On polynomials orthogonal on a circular arc*

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Abstract: Polynomials $\{\pi_k^R\}$ orthogonal on a circular arc with respect to the complex inner product $(f,g)=\int_\varphi^{\pi-\varphi} f_1(\theta)g_1(\theta)w_1(\theta)\,d\theta$, where $\varphi\in(0,\pi/2)$, and for $f(z)$ the function $f_1(\theta)$ is defined by $f_1(\theta)=f(-iR+e^{i\theta}\sqrt{R^2+1})$, $R=\tan\varphi$, have been introduced by M. G. de Bruin [1]. In this paper the functions of the second kind, as well as the corresponding associated polynomials, are introduced. Some recurrence relations and identities of Christoffel-Darboux type are proved. Also, the corresponding Stieltjes' polynomials which are orthogonal to all lower-degree polynomials with respect to a complex measure on $\Gamma_R=\{z\in\mathbb{C}\,:\,z=-iR+e^{i\theta}\sqrt{R^2+1},\ \varphi\le\theta\le\pi-\varphi,\ \tan\varphi=R\}$ are investigated. A class of polynomials orthogonal on a symmetrical circular arc in the down half plane is also introduced. Finally, in the Jacobi case $w(z)=(1-z)^\alpha(1+z)^\beta,\ \alpha,\beta>-1$, a linear second-order differential equation for $\pi_n^R(z)$ is obtained.

Keywords: Complex orthogonal polynomials, recurrence relations, differential equation.

1. Introduction

Polynomials orthogonal on the semicircle $\Gamma_0=\{z\in\mathbb{C}\,:\,z=e^{i\theta},\ 0\le\theta\le\pi\}$ have been introduced by Gautschi and Milovanović [5], [6]. The inner product is given by

$$(f,g)=\int_\Gamma f(z)g(z)(iz)^{-1}\,dz, \tag{1.1}$$

where Γ is the semicircle $\Gamma=\{z\in\mathbb{C}\,:\,z=e^{i\theta},\ 0\le\theta\le\pi\}$. Alternatively,

$$(f,g)=\int_0^\pi f(e^{i\theta})g(e^{i\theta})\,d\theta. \tag{1.2}$$

This inner product is not Hermitian, but the corresponding (monic) orthogonal polynomials $\{\pi_k\}$ exist uniquely and satisfy a three-term recurrence relation of the form

$$\begin{aligned}&\pi_{k+1}(z)=(z-i\alpha_k)\pi_k(z)-\beta_k\pi_{k-1}(z),\qquad k=0,1,2,\dots,\\ &\pi_{-1}(z)=0,\quad \pi_0(z)=1.\end{aligned} \tag{1.3}$$

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Notice that the inner product (1.1) possesses the property $(zf, g) = (f, zg)$.

Later, Gautschi, Landau and Milovanović [7] considered a general case of complex polynomials orthogonal with respect to a *complex weight function*. Namely, let $w : (-1, 1) \mapsto \mathbb{R}_+$ be a weight function which can be extended to a function $w(z)$ holomorphic in the half disc $D_+ = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$, and

$$(1.4) \qquad (f, g) = \int_{\Gamma} f(z)g(z)w(z)(iz)^{-1}\,dz = \int_0^{\pi} f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\,d\theta.$$

Under the assumption

$$(1.5) \qquad \operatorname{Re}(1,1) = \operatorname{Re}\int_0^{\pi} w(e^{i\theta})\,d\theta \neq 0,$$

the monic, complex polynomials $\{\pi_k\}$ orthogonal with respect to the inner product (1.4) exist and satisfy a recurrence relation like (1.3).

Several interesting properties of such polynomials and some applications in numerical integration were given in [6] and [9]. Also, differentiation formulas for higher derivatives of analytic functions, using quadratures on the semicircle, were considered in [2].

Recently M. G. de Bruin [1] has given a generalization of such orthogonal polynomials. Namely, he considered the polynomials $\{\pi_k^R\}$ orthogonal on a circular arc with respect to the complex inner product

$$(1.6) \qquad (f, g) = \int_{\varphi}^{\pi-\varphi} f_1(\theta)g_1(\theta)w_1(\theta)\,d\theta,$$

where $\varphi \in (0, \pi/2)$, and for $f(z)$ the function $f_1(\theta)$ is defined by

$$f_1(\theta) = f\big(-iR + e^{i\theta}\sqrt{R^2+1}\big), \qquad R = \tan\varphi.$$

Alternatively, the inner product (1.6) can be expressed in the form

$$(1.7) \qquad (f, g) = \int_{\Gamma_R} f(z)g(z)w(z)(iz - R)^{-1}\,dz,$$

where $\Gamma_R = \{z \in \mathbb{C} : z = -iR + e^{i\theta}\sqrt{R^2+1},\ \varphi \le \theta \le \pi - \varphi,\ \tan\varphi = R\}$.

