ON MARKOV-DUFFIN-SCHAEFFER INEQUALITIES

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Abstract. In this survey paper we consider extremal problems for algebraic and trigonometric polynomials in the uniform norm. We study the classical inequalities of the brothers Markov, several variants of Bernstein inequality, as well as refinements of these inequalities.

1. Introduction

The first result in the theory of extremal problems for polynomials was connected with some investigations of the well-known Russian chemist Mendeleev [25]. In mathematical terms, Mendeleev's problem was: *If $P(t)$ is an arbitrary quadratic polynomial defined on an interval $[a,b]$, with $\max_{t\in[a,b]} P(t) - \min_{t\in[a,b]} P(t) = L$, how large $P'(t)$ can be on $[a,b]$?* This problem can also be stated for polynomials of degree n in the following form: *If $P(t)$ is an arbitrary polynomial of degree n and $|P(t)| \le 1$ on $[-1,1]$, how large can $|P'(t)|$ be on $[-1,1]$?* Such a problem was solved by A. A. Markov [22]. His brother V. A. Markov [23] investigated the upper bound of $|P^{(k)}(t)|$, where $k \le n$. An analogue of Markov's theorem for the unit disk in the complex plane instead of for the interval $[-1,1]$ was investigated by Bernstein [2]. Markov's and Bernstein's inequalities are fundamental for the proof of many inverse theorems in polynomial approximation theory (see Dzyadyk [13], Lorentz [20], Meinardus [24], Ivanov [18]). There are many results on Markov's and Bernstein's theorems and their generalizations in various norms and restricted classes of

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polynomials. Several monographs and papers have been published in this area (cf. Boas [6], Durand [12], Mamedhanov [21], Milovanović [26], Milovanović, Mitrinović and Rassias [27−28], Rahman and Schmeisser [32], Rassias [34], Voronovskaja [47]).

In this paper we will give a short account of the classical results of the brothers Markov and Bernstein and refinements of their inequalities.

2. Classical Inequalities

We begin by considering the following extremal problem: *Let $\mathcal{P}_n$ be the set of all algebraic polynomials P ($\not\equiv 0$) of degree at most n. For a given norm $\|.\|$, determine the best constant $A_{n,k}$ such that*

$$\|P^{(k)}\| \le A_{n,k}\|P\| \qquad (P \in \mathcal{P}_n), \tag{2.1}$$

i.e.,

$$A_{n,k} = \sup_{P\in\mathcal{P}_n} \frac{\|P^{(k)}\|}{\|P\|}. \tag{2.2}$$

In 1889, A. A. Markov [22] solved this extremal problem in the uniform norm on $[-1, 1]$,

$$\|f\| = \|f\|_\infty = \max_{-1\le t\le 1} |f(t)|. \tag{2.3}$$

Theorem 2.1. *Let $k = 1$. In the uniform norm* (2.3)*, we have $A_{n,1} = n^2$. The equality in* (2.1) *holds only at ± 1 and only when $P(t) = cT_n(t)$, where T_n is the Chebyshev polynomial of the first kind of degree n and c is an arbitrary constant.*

The best possible constant for the k-th derivative was found by V. A. Markov [23] in 1892. A German version of his remarkable paper was published in 1916.

Theorem 2.2. *In the uniform norm* (2.3)*, for each $k = 1, \ldots, n$, we have $A_{n,k} = T_n^{(k)}(1)$. The extremal polynomial is T_n.*

Without loss of generality, we can suppose that $\|P\|_\infty = 1$.

Markov's proof of this result is based on a complicated variational method. He investigated a more general problem: *If $\lambda_0, \lambda_1, \ldots, \lambda_n$ are given constants and $P(t) = \sum_{\nu=0}^{n} a_\nu t^\nu$ satisfies the condition $\|P\|_\infty = 1$, what is the precise bound for the linear form $\sum_{\nu=0}^{n} a_\nu\lambda_\nu$?* By suitably choosing the constants λ_ν the linear form can be made equal to any derivative of $P(t)$ at any preassigned point.

