^n \, {}_{m+\rho}F_\sigma \left[\begin{array}{c} \Delta(m;-n), \Delta(\rho; \alpha); \\ \\ \Delta(\sigma; \gamma); \end{array} \frac{m^m \rho^\rho}{\sigma^\sigma} \left\{ x_1 \left(\frac{t_1}{T}\right)^m + \cdots + x_r \left(\frac{t_r}{T}\right)^m \right\} \right] \tag{3.12}$$

$$(T := t_1 + \cdots + t_r;\, n, n_j \in \mathbb{N}_0;\, j = 1, \ldots, r;\, m, \rho, \sigma \in \mathbb{N};\, \gamma \notin \mathbb{Z}_0^-),$$

where $\Delta(m; \lambda)$ abbreviates the array of m parameters:

$$\frac{\lambda}{m}, \frac{\lambda+1}{m}, \ldots, \frac{\lambda+m-1}{m} \quad (m \in \mathbb{N}).$$

For $m = \rho = \sigma = 1$, (3.12) would obviously correspond to the functional relationship (3.7).

Lastly, for Erdélyi's *multivariable* extension of the classical Laguerre polynomials defined by [cf., *e.g*., Ref. 2, Part II, p. 144, Eq. (29)]

$$L^{(\alpha)}_{n_1,\ldots,n_r}(x_1,\ldots,x_r) := \frac{(\alpha+1)_{n_1+\cdots+n_r}}{n_1!\cdots n_r!} \cdot F^{0:1;\ldots;1}_{1:0;\ldots;0}\left[\begin{matrix} \text{——}: & -n_1;\ldots; & -n_r; & \\ & & & x_1,\ldots,x_r \\ \alpha+1: & \text{——};\ldots; & \text{——}; & \end{matrix}\right], \tag{3.13}$$

it is easily seen from (2.6) with

$$p = q - 1 = 0 \ \ (\beta_1 = \alpha + 1) \quad \text{and} \quad p_j = q_j = 0 \quad (j = 1,\ldots,r)$$

that

$$\sum_{n_1+\cdots+n_r=n} L^{(\alpha)}_{n_1,\ldots,n_r}(x_1,\ldots,x_r) t_1^{n_1}\cdots t_r^{n_r} = \binom{\alpha+n}{n} T^n\, {}_1F_1\left(-n;\, \alpha+1;\, \frac{x_1t_1+\cdots+x_rt_r}{T}\right) \tag{3.14}$$

$$(T := t_1+\cdots+t_r;\ n, n_j \in \mathbb{N}_0; j = 1,\ldots,r;\ \alpha \in \mathbb{Z}^- := \mathbb{Z}_0^- \backslash \{0\}).$$

Clearly, since [cf. Eqs. (3.4) and (3.13)]

$$L^{(\alpha)}_{n_1,\ldots,n_r}(x_1,\ldots,x_r) = \lim_{|\lambda|\to\infty}\left\{F_D^{(r)}\left[\lambda, -n_1,\ldots,-n_r;\, \alpha+1;\, \frac{x_1}{\lambda},\ldots,\frac{x_r}{\lambda}\right]\right\}, \tag{3.15}$$

our last functional relationship (3.14) can be deduced as a limit case of (3.7) when

$$c = \alpha + 1, \quad x_j \mapsto \frac{x_j}{a} \ \ (j = 1,\ldots,r), \quad \text{and} \quad |a| \to \infty.$$

4 FURTHER EXTENSIONS AND CONSEQUENCES

The multinomial theorem (2.1), which provided one of the tools used in our present investigation, is itself a limit case of the following well-known multiple sum [Ref. 3, p. 13, Eq. 2.3(5)]:

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1,\ldots,n_r} (\lambda_1)_{n_1}\cdots(\lambda_r)_{n_r} = (\lambda_1+\cdots+\lambda_r)_n \tag{4.1}$$

$$(n, n_j \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C}; j = 1,\ldots,r),$$

which, for $r = 2$, is equivalent to the Chu-Vandermonde summation theorem:

$$\sum_{k=0}^{n} \binom{\lambda}{n-k} \binom{\mu}{k} = \binom{\lambda+\mu}{n} \quad (n \in \mathbb{N}_0;\ \lambda,\ \mu \in \mathbb{C}). \tag{4.2}$$

In fact, since

$$\lim_{|\lambda|\to\infty} \left\{ (\lambda)_n \left(\frac{z}{\lambda}\right)^n \right\} = z^n = \lim_{|\mu|\to\infty} \left\{ \frac{(\mu z)^n}{(\mu)_n} \right\} \quad (n \in \mathbb{N}_0;\ |z| < \infty), \tag{4.3}$$

upon setting $\lambda_j = \lambda x_j$ $(j = 1, \ldots, r)$, dividing both sides by λ^n, and letting $|\lambda| \to \infty$, (4.1) yields the multinomial theorem (2.1).