For $R = 0$ the arc Γ_R reduces to the semicircle Γ.

Another type of orthogonality of these polynomials, so-called Geronimus' version of orthogonality on a contour with respect to a complex weight, was investigated in [10].

For polynomials $\{\pi_k^R\}$ orthogonal on a circular arc with respect to the complex inner product (1.6), in this paper, we introduce the functions of the second kind, as well as the corresponding associated polynomials, and prove some recurrence relations and identities of Christoffel-Darboux type. Also, we study the corresponding Stieltjes' polynomials which are orthogonal on Γ_R to all lower-degree polynomials with respect to a complex measure. In Section 3 we consider a class of polynomials orthogonal on a symmetrical circular arc in the down half plane. Sometimes, these *dual* polynomials can be used to shorten certain proofs for polynomials $\{\pi_k^R\}$ (see Theorem 3.6). Finally, in Section 4 we obtain a linear second-order differential equation for $\pi_n^R(z)$, when $w(z) = (1-z)^\alpha(1+z)^\beta,\ \alpha, \beta > -1$,

2. Functions of the second kind, Stieltjes' polynomials and associated polynomials

Let the inner product $(\cdot,\cdot)$ be given by (1.6), i.e., (1.7). Under suitable integrability conditions on w and assuming the existence of an analytic continuation to the moon-shaped region $M_+ = \{z \in \mathbb{C} : |z + iR| < \sqrt{R^2+1},\, \operatorname{Im} z > 0\}$, where $R > 0$, the orthogonal polynomials $\{\pi_k^R\}$ always exist, because (see [1, Lemma 2.2])

$$\mu_0 = (1,1) = R\int_{-1}^{1} \frac{w(x)}{x^2+R^2}\,dx + i\int_{-1}^{1} \frac{xw(x)}{x^2+R^2}\,dx \neq 0.$$

In connection with polynomials $\{\pi_k^R\}$ orthogonal with respect to $(\cdot,\cdot)$ on Γ_R, we can introduce the functions, so-called *functions of the second kind*,

$$\varrho_k^R(z) = \int_{\Gamma_R} \frac{\pi_k^R(\zeta)}{z-\zeta}\cdot\frac{w(\zeta)}{i\zeta - R}\,d\zeta, \qquad k = 0,1,2,\dots\ . \tag{2.1}$$

It is easily seen that they also satisfy the same recurrence relation as the polynomials π_k^R. Indeed, from recurrence relation (1.3) for $z = \zeta$, multiplying by $w(\zeta)/((i\zeta - R)(z-\zeta))$ and integrating, we obtain

$$\varrho_{k+1}^R(z) = (z - i\alpha_k)\varrho_k^R(z) - \int_{\Gamma_R} \pi_k^R(\zeta)\frac{w(\zeta)}{i\zeta - R}\,d\zeta - \beta_k\varrho_{k-1}^R(z).$$

By orthogonality, the integral on the right side in the above equality vanishes if $k \geq 1$, and equals μ_0 if $k = 0$. If we define $\varrho_{-1}^R(z) = 1$ (and $\beta_0 = \mu_0$), we have

$$\varrho_{k+1}^R(z) = (z - i\alpha_k)\varrho_k^R(z) - \beta_k\varrho_{k-1}^R(z), \qquad k = 0,1,2,\dots\ . \tag{2.2}$$

Theorem 2.1. *For $|z|$ sufficiently large, we have*

$$\varrho_n^R(z) = \frac{\|\pi_n^R\|^2}{z^{n+1}}\Big(1 + O\big(\frac{1}{z}\big)\Big), \tag{2.3}$$

where $\|\pi_n^R\|^2 = (\pi_n^R, \pi_n^R)$.

Proof. Let $\zeta \in \Gamma_R$ and $z \in \mathbb{C}$, such that $|z| > 1$. Since

$$\frac{1}{z-\zeta} = \frac{1}{z} \cdot \frac{1}{1-\zeta/z} = \frac{1}{z} \sum_{k=0}^{n} \left(\frac{\zeta}{z}\right)^k + \frac{\zeta^{n+1}}{(z-\zeta)z^{n+1}} \qquad (n \in \mathbb{N}),$$

we have

$$\varrho_n^R(z) = \sum_{k=0}^{n} \frac{1}{z^{k+1}} \int_{\Gamma_R} \zeta^k \pi_n^R(\zeta) \frac{w(\zeta)}{i\zeta - R}\, d\zeta + \frac{1}{z^{n+1}} e_n(z),$$

where

$$e_n(z) = \int_{\Gamma_R} \frac{\zeta^{n+1}\pi_n^R(\zeta)}{z-\zeta} \cdot \frac{w(\zeta)}{i\zeta - R}\, d\zeta.$$

For $|z|$ sufficiently large, there exists a constant $C > 0$ such that $|e_n(z)| < C/|z|$ and $e_n(z) \to 0$, when $|z| \to \infty$.