For fixed k suppose that $t \mapsto \bar{P}(t)$ is an extremal polynomial, i.e., there exists $t^* \in [-1, 1]$ such that

$$|\bar{P}^{(k)}(t^*)| = \sup\left\{|P^{(k)}(t^*)| \,:\, P \in \mathcal{P}_n,\ \|P\|_\infty = 1\right\}.$$

It is easily shown that such a polynomial exists. Markov's variational approach was to show that $|\bar{P}^{(k)}(t^*)|$ must be equal to 1 at either n or $n+1$ different points

$-1 < \tau_1 < \tau_2 < \cdots < \tau_m < 1$ $(m = n$ or $m = n+1)$. In the latter case $\pm\bar{P}(t)$ is the Chebyshev polynomial $T_n(t)$, whose derivatives satisfy

$$(2.4) \qquad |P^{(k)}(t)| \le \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdot 5\cdot \ldots \cdot (2k-1)} \quad (-1 \le t \le 1).$$

In the former case it can be proved that the extremal polynomial satisfies a differential equation of the form

$$1 - y^2 = \frac{(1-t^2)(t-b)(t-c)}{n^2(t-a)^2}\,(y')^2,$$

where a, b, c are real constants which depend upon one parameter. Markov thus proved that derivatives of this class of polynomials satisfy also (2.4).

A simple proof of Theorem 2.2 was given by Bernstein [4] and an elementary proof by Mohr [29].

Another type of these inequalities goes back to Bernstein [2] in 1912, who considered the following problem: *Let $P(z)$ be a polynomial of degree n and $|P(z)| \le 1$ in the unit disk $|z| \le 1$. Determine how large can $|P'(z)|$ be for $|z| \le 1$.*

If we take $\|f\| = \max\limits_{|z|\le 1} |f(z)|$, this problem can be reduced to the inequality (2.1) for $k = 1$. Thus, Bernstein's theorem gives:

Theorem 2.3. *$A_{n,1} = n$. The extremal polynomial is $P(z) = cz^n$, $c =$ const.*

Bernstein's theorem 2.3 can be stated in several different forms. One of them is known as Bernstein's theorem for trigonometric polynomials:

Theorem 2.4. *Let $T(\theta)$ be a trigonometric polynomial of degree n and $|T(\theta)| \le 1$, then*

$$(2.5) \qquad |T'(\theta)| \le n.$$

The equality holds for $T(\theta) = \gamma \sin n(\theta - \theta_0)$, where $|\gamma| = 1$.

A standard form of Bernstein's theorem is as follows:

Theorem 2.5. *Let $P \in \mathcal{P}_n$ and $|P(t)| \le 1$ $(-1 \le t \le 1)$, then*

$$(2.6) \qquad |P'(t)| \le \frac{n}{\sqrt{1-t^2}}, \qquad -1 < t < 1.$$

The equality is attained at the points $t = t_\nu = \cos\frac{(2\nu-1)\pi}{2n}$, $1 \le \nu \le n$, if and only if $P(t) = \gamma T_n(t)$, where $|\gamma| = 1$.

This result was proved by Bernstein [2] at the same time as Theorem 2.4, except that in (2.5) he had $2n$ in place of n. Inequality (2.6) in the present form first appeared in print in a paper of Fekete [15] who attributes the proof to Fejér [14]. Bernstein [3] attributes the proof to E. Landau.

Bernstein's proof of Theorem 2.4 was based on a variational method. Simpler proofs of this theorem have been obtained by M. Riesz [36], F. Riesz [35], and de la Vallée Poussin [41]. Now, we give an adapted proof of de la Vallée Poussin (cf. Lorentz [20]).

A similar method can be used to show that under the condition of Theorem 2.4 a sharper inequality

$$n^2 T(\theta)^2 + T'(\theta)^2 \le n^2 \tag{2.7}$$

holds, where $T(\theta)$ is assumed to be real. In the general case in which the polynomial $T(\theta)$ is a complex-valued function we cannot say that the sum of the absolute magnitudes of the two terms on the left in (2.7) is less than n^2. This is shown by the example $T(\theta) = e^{in\theta}$. Inequality (2.7) was first explicitly stated by van der Corput and Schaake [7–8], although it is implicit in an earlier inequality due to Szegö (cf. Schaeffer [37]).

Iterating the Bernstein's inequality (2.5) we can obtain the corresponding inequality for the k-th derivative

$$|T^{(k)}(\theta)| \le n^k.$$

This result can be stated in the following form:

Theorem 2.6. *Let $T(\theta)$ be a trigonometric polynomial of degree n. Then*

$$\max_\theta |T^{(k)}(\theta)| \le n^k \max_\theta |T(\theta)| \quad (k = 1, 2, \ldots), \tag{2.8}$$

with equality only if $T(\theta) = a\cos n\theta + b \sin n\theta$, where a and b are constants.

