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Extremal problems, inequalities, and classical orthogonal polynomials ☆

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Abstract

In this survey paper we give a short account on characterizations for very classical orthogonal polynomials via extremal problems and the corresponding inequalities. Besides the basic properties of the classical orthogonal polynomials, we consider polynomial inequalities of Landau and Kolmogoroff type, some weighted polynomial inequalities in L^2-norm of Markov–Bernstein type, as well as the corresponding connections with the classical orthogonal polynomials. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let $\mathscr{P}_n$ be the set of all algebraic polynomials of degree at most n and let $\hat{\mathscr{P}}_n$ be its subset containing only monic polynomials of degree n, i.e.,

$$\hat{\mathscr{P}}_n = \{z^n + q(z) | q(z) \in \mathscr{P}_{n-1}\}.$$

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A general result on orthogonal polynomials with respect to a given inner product $(\cdot,\cdot)$ defined by

$$(f,g)=\int f(z)\overline{g(z)}\,\mathrm{d}\mu(z)\quad (f,g\in L^2(\mathrm{d}\mu)),$$

where $\mathrm{d}\mu$ is a finite positive Borel measure in the complex plane $\mathbb{C}$, with an infinite set as its support, can be expressed as an extremal problem.

Let $\{p_n\}$ be a system of orthonormal polynomials, i.e.,

$$p_n(z)=\gamma_n z^n+\text{lower degree terms},\quad \gamma_n>0,$$
$$(p_n,p_m)=\delta_{nm},\quad n,m\geqslant 0,$$

and $\pi_n(z)=p_n(z)/\gamma_n=z^n+\text{lower degree terms}$ $(n\in\mathbb{N}_0)$ be the corresponding monic orthogonal polynomials.

Theorem 1.1. *The polynomial* $\pi_n(z)=p_n(z)/\gamma_n=z^n+\cdots$ *is the unique monic polynomial of degree n of the minimal* $L^2(\mathrm{d}\mu)$*-norm, i.e.*,

$$\min_{p\in\hat{\mathscr{P}}_n}\int |p(z)|^2\,\mathrm{d}\mu(z)=\int|\pi_n(z)|^2\,\mathrm{d}\mu(z)=\frac{1}{\gamma_n^2}.$$

This extremal property is completely equivalent to orthogonality, so that it characterizes orthogonal polynomials. Many questions regarding orthogonal polynomials can be answered by using only this extremal property (cf. [36,40,47]). Notice also that the previous theorem gives the polynomial of the best approximation to the monomial z^n in the class $\mathscr{P}_{n-1}$. It is evidently expressed in the form $z^n-\pi_n(z)$.

A survey on characterization theorems for orthogonal polynomials on the real line was given by Al-Salam [3]. The most important orthogonal polynomials on the real line are the so-called *very* classical orthogonal polynomials (cf. [47]). An extension of the very classical orthogonal polynomials using difference operators and q-difference operators is known nowadays as the classical orthogonal polynomials (see [5–7]). Such a much larger class of orthogonal polynomials can be arranged in a table, which is known as the Askey table and its q-extension (cf. [24]).

In this survey paper we give a short account on characterizations for very classical orthogonal polynomials via extremal problems and the corresponding inequalities. In the sequel we will omit the term "very" and call such polynomials the classical orthogonal polynomials. The paper is organized as follows. In Section 2 we give the basic properties of the classical orthogonal polynomials. Section 3 is devoted to polynomial inequalities of Landau and Kolmogoroff type. Finally, some weighted polynomial inequalities in L^2-norm of Markov–Bernstein type and connections with the classical orthogonal polynomials are studied in Section 4.

2. The basic properties of the classical orthogonal polynomials

A very important class of orthogonal polynomials on an interval of orthogonality $(a,b) \in \mathbb{R}$, with respect to the inner product

$$(f,g)_w = \int_a^b w(t)f(t)g(t)\,\mathrm{d}t, \tag{2.1}$$

is constituted by the classical orthogonal polynomials. They are distinguished by several particular properties.