Because of orthogonality $(\zeta^k, \pi_n^R(\zeta)) = 0,\ k < n$, we obtain

$$\varrho_n^R(z) = \frac{\|\pi_n^R\|^2}{z^{n+1}} + \frac{1}{z^{n+1}} e_n(z),$$

i.e., (2.3). $\square$

Based on an idea by Stieltjes (see Monegato [11], Gautschi [4]) we can consider an expansion of $1/\varrho_n^R(z)$ into descending powers of z. So, using (2.3) we have

$$\begin{aligned}\frac{1}{\varrho_n^R(z)} &= \frac{z^{n+1}}{\|\pi_n^R\|^2}\left(1 + c_1 z^{-1} + c_2 z^{-2} + \cdots\right) \\ &= E_{n+1}^R(z) + d_1 z^{-1} + d_2 z^{-2} + \cdots\,,\end{aligned}$$

where

$$E_{n+1}^R(z) = \frac{1}{\|\pi_n^R\|^2}\left(z^{n+1} + c_1 z^n + \cdots + c_{n+1}\right)$$

and $d_k = c_{n+k+1}/\|\pi_n^R\|^2$, $k = 1, 2, \ldots$.

We call E_{n+1}^R the *Stieltjes polynomial associated with polynomials* $\{\pi_k^R\}$ orthogonal with respect to $(\cdot,\cdot)$ on Γ_R. By a residue calculation, this polynomial of exact degree $n+1$, can be expressed in the form

$$E_{n+1}^R(z) = \frac{1}{2\pi i} \oint_C \frac{d\zeta}{(\zeta - z)\varrho_n^R(\zeta)}, \tag{2.4}$$

where C is a sufficiently large contour with z in its interior.

Multiplying (2.4) by $z^k \pi_n^R(z) w(z)(iz - R)^{-1} dz$, $k = 0, 1, \ldots, n$, and integrating over Γ_R, we obtain

$$I_{n,k} = \int_{\Gamma_R} E_{n+1}^R(z) z^k \pi_n^R(z) \frac{w(z)}{iz - R}\, dz = \frac{1}{2\pi i} \oint_C \frac{d\zeta}{\varrho_n^R(\zeta)} \int_{\Gamma_R} \pi_n^R(z) \frac{z^k}{\zeta - z} \cdot \frac{w(z)}{iz - R}\, dz,$$

i.e,

$$I_{n,k} = \frac{1}{2\pi i}\oint_C \frac{d\zeta}{\varrho_n^R(\zeta)}\int_{\Gamma_R} \pi_n^R(z)\frac{\zeta^k-(\zeta^k-z^k)}{\zeta-z}\cdot\frac{w(z)}{iz-R}\,dz.$$

Because of orthogonality

$$\int_{\Gamma_R} \pi_n^R(z)\frac{\zeta^k-z^k}{\zeta-z}\cdot\frac{w(z)}{iz-R}\,dz = 0, \qquad k=0,1,\dots,n,$$

we have

$$I_{n,k} = \frac{1}{2\pi i}\oint_C \frac{\zeta^k d\zeta}{\varrho_n^R(\zeta)}\int_{\Gamma_R} \frac{\pi_n^R(z)}{\zeta-z}\cdot\frac{w(z)}{iz-R}\,dz = \frac{1}{2\pi i}\oint_C \zeta^k\,d\zeta = 0,$$

for $k=0,1,\dots,n$.

Thus, we have proved:

Theorem 2.2. *Stieltjes' polynomial E_{n+1}^R is orthogonal to all lower-degree polynomials with respect to the complex measure $d\lambda(z)=\pi_n^R(z)w(z)(iz-R)^{-1}dz$, i.e.,*

$$\int_{\Gamma_R} E_{n+1}^R(z)p(z)\pi_n^R(z)\frac{w(z)}{iz-R}\,dz = 0, \qquad \forall p\in\mathcal{P}_n,$$

where $\mathcal{P}_n$ is the set of all polynomials of degree at most n.

The quantities $\rho_n^R(z)/\pi_n^R(z)$, $|z|>1$, are important in getting error bounds for Gaussian quadrature formulas over Γ_R, applied to analytic functions (cf. Gautschi and Varga [8]). Stieltjes' polynomials appear in quadrature formulas of Gauss-Kronrod's type (cf. Gautschi [4]).