Using the forward-difference operator Δ_h, defined by

$$\Delta_h f(\theta) = f(\theta + h) - f(\theta) \qquad (h > 0),$$

Stečkin [40] gave an interesting analogue of the Bernstein inequality (2.8):

Theorem 2.7. *Let $0 < h < 2\pi/n$. Then*

$$\max_\theta |T^{(k)}(\theta)| \le \left(\frac{n}{2 \sin \frac{nh}{2}} \right)^k \max_\theta |\Delta_h^k T(\theta)| \quad (k = 1, 2, \ldots), \tag{2.9}$$

where

$$\Delta_h^k T(\theta) = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} T(\theta + \nu h).$$

The equality in (2.9) is attained if and only if

$$T(\theta) = a\cos n\theta + b \sin n\theta + c,$$

where a, b, c are constants.

For $k = 1$, Stečkin inequality (2.9) reduces to

$$|T'(\theta)| \le \frac{n}{2 \sin \frac{nh}{2}} \max_\theta |T(\theta + h) - T(\theta)| \qquad (0 < nh < 2\pi).$$

If $h=\pi/n$, inequality (2.9) becomes

$$\max_{\theta}|T^{(k)}(\theta)|\le\left(\frac{n}{2}\right)^{k}\max_{\theta}|\Delta^{k}_{\pi/n}T(\theta)|. \tag{2.10}$$

This result was also obtained by S. N. Nikol'skiĭ at the same time, but with a different method.

Since

$$\max_{\theta}|\Delta^{k}_{\pi/n}T(\theta)|\le 2^{k}\max_{\theta}|T(\theta)|,$$

using (2.10) we obtain the Bernstein's inequality (2.8).

Bernstein's result (2.6) can also be interpreted in the following way: *If a polynomial $P(t)$ of degree n satisfies the inequality*

$$|P|\le 1\equiv\left(T_n(t)^2+S_n(t)^2\right)^{1/2}\qquad(-1\le t\le 1),$$

where $T_n(t)=\cos(n\arccos t)$ *and* $S_n(t)=\sin(n\arccos t)$, *then*

$$|P'(t)|\le\frac{n}{\sqrt{1-t^2}}=\left|\ \frac{d}{dt}\big(T_n(t)+iS_n(t)\big)\ \right|\qquad(-1<t<1).$$

Similarly, Schaeffer and Duffin [38] proved the corresponding inequality for the k-th derivative of a polynomial P of degree n for which $|P|\le 1$ on $[-1,1]$. Namely,

$$|P^{(k)}(t)|\le\left|\ \frac{d^k}{dt^k}\big(T_n(t)+iS_n(t)\big)\ \right|\qquad(-1<t<1),$$

i.e.,

$$|P^{(k)}(t)|^2\le M_k(t)\qquad(-1<t<1;\ k=1,\ldots,n),$$

where

$$M_k(t)=\left(\frac{d^k}{dt^k}\cos n\theta\right)^2+\left(\frac{d^k}{dt^k}\sin n\theta\right)^2,\qquad t=\cos\theta.$$

Using this result, Schaeffer and Duffin [38] obtained an elegant proof of V. A. Markov's inequality. Namely, for $P\in\mathcal{P}_n$ and such that $\|P\|_\infty\le 1$, they proved that

$$\|P^{(k)}\|_\infty\le\frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1)} \tag{2.11}$$

for $k=1,2,\ldots,n$. The equality can occur only at $t=\pm1$ and here only if $P(t)=\gamma T_n(t)$, where $|\gamma|=1$.

Now, we give an interesting result of Schur [39].

Theorem 2.8. *Let* $Q\in\mathcal{P}_{n-1}$ *and*

$$|Q(t)|\le M(1-t^2)^{-1/2}\qquad(-1<t<1),$$

then

$$|Q(t)|\le Mn\qquad(-1\le t\le 1).$$

Using Bernstein's and Schur's theorems 2.5 and 2.8, we can give a simple proof of Markov's theorem 2.1. Namely, if $P \in \mathcal{P}_n$ and $|P(t)| \le 1 \quad (-1 \le t \le 1)$, then $|P'(t)| \le n(1-t^2)^{-1/2}$ and finally, by Theorem 2.8, we obtain $|P'(t)| \le n^2$ $(-1 \le t \le 1)$, i.e. $A_{n,1} = n^2$.