If we apply the general result (4.1) in place of the multinomial theorem (2.1), we find from the definition (1.1) that

$$\begin{aligned}
&\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} (\lambda_1)_{n_1} \cdots (\lambda_r)_{n_r} \Theta_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}(x_1, \ldots, x_r) \\
&= (\lambda_1 + \cdots + \lambda_r)_n \sum_{k_1,\ldots,k_r=0}^{m_1k_1+\cdots+m_rk_r \le n} \frac{(-n)_{m_1k_1+\cdots+m_rk_r} (\lambda_1)_{m_1k_1} \cdots (\lambda_r)_{m_rk_r}}{(\lambda_1 + \cdots + \lambda_r)_{m_1k_1+\cdots+m_rk_r}} \\
&\cdot \Omega(k_1, \ldots, k_r) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!} \quad (n,\ n_j \in \mathbb{N}_0;\ m_j \in \mathbb{N};\ \lambda_j \in \mathbb{C};\ j = 1, \ldots, r),
\end{aligned} \tag{4.4}$$

which provides a seemingly interesting extension of (2.4).

Finally, in order to give a conveniently simple hypergeometric form of (4.4), we set

$$\Lambda := \lambda_1 + \cdots + \lambda_r \quad \text{and} \quad m_j = 1 \quad (j = 1, \ldots, r),$$

and choose the multiple sequence $\{\Omega(k_1, \ldots, k_r)\}$ as in the definition (2.5). We thus obtain

$$\begin{aligned}
&\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} (\lambda_1)_{n_1} \cdots (\lambda_r)_{n_r} \\
&\cdot F_{q:\ q_1;\ldots;q_r}^{p:p_1+1;\ldots;p_r+1} \left[\begin{array}{l} (\alpha_p){:}-n_1,\ (\gamma'_{p_1});\ldots;\ -n_r,\ (\gamma^{(r)}_{p_r}); \\ (\beta_q){:}\qquad (\delta'_{q_1});\ldots;\qquad (\delta^{(r)}_{q_r}); \end{array} x_1, \ldots, x_r \right] \\
&= (\Lambda)_n\, F_{q+1:\ q_1;\ldots;q_r}^{p+1:p_1+1;\ldots;p_r+1} \left[\begin{array}{l} -n,\ (\alpha_p){:}\ \lambda_1,\ (\gamma'_{p_1});\ldots;\ \lambda_r,\ (\gamma^{(r)}_{p_r}); \\ \Lambda,\ (\beta_q){:}\qquad (\delta'_{q_1});\ldots;\qquad (\delta^{(r)}_{q_r}); \end{array} x_1, \ldots, x_r \right]
\end{aligned} \tag{4.5}$$

$$(\Lambda := \lambda_1 + \cdots + \lambda_r;\ n,\ n_j \in \mathbb{N}_0;\ j = 1, \ldots, r),$$

provided that each side of (4.5) exists. In its special case when

$$p = q = 0 \quad \text{and} \quad p_j = q_j = 1 \quad (\gamma^{(j)}_{p_j} = \mu_j;\ \delta^{(j)}_{p_j} = \nu_j;\ j = 1, \ldots, r),$$

the hypergeometric summation formula (4.5) reduces immediately to the elegant form:

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1,\ldots,n_r} (\lambda_1)_{n_1}\cdots(\lambda_r)_{n_r}\cdot {}_2F_1(-n_1,\mu_1;\nu_1;x_1)\cdots{}_2F_1(-n_r,\mu_r;\nu_r;x_r)$$
$$= (\Lambda)_n\, F^{1:2;\ldots;2}_{1:1;\ldots;1}\left[\begin{matrix} -n\colon \lambda_1,\mu_1;\ldots;\lambda_r,\mu_r; \\ \Lambda\colon \quad \nu_1;\ldots;\quad \nu_r; \end{matrix}\; x_1,\ldots,x_r\right] \tag{4.6}$$
$$(\Lambda := \lambda_1+\cdots+\lambda_r;\, n, n_j\in\mathbb{N}_0;\, \Lambda, \nu_j\notin\mathbb{Z}_0^-;\, j=1,\ldots,r).$$

A *further* special case of (4.6) when

$$\nu_j = \lambda_j \quad (j=1,\ldots,r)$$

was proven, in a markedly different way, by Toscano [11, p. 241, Eq. (11.4)]; indeed, in this special case, the right-hand side of (4.6) would involve the Lauricella function $F_D^{(r)}$ defined by (3.4).