We can also introduce the polynomials

$$\sigma_k^R(z) = \int_{\Gamma_R} \frac{\pi_k^R(z)-\pi_k^R(\zeta)}{z-\zeta}\cdot\frac{w(\zeta)}{i\zeta-R}\,d\zeta, \qquad k=0,1,2,\dots,$$

which are called the *polynomials associated with the orthogonal polynomials* π_k^R. It is easy to see that

$$\varrho_k^R(z) = \pi_k^R(z)\varrho_0^R(z) - \sigma_k^R(z).$$

The polynomials $\{\sigma_k^R\}$ satisfy the same three-term recurrence relation

$$\begin{aligned} &\sigma_{k+1}^R(z) = (z-i\alpha_k)\sigma_k^R(z) - \beta_k\sigma_{k-1}^R(z), \qquad k=0,1,2,\dots, \\ &\sigma_0^R(z)=0, \quad \sigma_1^R(z)=\mu_0. \end{aligned} \tag{2.5}$$

If we define $\sigma_{-1}^R(z)=-1$ and $\beta_0=\mu_0$, we can note that (2.5) also holds for $k=0$ (see Gautschi [3]).

Using the recurrence relations for $\{\pi_k^R\}$, $\{\varrho_k^R\}$, and $\{\sigma_k^R\}$ ((1.3), (2.2), and (2.5), respectively), where

1° $\pi_{-1}^R(z)=0, \quad \pi_0^R(z)=1\,;$

2° $\varrho_{-1}^R(z)=1, \quad \varrho_0^R(z)=F(z)$ (defined by (2.1));

3° $\sigma_{-1}^R(z)=-1, \quad \sigma_0^R(z)=0\,,$

we can prove the following identity of Christoffel-Darboux type:

Theorem 2.3. *Let $\{f_k\}$ and $\{g_k\}$ satisfy the three-term recurrence relation of the form* (2.5), *and*

$$S_k(z,w)=f_{k+1}(z)g_k(w)-g_{k+1}(w)f_k(z).$$

Then the identity

$$(z-w)\sum_{k=0}^{n}\frac{f_k(z)g_k(w)}{\beta_0\beta_1\cdots\beta_k}=\frac{S_n(z,w)}{\beta_0\beta_1\cdots\beta_n}-S_{-1}(z,w) \tag{2.6}$$

holds, where β_k $(k=0,1,2,\dots)$ are the recursion coefficients in (2.5). *Under conditions* 1°–3°, *we have the following special cases*

(a) $f_k:=\pi_k^R,\quad g_k:=\pi_k^R,\quad S_{-1}=0\,;$
(b) $f_k:=\pi_k^R,\quad g_k:=\varrho_k^R,\quad S_{-1}=1\,;$
(c) $f_k:=\pi_k^R,\quad g_k:=\sigma_k^R,\quad S_{-1}=-1\,;$
(d) $f_k:=\varrho_k^R,\quad g_k:=\varrho_k^R,\quad S_{-1}=F(z)-F(w)\,;$
(e) $f_k:=\varrho_k^R,\quad g_k:=\sigma_k^R,\quad S_{-1}=-F(z)\,;$
(f) $f_k:=\sigma_k^R,\quad g_k:=\sigma_k^R,\quad S_{-1}=0\,.$

Proof. Multiplying

$$f_{k+1}(z)=(z-i\alpha_k)f_k(z)-\beta_kf_{k-1}(z),\qquad k=0,1,2,\dots\,,$$

by $g_k(w)$, and

$$g_{k+1}(w)=(w-i\alpha_k)g_k(w)-\beta_kg_{k-1}(w),\qquad k=0,1,2,\dots\,,$$

by $f_k(z)$, and substracting we obtain

$$S_k(z,w)-\beta_kS_{k-1}(z,w)=(z-w)f_k(z)g_k(w).$$

Dividing this equality by $\beta_0\beta_1\cdots\beta_k$ and summing over $k=0,1,\dots,n$, we find

$$\sum_{k=0}^{n}\left(\frac{S_k(z,w)}{\beta_0\beta_1\dots\beta_k}-\frac{S_{k-1}(z,w)}{\beta_0\beta_1\cdots\beta_{k-1}}\right)=(z-w)\sum_{k=0}^{n}\frac{f_k(z)g_k(w)}{\beta_0\beta_1\cdots\beta_k},$$

i.e., (2.6). The first term in the sum on the left-hand side of the above equality (for $k=0$) is equal to $S_0(z,w)/\beta_0-S_{-1}(z,w)$.