We can ask how large can $|P'(t)|$ be, for a given t, when $|P(t)| \le 1$ on $[-1,1]$? Let this maximum be $M_n(t)$. It is easy to see that the function M_n is even, i.e. $M_n(-t) = M_n(t)$.

The problem of finding $M_n(t)$ was stated by Markov himself, and solved for $n = 2$ and $n = 3$. So he determined that

$$M_2(t) = \begin{cases} \dfrac{1}{1-t}, & 0 \le t \le \frac{1}{2}, \\ 4t, & \frac{1}{2} \le t \le 1, \end{cases}$$

and

$$M_3(t) = \begin{cases} 3(1-4t^2), & t \in [t_0, t_1], \\ \dfrac{7\sqrt{7}+10}{9(1+t)}, & t \in [t_1, t_2], \\ \dfrac{16t^3}{(1-9t^2)(1-t^2)}, & t \in [t_2, t_3], \\ \dfrac{7\sqrt{7}-10}{9(1-t)}, & t \in [t_3, t_4], \\ 3(4t^2-1), & t \in [t_4, t_5], \end{cases}$$

where $t_0 = 0$, $t_1 = \frac{1}{6}(\sqrt{7}-2) \cong 0.1076$, $t_2 = \frac{1}{9}(2\sqrt{7}-1) \cong 0.4768$, $t_3 = \frac{1}{9}(2\sqrt{7}+1) \cong 0.6991$, $t_4 = \frac{1}{6}(\sqrt{7}+2) \cong 0.7743$, and $t_5 = 1$.

The determination of $M_n(t)$, $n \ge 4$, is very complicated and it can be given by a technique of Voronovskaja (see [47]). Using the same method, Gusev [16] found the corresponding function $M_{n,k}(t)$ in the inequality

$$|P^{(k)}(t)| \le M_{n,k}(t), \qquad 1 < k < n.$$

Instead of the condition $|P(t)| \le 1$ on $[-1,1]$, Bernstein [5] used a more general condition

$$|P(t)| \le \sqrt{H(t)} \qquad (-1 \le t \le 1), \tag{2.12}$$

where H is an arbitrary positive polynomial on $[-1,1]$ of degree s. If $n \ge s/2$, the polynomial H can be uniquely represented in the form

$$H(t) = M_n(t)^2 + (1-t^2)N_{n-1}(t)^2, \tag{2.13}$$

where M_n and N_{n-1} are polynomials of degree n and $n-1$, respectively, such that all their zeros belong to $(-1,1)$ satisfying an interlacing property, and $M_n(1) > 0$, $N_{n-1}(1) > 0$.

Theorem 2.9. *Let $P \in \mathcal{P}_n$. Under the condition* (2.12)*, where H is given by* (2.13)*, the inequality*

$$
|P'(t)| \le \Big| \frac{d}{dt}\big(M_n(t) + i\sqrt{1-t^2}N_{n-1}(t)\big) \Big| \qquad (-1 < t < 1). \tag{2.14}
$$

holds. The equality is attained in (2.14) *for $P(t) = \gamma M_n(t)$, where $|\gamma| = 1$.*

Videnskiĭ [44] proved the corresponding inequality for the k-th derivative of P:

Theorem 2.10. *Let $P \in \mathcal{P}_n$. Under the condition* (2.12)*, where H is given by* (2.13)*, the following inequality*

$$
|P^{(k)}(t)| \le \Big| \frac{d^k}{dt^k}\big(M_n(t) + i\sqrt{1-t^2}N_{n-1}(t)\big) \Big| \qquad (-1 < t < 1), \tag{2.15}
$$

holds, for $k = 1, \dots, n$. The equality is attained in (2.15) *for $P(t) = \gamma M_n(t)$, where $|\gamma| = 1$.*

Several inequalities of this type were given by Videnskiĭ [42–46], and others.

3. Markov-Duffin-Schaeffer Inequalities

As we mentioned in the previous section, Schaeffer and Duffin [38] in 1938 found a shorter proof of Theorem 2.2 due to V. Markov, proving the inequality (2.11). Duffin and Schaeffer [11] proved also that for this inequality to hold it is only necessary to assume that $|P(t)| \le 1$ at $n+1$ selected points in $[-1, 1]$. This inequality is a consequence of a more general inequality concerning Lagrange interpolation.