For the Lauricella hypergeometric function $F_A^{(r)}$ defined by (3.1), our summation formula (4.5) with

$$p-1=q=0 \;\; (\alpha_1=\alpha) \quad \text{and} \quad p_j=q_j-1=0 \;\; (\delta^{(j)}_{q_j}=\mu_j;\, j=1,\ldots,r)$$

would readily yield

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1,\ldots,n_r} (\lambda_1)_{n_1}\cdots(\lambda_r)_{n_r}\cdot F_A^{(r)}[\alpha,-n_1,\ldots,-n_r;\mu_1,\ldots,\mu_r;x_1,\ldots,x_r]$$
$$= (\Lambda)_n\, F^{2:1;\ldots;1}_{1:1;\ldots;1}\left[\begin{matrix} -n,\alpha\colon \lambda_1;\ldots;\lambda_r; \\ \Lambda\colon \mu_1;\ldots;\mu_r; \end{matrix}\; x_1,\ldots,x_r\right] \tag{4.7}$$
$$(\Lambda := \lambda_1+\cdots+\lambda_r;\, n, n_j\in\mathbb{N}_0;\, \Lambda, \lambda_j\notin\mathbb{Z}_0^-;\, j=1,\ldots,r),$$

which, in the special case when

$$\mu_j = \lambda_j \quad (j=1,\ldots,r),$$

reduces at once to the form:

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1,\ldots,n_r} (\lambda_1)_{n_1}\cdots(\lambda_r)_{n_r}\cdot F_A^{(r)}[\alpha,-n_1,\ldots,-n_r;\lambda_1,\ldots,\lambda_r;x_1,\ldots,x_r]$$
$$= (\Lambda)_n\, {}_2F_1(-n,\alpha;\Lambda;x_1+\cdots+x_r) \tag{4.8}$$
$$(\Lambda := \lambda_1+\cdots+\lambda_r;\, n, n_j\in\mathbb{N}_0;\, \Lambda, \lambda_j\notin\mathbb{Z}_0^-;\, j=1,\ldots,r).$$

Lastly, we consider the basic multivariable Appell polynomials

$$E_{n_1,\ldots,n_r}(a, b_1,\ldots,b_r;x_1,\ldots,x_r)$$

defined by [cf. Ref. 4, p. 26; see also Ref. 5]

$$E_{n_1,\ldots,n_r}(a, b_1, \ldots, b_r; x_1, \ldots, x_r) := (-1)^n \frac{(b_1)_{n_1} \cdots (b_r)_{n_r}}{(a + b_1 + \cdots + b_r + n)_n}$$

$$\cdot F_A^{(r)}[a + b_1 + \cdots + b_r + n, -n_1, \ldots, -n_r; b_1, \ldots, b_r; x_1, \ldots, x_r] \tag{4.9}$$

$$(n := n_1 + \cdots + n_r; n, n_j \in \mathbb{N}_0; j = 1, \ldots, r)$$

or, equivalently, by

$$E_{n_1,\ldots,n_r}(a, b_1, \ldots, b_r; x_1, \ldots, x_r) := x_1^{n_1} \cdots x_r^{n_r}$$

$$\cdot F_B^{(r)}\Bigg[-n_1, \ldots, -n_r, 1 - b_1 - n_1, \ldots, 1 - b_r - n_r;$$

$$1 - a - b_1 - \cdots - b_r - 2n; \frac{1}{x_1}, \ldots, \frac{1}{x_r}\Bigg] \tag{4.10}$$

$$(n := n_1 + \cdots + n_r; n, n_j \in \mathbb{N}_0; j = 1, \ldots, r).$$

Indeed these basic multivariable Appell polynomials are *orthogonal* over the simplex

$$\mathcal{T}_r := \{(x_1, \ldots, x_r) : x_1 + \cdots + x_r \leq 1 \ (x_j \geq 0; j = 1, \ldots, r)\} \tag{4.11}$$

with the weight function

$$w(x_1, \ldots, x_r) := x_1^{b_1 - 1} \cdots x_r^{b_r - 1}(1 - x_1 - \cdots - x_r)^a. \tag{4.12}$$