Since $S_{-1}(z,w)=f_0(z)g_{-1}(w)-g_0(w)f_{-1}(z)$, using the conditions 1°–3°, we prove the cases (a)–(f). $\square$

3. Dual orthogonal polynomials

Let $\{\pi_n\}$ be the set of polynomials orthogonal on the circular arc Γ_R, with respect to the inner product (1.6), i.e., (1.7) (The upper index R is omitted). In this section we introduce the polynomials $\{\pi_n^*\}$ orthogonal on the symmetric down circular arc Γ_R^* with respect to the inner product defined by

$$(f,g)^* = \int_{\Gamma_R^*} f(z)g(z)w(z)(iz+R)^{-1}\,dz, \tag{3.1}$$

where $\Gamma_R^* = \{z \in \mathbb{C} \,:\, z = iR + e^{-i\theta}\sqrt{R^2+1},\ \varphi \le \theta \le \pi - \varphi,\ \tan\varphi = R\}$. Such polynomials we will call *dual orthogonal polynomials* with respect to polynomials $\{\pi_n\}$.

Also, we use the inner product

$$[f,g] = \int_{-1}^{1} f(x)\overline{g(x)}w(x)\,dx. \tag{3.2}$$

Let M be a lentil-shaped region with the boundary $\partial M = \Gamma_R \cup \Gamma_R^*$, i.e.,

$$M = \bigl\{z \in \mathbb{C} \,:\, |z \pm iR| < \sqrt{R^2+1}\bigr\},$$

where $R > 0$.

We assume that w is a weight function, positive on $(-1,1)$, holomorphic in M, and such that the integrals in (1.7), (3.1), and (3.2) exist for smooth functions f and g (possibly) as improper integrals. Under the same additional conditions on w and f, like in [7] and [1], we have

$$0 = \int_{\Gamma} f(z)w(z)\,dz + \int_{-1}^{1} f(x)w(x)\,dx, \tag{3.3}$$

where $\Gamma = \Gamma_R$ or Γ_R^*. Then both systems of the orthogonal polynomials $\{\pi_n\}$ and $\{\pi_n^*\}$ exist uniquely.

The inner products in (1.7) and (3.1) define the moment functionals

$$\mathcal{L}z^k = \mu_k, \qquad \mu_k = (z^k,1) = \int_{\Gamma_R} z^k w(z)(iz-R)^{-1}\,dz \tag{3.4}$$

and

$$\mathcal{L}^* z^k = \mu_k^*, \qquad \mu_k^* = (z^k,1)^* = \int_{\Gamma_R^*} z^k w(z)(iz+R)^{-1}\,dz, \tag{3.5}$$

respectively.

Using the moment determinants, we can express the (monic) polynomials π_n and π_n^* as

$$\pi_n(z) = \frac{1}{\Delta_n}\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & & \mu_{n+1} \\ \vdots & & & \\ \mu_{n-1} & \mu_n & & \mu_{2n-1} \\ 1 & z & & z^n \end{vmatrix}, \quad \pi_n^*(z) = \frac{1}{\Delta_n^*}\begin{vmatrix} \mu_0^* & \mu_1^* & \cdots & \mu_n^* \\ \mu_1^* & \mu_2^* & & \mu_{n+1}^* \\ \vdots & & & \\ \mu_{n-1}^* & \mu_n^* & & \mu_{2n-1}^* \\ 1 & z & & z^n \end{vmatrix},$$

respectively, where

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & & \mu_n \\ \vdots & & & \\ \mu_{n-1} & \mu_n & & \mu_{2n-2} \end{vmatrix}, \qquad \Delta_n^* = \begin{vmatrix} \mu_0^* & \mu_1^* & \cdots & \mu_{n-1}^* \\ \mu_1^* & \mu_2^* & & \mu_n^* \\ \vdots & & & \\ \mu_{n-1}^* & \mu_n^* & & \mu_{2n-2}^* \end{vmatrix}.$$

Lemma 3.1. *For the moments μ_k and μ_k^* the following equality*

$$\mu_k^* = -\bar{\mu}_k\,, \qquad k \geq 0, \tag{3.6}$$

holds.

Proof. Inserting $f(z) = z^k(iz \pm R)^{-1}$ into (3.3), the integrals (3.4) and (3.5) reduce to integrals on $[-1, 1]$, implying (3.6). $\square$

Theorem 3.2. *We have*

$$\pi_n^*(\overline{z}) = \overline{\pi_n(z)}. \tag{3.7}$$

Proof. Conjugating $\pi_n(z)$ and using (3.6), we get

$$\overline{\pi_n(z)} = \frac{1}{\overline{\Delta}_n}\begin{vmatrix} -\mu_0^* & -\mu_1^* & \cdots & -\mu_n^* \\ -\mu_1^* & -\mu_2^* & & -\mu_{n+1}^* \\ \vdots & & & \\ -\mu_{n-1}^* & -\mu_n^* & & -\mu_{2n-1}^* \\ 1 & \overline{z} & & \overline{z}^n \end{vmatrix} = \frac{(-1)^n}{\overline{\Delta}_n}\,\Delta_n^*\pi_n^*(\overline{z})\,.$$

Since $\overline{\Delta}_n = (-1)^n\Delta_n^*$, we obtain (3.7). $\square$

In the same way as in [1] we can prove a representation of the dual polynomials in terms of the monic real polynomials $\{p_n\}$ orthogonal with respect to the inner product (3.2).