Let $z \mapsto Q(z)$ be a polynomial of degree n with n distinct real zeros ζ_ν lying in the real interval (a, b). Then

$$
Q(z) = A \prod_{\nu=1}^{n} (z - \zeta_\nu) \qquad (A \ne 0), \tag{3.1}
$$

which implies

$$
Q'(z) = \sum_{\nu=1}^{n} \frac{Q(z)}{z - \zeta_\nu}.
$$

Duffin and Schaeffer [11] studied the problem of determining the maximum of the non-analytic function

$$
t \mapsto F(t) = \sum_{\nu=1}^{n} \Big| \frac{Q(t)}{t - \zeta_\nu} \Big|,
$$

when t belongs to the interval (a, b). In fact, by making the hypothesis

$$
|Q(t + is)| \le |Q(b + is)| \qquad (a \le t \le b,\ -\infty < s < +\infty), \tag{3.2}
$$

they obtained that

$$
F(t) \le F(b) = |Q'(b)| \qquad (a \le t \le b),
$$

which is equivalent to the following statement: *Suppose that $z \mapsto P(z)$ is an arbitrary polynomial of degree at most n, such that*

$$|P'(t)| \le |Q'(t)| \quad \text{wherever} \quad Q(t) = 0 \tag{3.3}$$

and let (3.2) *be satisfied. Then*

$$|P'(t)| \le |Q'(b)| \qquad (a \le t \le b).$$

The equivalence is a consequence of the Lagrange interpolation formula. Namely,

$$P'(z) = \sum_{\nu=1}^{n} \frac{P'(\zeta_\nu)}{Q'(\zeta_\nu)} \frac{Q(z)}{z - \zeta_\nu}.$$

Thus, if t is an arbitrary point in (a, b) then

$$\max_{a<t<b} |P'(t)| = \sum_{\nu=1}^{n} \left| \frac{Q(t)}{t - \zeta_\nu} \right|.$$

Duffin and Schaeffer [11] proved the following results:

Theorem 3.1. *Let Q be a polynomial of degree n with n distinct real zeros ζ_ν $(\nu = 1, \dots, n)$, given by* (3.1)*, and let $P \in \mathcal{P}_n$ satisfies* (3.3)*. Then*

$$|P^{(k)}(z)| \le |Q^{(k)}(z)|$$

at the zeros of $Q^{(k-1)}(z)$, where $k = 1, \dots, n$.

Theorem 3.2. *Let Q be a polynomial of degree n with n distinct real zeros ζ_ν $(\nu = 1, \dots, n)$ which lie to the left of the point b on the real axis and suppose that in a strip of the complex plane, the polynomial Q satisfies the inequality* (3.2)*. If $P \in \mathcal{P}_n$ is a polynomial with real coefficients satisfying* (3.3)*, then the derivatives of P and Q must satisfy*

$$|P^{(k)}(t + is)| \le |Q^{(k)}(b + is)| \qquad (k = 1, \dots, n)$$

in the strip $a \le t \le b$, $-\infty < s < +\infty$.

Theorem 3.3. *The Chebyshev polynomials T_n satisfy the inequality*

$$|T_n(t + is)| \le |T_n(1 + is)| \qquad (-1 \le t \le 1, \ -\infty < s < +\infty).$$

According to Schur's result given by Theorem 2.8, if a polynomial R of degree at most $n - 1$ satisfies

$$(1 - t^2)^{1/2} |R(t)| \le 1 \qquad (-1 \le t \le 1), \tag{3.4}$$

then

$$|R(t)| \le n \qquad (-1 \le t \le 1). \tag{3.5}$$

Using Theorem 3.2, with $Q(t) = T_n(t)$, where T_n is the Chebyshev polynomial of degree n, we can see that if (3.4) is satisfied at the n points

$$t = \tau_\nu = \cos((2\nu - 1)\pi/(2n)) \qquad (\nu = 1, \dots, n),$$

then (3.5) is still true. Indeed, when $T_n(t) = 0$, $T_n'(t) = n(1 - t^2)^{-1/2}$ we may write (in Theorem 3.2) $P'(t) = nR(t)$. Therefore,

$$|nR(t)| \le T_n'(1) = n^2 \qquad (-1 \le t \le 1).$$

Theorem 3.4. *Let $P \in \mathcal{P}_n$ be a polynomial with real coefficients such that*

$$|P(\cos \nu\pi/n)| \le 1 \qquad (\nu = 0, 1, \ldots, n), \tag{3.6}$$

then

$$|P^{(k)}(t+is)| \le |T_n^{(k)}(1+is)| \quad (-1 \le t \le 1,\ -\infty < s < +\infty)$$

for $k = 1, 2, \ldots, n$. The equality holds only if $P(z) = \pm T_n(z)$.