Making use of the definition (4.10), it is not difficult to deduce the following summation formula from the functional relationship (3.6):

$$\sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} E_{n_1,\ldots,n_r}\left(a + 1, b_1 - n_1 + 1, \ldots, b_r - n_r + 1; \frac{t_1}{\tau}, \ldots, \frac{t_r}{\tau}\right)$$

$$= \binom{a + B + r + n}{n}^{-1} P_n^{(a+r, B-n)}\left(\frac{2T}{\tau} - 1\right) \tag{4.13}$$

$$(n, n_j \in \mathbb{N}_0; j = 1, \ldots, r; B := b_1 + \cdots + b_r; T := t_1 + \cdots + t_r),$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the classical Jacobi polynomials defined by [cf., *e.g.*, Ref. 10, p. 68]

$$P_n^{(\alpha,\beta)}(x) := \sum_{k=0}^{n} \binom{n + \alpha}{n - k} \binom{n + \beta}{k} \left(\frac{x - 1}{2}\right)^k \left(\frac{x + 1}{2}\right)^{n-k} \tag{4.14}$$

or, in terms of the Gauss hypergeometric function, by

$$P_n^{(\alpha,\beta)}(x) := \binom{\alpha + \beta + 2n}{n} \left(\frac{x + 1}{2}\right)^n {}_2F_1\left(-n, -\beta - n; -\alpha - \beta - 2n; \frac{2}{x + 1}\right). \tag{4.15}$$

Many other interesting corollaries and consequences of our main summation formulas can be deduced in a similar manner.

Acknowledgements

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353 (H. M. Srivastava) and by the *Serbian Ministry of Science, Technology, and Development* under Grant 2000: *Applied Orthogonal Systems, Constructive Approximation and Numerical Methods* (L. N. Djordjević and G. V. Milovanović).

References

[1] Appell, P. and Kampé de Fériet, J. (1926). *Fonctions Hypergéométriques et Hypersphériques; Polynômes d'Hermite.* Gauthier-Villars, Paris.

[2] Carlitz, L. and Srivastava, H. M. (1976). Some hypergeometric polynomials associated with the Lauricella function F_D of several variables: I and II. *Mat. Vesnik*, **13**(28), 41–47 and 143–152.

[3] Carlson, B. C. (1977). *Special Functions of Applied Mathematics.* Academic Press, New York, San Francisco, and London.

[4] Djordjević, L. N. (1977). On a new class of cubature formulas. *Doctoral thesis*, University of Niš, Niš (Serbian).

[5] Djordjević, L. N., Savić, D. S. and Djordjević, Dj. R. (2000). Addition formulas for multivariate hypergeometric polynomials. *Novi Sad J. Math.*, **30**, 31–39.

[6] Kalnins, E. G., Miller, W., Jr. and Tratnik, M. V. (1991). Families of orthogonal and biorthogonal polynomials on the *N*-sphere. *SIAM J. Math. Anal.*, **22**, 272–294.

[7] Lam, C. S. and Tratnik, M. V. (1985). Conformally invariant operator-product expansions of any number of operators of arbitrary spin. *Canad. J. Phys.*, **63**, 1427–1437.

[8] Panda, R. (1974). Some multiple series transformations. *Jñānābha Sect. A*, **4**, 165–168.

[9] Srivastava, H. M. and Karlsson, P. W. (1985). *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto.

[10] Szegö, G. (1975). *Orthogonal Polynomials*, 4th ed., Vol. 23. American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, Rhode Island.

[11] Toscano, L. (1972). Sui polinomi ipergeometrici a più variabili del tipo F_D di Lauricella. *Matematiche* (*Catania*), **27**, 219–250.

[12] Tratnik, M. V. (1988). Multivariable continuous Hahn polynomials. *J. Math. Phys.*, **29**, 1529–1534.

[13] Tratnik, M. V. (1989). Multivariable biorthogonal Hahn polynomials. *J. Math. Phys.*, **30**, 627–634.

[14] Tratnik, M. V. (1989). Multivariable Wilson polynomials. *J. Math. Phys.*, **30**, 2001–2011.

[15] Tratnik, M. V. (1989). Multivariable Meixner, Krawtchouk, and Meixner-Pollaczek polynomials, *J. Math. Phys.*, **30**, 2740–2749.

[16] Tratnik, M. V. (1990). Multivariable biorthogonal continuous-discrete Wilson and Racah polynomials. *J. Math. Phys.*, **31**, 1559–1575.

[17] Tratnik, M. V. (1991). Some multivariable orthogonal polynomials of the Askey tableau – continuous families. *J. Math. Phys.*, **32**, 2065–2073.

[18] Tratnik, M. V. (1991). Some multivariable orthogonal polynomials of the Askey tableau – discrete families. *J. Math. Phys.*, **32**, 2337–2342.