Theorem 3.3. *We have*

$$\pi_n^*(z) = p_n(z) - i\theta_{n-1}^* p_{n-1}(z), \qquad n = 0, 1, 2, \dots , \tag{3.8}$$

where

$$\theta_{n-1}^* = \frac{(\pi_n^*, \pi_n^*)^*}{[p_{n-1}, p_{n-1}]}, \quad n = 1, 2, \dots , \qquad \theta_{-1}^* = \mu_0^*.$$

Theorem 3.4. *We have*

$$\theta_{n-1}^* = -\overline{\theta}_{n-1}, \tag{3.9}$$

where θ_{n-1} is the corresponding coefficient in the polynomial π_n.

Proof. Conjugating (3.7) and using (3.8) we find

$$\pi_n(z) = \overline{\pi_n^*(\overline{z})} = \overline{p_n(\overline{z}) - i\theta_{n-1}^* p_{n-1}(\overline{z})},$$

i.e.,

$$\pi_n(z) = \overline{p_n(\overline{z})} + i\,\overline{\theta}_{n-1}^*\ \overline{p_{n-1}(\overline{z})} = p_n(z) + i\,\overline{\theta}_{n-1}^* p_{n-1}(z).$$

Comparing with $\pi_n(z) = p_n(z) - i\theta_{n-1}p_{n-1}(z)$ we obtain (3.9). □

Also, we can prove:

Theorem 3.5. *The dual (monic) orthogonal polynomials $\{\pi_n^*\}$ satisfy the three-term recurence relation*

$$\pi_{n+1}^*(z) = (z - i\alpha_n^*)\pi_n^*(z) - \beta_n^*\pi_{n-1}^*(z), \qquad n = 0, 1, 2, \dots ,$$
$$\pi_{-1}^*(z) = 0, \quad \pi_0^*(z) = 1,$$

with

$$\alpha_n^* = -\overline{\alpha}_n \qquad \text{and} \qquad \beta_n^* = \overline{\beta}_n,$$

where α_n and β_n are the coefficients in the corresponding recurrence relation for the polynomials $\{\pi_n\}$.

At the end of this section we will give a short proof of Theorem 4.1 from [1] using dual polynomials.

Theorem 3.6. *Let $w(z) = w(-z)$. Then $\theta_{n-1} > 0$, for $n \ge 0$.*

Proof. Since $(\pi_n, \pi_n) = \theta_{n-1}[p_{n-1}, p_{n-1}]$ it is enough to prove inequality

$$(\pi_n, \pi_n) > 0.$$

In this *symmetric case*, θ_{n-1} is real and we have $\theta_{n-1}^* = -\theta_{n-1}$ and

$$(\pi_n, \pi_n) = (\pi_n, \pi_n^*) = \int_{\Gamma_R} G(z)w(z)(iz - R)^{-1}\,dz = -\int_{-1}^{1} G(x)\frac{w(x)}{ix - R}\,dx,$$

where $G(z) = p_n(z)^2 + \theta_{n-1}^2 p_{n-1}(z)^2$. Then

$$(\pi_n, \pi_n) = R\int_{-1}^{1} G(x)\frac{w(x)}{R^2 + x^2}\,dx + i\int_{-1}^{1} xG(x)\frac{w(x)}{R^2 + x^2}\,dx.$$

Since $x \mapsto G(x)$ is an even positive function, the second integral on the right-hand side vanishes and $(\pi_n, \pi_n) > 0$. □

4. Differential equation

We consider the Jacobi weight function

$$w(z) = w^{\alpha,\beta}(z) = (1-z)^\alpha (1+z)^\beta, \qquad \alpha, \beta > -1,$$

where fractional powers are understood in terms of their principal branches.

The corresponding (monic) polynomials $\{\pi_n^R\}$ orthogonal on the circular arc Γ_R, with respect to the inner product (1.6), i.e., (1.7), where $w(z) = w^{\alpha,\beta}(z)$, can be expressed (see [1, Thm. 2.1]) in the form

$$\pi_n^R(z) = \pi_n(z) = p_n(z) - i\theta_{n-1} p_{n-1}(z), \tag{4.1}$$

where $p_k(z) = \hat{P}_k^{\alpha,\beta}(z)$ are the monic Jacobi polynomials and $\theta_{n-1} = \theta_{n-1}^{\alpha,\beta}$ is given by

$$\theta_{n-1} = \frac{1}{i} \frac{\varrho_n(-iR)}{\varrho_{n-1}(-iR)}, \qquad n \geq 1,$$

where

$$\varrho_n(z) = \int_{-1}^{1} \frac{p_n(x)}{z-x}\, w(x)\, dx, \qquad n \geq 0.$$