In this theorem the restriction that P have real coefficients is essential. For points on the real axis the same estimate holds for $|P^{(k)}(z)|$ even if P has complex coefficients.

Under the same condition for P, it can be proved that

$$|P^{(k)}(t+is)| \le |T_n^{(k)}(b+is)| \qquad (k = 0, 1, \ldots, n),$$

when t, s, b are real and $|t| \le b,\ b \ge 1$ (cf. Zinger [48–50] and Voronovskaja [47]).

In connection with the above inequalities, Kemperman [19] obtained the following results:

Theorem 3.5. *Let $P \in \mathcal{P}_n$ be a polynomial with real coefficients such that $|P(t)| \le 1$ for $-1 \le t \le 1$. Then, for all complex numbers z,*

$$|P(z)|^2 + (1 - P(\tilde{z})^2) \le T_n(z^*)^2, \tag{3.7}$$

where z^* denotes the associated point on $[1, +\infty)$, where the ellipse through z with foci -1 and $+1$ intersects the positive real axis, i.e. $z^* = \cosh v = \frac{1}{2}(|z+1|+|z-1|)$, and $\tilde{z} = \cos u$ on $[-1, 1]$, where the hyperbola through $z = \cos(u+iv)$ with foci -1 and $+1$ intersects the real axis.

The equality holds in (3.7) when $z \in [-1, +1]$. Otherwise, it can only hold when $P(z) = \pm T_n(z)$, for all complex z.

Theorem 3.6. *Under the same conditions on P as in the above theorem, the following inequality*

$$|P'(z)| \le T_n'(z^*)$$

holds, for all complex z. The equality occurs only when $P(z) = \pm T_n(z)$ and $z = \pm z^*$.

Theorem 3.7. *Let $P \in \mathcal{P}_n$ be a polynomial with real coefficients satisfying (3.6), then*

$$|P^{(k)}(z)| \le |T_n^{(k)}(z)| \le T_n^{(k)}(z^*),$$

for $k = 0, 1, \ldots, n$, provided $|z| \ge \beta_{n,k}$, where $\beta_{n,k}$ denotes the largest zero of $T_n^{(k-1)}(z)$.

Regarding the Markov's inequality, Duffin and Schaeffer [11] proved the following refinement:

Theorem 3.8. *Let $P \in \mathcal{P}_n$ such that* (3.6) *holds, then inequality*

$$\|P^{(k)}\|_\infty \le T_n^{(k)}(1) = \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdot 5\cdot \ldots \cdot (2k-1)} \tag{3.8}$$

is satisfied for $k = 1, \ldots, n$. The equality occurs only if $P(t) = \gamma T_n(t)$, where $|\gamma| = 1$.

An interesting question appears whether or not there are $n+1$ other points in the interval $(-1,1)$ satisfying the same property. Duffin and Schaeffer [11] gave a negative answer to this question. In fact, they showed that if E is any closed subset of $(-1,1)$ which does not contain all the points $\tau_\nu = \cos(\nu\pi/n)$, then there is a polynomial $P \in \mathcal{P}_n$ which is bounded by 1 in E, but (3.8) is not satisfied.

The above refined inequalities of Markov are known as *Markov-Duffin-Schaeffer inequalities*. These inequalities have interesting applications in numerical analysis (cf. Berman [1] and Haverkamp [17]). In particular one can use these inequalities to construct optimally stable formulae for numerical differentiation of functions f satisfying $\|f\|_\infty < +\infty$.

The proof of Duffin and Schaeffer is based on continuation of the functions to the complex plane and on using various techniques from complex analysis. Duffin and Karlovitz [9] have given an elementary variational approach that, simultaneously, proves the Markov inequalities and the Duffin-Schaeffer refinement as well as generalizes the result to a larger class of functions. They have given explicit bounds on the first and second derivatives. Their method can be extended to higher derivatives of polynomials.

Recently, Rahman and Schmeisser [33] considered such a refinement of the inequalities which is connected with a question that was posed by the late Professor Paul Turán in 1970 at a conference on *Constructive Function Theory* held in Varna, Bulgaria (cf. Pierre and Rahman [30–31]).

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