The monic polynomials $p_k(z)$ satisfy Jacobi's differential equation

$$A(z)u'' + B(z)u' + \lambda_k u = 0, \tag{4.2}$$

and differentiation formula

$$A(z)p_k'(z) = [(k+\alpha+\beta+1)z + v_k]p_k(z) - (2k+\alpha+\beta+1)p_{k+1}(z), \tag{4.3}$$

where

$$A(z) = 1 - z^2, \quad B(z) = \beta - \alpha - (\alpha+\beta+2)z, \quad \lambda_k = k(k+\alpha+\beta+1),$$

and

$$v_k = (\alpha - \beta)\frac{k+\alpha+\beta+1}{2k+\alpha+\beta+2}.$$

Now, we consider the following problem: *Find a function $z \mapsto \Omega(z)$ such that*

$$G(z) \equiv (z^2-1)[\Omega(z)u(z)]' = \Omega(z)v(z), \tag{4.4}$$

for $u(z) = p_{n-1}(z)$ and $v(z) = \gamma_n \pi_n(z)$, where γ_n is a constant.

We will suppose that $\Omega(z)$ has the following form

$$\Omega(z) = (z-1)^{r_n - it_n}(z+1)^{s_n + it_n},$$

where r_n, s_n, and t_n are some real constants.

Putting this expression for $\Omega(z)$ into (4.4) and using (4.3), for $k = n-1$, we obtain

$$G(z) = (z^2-1)\big\{\Omega'(z)u(z)+\Omega(z)u'(z)\big\}$$
$$= \Omega(z)(z^2-1)p'_{n-1}(z) + \big\{(r_n - it_n)(z+1) + (s_n+it_n)(z-1)\big\}\Omega(z)p_{n-1}(z),$$

i.e.,

$$G(z) = \Omega(z)\big\{[r_n+s_n-(n+\alpha+\beta)]zp_{n-1}(z)$$
$$+ [(r_n-s_n-2it_n)-v_{n-1}]p_{n-1}(z) + (2n+\alpha+\beta-1)p_n(z)\big\}.$$

In order to make the coefficient of $zp_{n-1}(z)$ vanish on the right side in the last equality, we put

$$r_n + s_n = n+\alpha+\beta. \tag{4.5}$$

So we find

$$G(z) = \Omega(z)\big\{(r_n-s_n-v_{n-1})p_{n-1}(z)+(2n+\alpha+\beta-1)p_n(z)-2it_np_{n-1}(z)\big\}. \tag{4.6}$$

Because of further reduction of $G(z)$ to the form $\Omega(z)v(z)$, we take

$$r_n - s_n = v_{n-1}. \tag{4.7}$$

Therefore, from (4.5) and (4.7) follows

$$r_n = \frac{1}{2}\left(n+\alpha+\beta+v_{n-1}\right) \qquad \text{and} \qquad s_n = \frac{1}{2}\left(n+\alpha+\beta-v_{n-1}\right). \tag{4.8}$$

Finally, if we take

$$t_n = \frac{1}{2}(2n+\alpha+\beta-1)\theta_{n-1}, \tag{4.9}$$

(4.6) reduces to the form (4.4), where $\gamma_n = 2n+\alpha+\beta-1$, i.e.,

$$v(z) = (2n+\alpha+\beta-1)\pi_n(z).$$

Now, from (4.4) follows

$$u = \frac{1}{\Omega}\int \frac{\Omega}{z^2-1} v\,dz,$$
$$u' = \left(\frac{1}{\Omega}\right)'\int \frac{\Omega}{z^2-1} v\,dz + \frac{1}{z^2-1}v,$$
$$u'' = \left(\frac{1}{\Omega}\right)''\int \frac{\Omega}{z^2-1} v\,dz + \left(\frac{1}{\Omega}\right)'\frac{\Omega}{z^2-1}v + \left(\frac{1}{z^2-1}\right)'v + \frac{1}{z^2-1}v'.$$

Substituting u, u', u'' into Jacobi's differential equation (4.2), for $k = n-1$, we obtain

$$v' + a(z)v + b(z)\int \frac{\Omega}{z^2-1} v\,dz = 0, \tag{4.10}$$

where

$$\begin{aligned} a(z) &= \frac{1}{A(z)}(g(z) + B(z) - A'(z)),\\ b(z) &= -\frac{1}{A(z)\Omega(z)}\Big((\lambda_{n-1} + g'(z))A(z) + g(z)(g(z) + B(z) - A'(z))\Big), \end{aligned} \tag{4.11}$$

and

$$g(z) = (n+\alpha+\beta)z + c_{n-1}, \qquad c_{n-1} = v_{n-1} - i(2n+\alpha+\beta-1)\theta_{n-1}.$$

The parameters r_n, s_n, and t_n in the function $z \mapsto \Omega(z)$ are given by (4.8) and (4.9).

Theorem 4.1. *The polynomial $\pi_n^R(z)$ in (4.1) satisfies the differential equation*

$$P(z)y'' + Q(z)y' + R(z)y = 0, \tag{4.12}$$

with polynomial coefficients

$$\begin{aligned} P(z) &= -A(z)^2\Omega(z)b(z),\\ Q(z) &= A(z)^2\Omega(z)\big(b'(z) - a(z)b(z)\big),\\ R(z) &= A(z)\big[A(z)\Omega(z)\big(a(z)b'(z) - a'(z)b(z)\big) + \Omega(z)^2 b(z)^2\big], \end{aligned} \tag{4.13}$$

where $a(z)$ and $b(z)$ are given by (4.11).

Proof. Differentiating (4.10) and eliminating the integral term, we get (4.12) and (4.13) after some computation. □

Remark. All coefficients in (4.13) have the factor $A(z)\Omega(z)$, but it is because of this factor that the coefficients turn out to be polynomials.

Using (4.13) and (4.11) we find $P(z) = (1-z^2)C(z)$, where $z \mapsto C(z)$ is the following polynomial of the first degree

$$C(z) = n(n+\alpha+\beta) + (\beta-\alpha+c_{n-1})c_{n-1} - i(2n+\alpha+\beta)(2n+\alpha+\beta-1)\theta_{n-1}z,$$

where

$$c_{n-1} = (\alpha-\beta)\frac{n+\alpha+\beta}{2n+\alpha+\beta} - i(2n+\alpha+\beta-1)\theta_{n-1}.$$

After some calculation, $C(z)$ can be expressed in the form

$$C(z) = \gamma_0 + i\gamma_1\theta_{n-1} + \gamma_2\theta_{n-1}^2 - \eta_n z, \tag{4.14}$$

where

$$\gamma_0 = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2}, \quad \gamma_1 = (\beta^2-\alpha^2)\frac{2n+\alpha+\beta-1}{2n+\alpha+\beta},$$

$$\gamma_2 = -(2n+\alpha+\beta-1)^2, \quad \eta_n = (2n+\alpha+\beta)(2n+\alpha+\beta-1)i\theta_{n-1}.$$

We remark that the differential equation (4.12) has regular singular points at 1, -1, ∞, and an additional regular singular point ζ_n which depends on n and is given by

$$\zeta_n = \frac{\gamma_1 - i(\gamma_2\theta_{n-1} + \gamma_0/\theta_{n-1})}{(2n+\alpha+\beta)(2n+\alpha+\beta-1)}.$$

The polynomials P, Q, and R in (4.13) can be expressed by the the coefficients $A(z)$, $B(z)$, and λ_k in differential equation (4.2). Namely,

$$\begin{aligned} P(z) &= A(z)C(z), \\ Q(z) &= B(z)C(z) + A(z)\eta_n, \\ R(z) &= \lambda_n C(z) + A(z)a(z)\eta_n, \end{aligned} \tag{4.15}$$

where

$$a(z) = \frac{1}{A(z)}\left(nz + \frac{n(\beta-\alpha)}{2n+\alpha+\beta} - i(2n+\alpha+\beta-1)\theta_{n-1}\right)$$

and $C(z)$ is given by (4.14).

We can see that Q and R are the complex polynomials of degree two and one, respectively.

When $\alpha = \beta = \lambda - 1/2$ ($\lambda > -1/2$) we obtain the Gegenbauer case, which is considered for $R = 0$ in [7].

It is interesting to consider a case when $R \to +\infty$, i.e., when $\varGamma_R$ reduces to the interval $[-1,1]$. Since

$$\lim_{R\to+\infty} \theta_{n-1} = 0,$$

we have

$$\lim_{R\to+\infty} \eta_n = 0, \quad \lim_{R\to+\infty} C(z) = \gamma_0, \quad \lim_{R\to+\infty} a(z) = \frac{1}{A(z)}\left(nz + \frac{n(\beta-\alpha)}{2n+\alpha+\beta}\right).$$

Thus, the limit case of (4.15) gives

$$\lim_{R\to+\infty} P(z) = \gamma_0 A(z), \quad \lim_{R\to+\infty} Q(z) = \gamma_0 B(z), \quad \lim_{R\to+\infty} R(z) = \gamma_0\lambda_n.$$

In that case, dividing (4.12) by γ_0 we obtain Jacobi's differential equation, which is, in fact, the result expected.

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