Since every interval (a,b) can be transformed by a linear transformation to one of the following intervals: $(-1,1)$, $(0,+\infty)$, $(-\infty,+\infty)$, it is enough to restrict our consideration (without loss of generality) only to these three intervals.

Definition 2.1. The orthogonal polynomials $\{Q_n(t)\}$ on (a,b) with respect to the inner product (2.1) are called the classical orthogonal polynomials if their weight functions $t \mapsto w(t)$ satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(A(t)w(t)) = B(t)w(t),$$

where

$$A(t) = \begin{cases} 1-t^2 & \text{if } (a,b)=(-1,1), \\ t & \text{if } (a,b)=(0,+\infty), \\ 1 & \text{if } (a,b)=(-\infty,+\infty), \end{cases}$$

and $B(t)$ is a polynomial of the first degree. For such classical weights we will write $w \in CW$.

We note that if $w \in CW$, then $w \in C^1(a,b)$, and also the following property:

Theorem 2.2. *If $w \in CW$, then for each $m = 0,1,\ldots,$ we have*

$$\lim_{t\to a+} t^m A(t)w(t) = 0 \quad \textit{and} \quad \lim_{t\to b-} t^m A(t)w(t) = 0.$$

Based on the above definition, the classical orthogonal polynomials $\{Q_n(t)\}$ on (a,b) can be specificated as the *Jacobi polynomials* $P_n^{(\alpha,\beta)}(t)(\alpha,\beta > -1)$ on $(-1,1)$, the *generalized Laguerre polynomials* $L_n^s(t)(s > -1)$ on $(0,+\infty)$, and finally as the *Hermite polynomials* $H_n(t)$ on $(-\infty,+\infty)$. Their weight functions $t \mapsto w(t)$ and the corresponding polynomials $A(t)$ and $B(t)$ are given in Table 1.

Special cases of the Jacobi polynomials are the *Legendre polynomials* $P_n(t)$ for $\alpha=\beta=0$, the *Chebyshev polynomials of the first kind* $T_n(t)$ for $\alpha=\beta=-(1/2)$ and the *second kind* $S_n(t)$ for $\alpha=\beta=1/2$, etc.

There are many characterizations of the classical orthogonal polynomials. In sequel we give the basic common properties of these polynomials (cf. [32,33]).

Table 1
The classification of the classical orthogonal polynomials

(a,b)	$w(t)$	$A(t)$	$B(t)$	λ_n
$(-1,1)$	$(1-t)^{\alpha}(1+t)^{\beta}$	$1-t^2$	$\beta-\alpha-(\alpha+\beta+2)t$	$n(n+\alpha+\beta+1)$
$(0,+\infty)$	$t^s e^{-t}$	t	$s+1-t$	n
$(-\infty,+\infty)$	e^{-t^2}	1	$-2t$	$2n$

Theorem 2.3. *The mth derivatives* $\{Q_n^{(m)}(t)\}$ *of the classical orthogonal polynomials* $\{Q_n(t)\}$ *form also a sequence of the classical orthogonal polynomials on* (a,b) *with respect to the weight function* $t \mapsto w_m(t) = A(t)^m w(t)$. *The differential equation for this weight is* $(A(t)w_m(t))' = B_m(t)w_m(t)$, *where* $B_m(t) = mA'(t) + B(t)$.

Theorem 2.4. *The classical orthogonal polynomial* $Q_n(t)$ *is a particular solution of the second-order linear differential equation of hyphergeometric type*

$$L[y] = A(t)y'' + B(t)y' + \lambda_n y = 0, \tag{2.2}$$

where

$$\lambda_n = -n\left[\frac{1}{2}(n-1)A''(0) + B'(0)\right].$$

Eq. (2.2) can be written in the Sturm–Liouville form

$$\frac{d}{dt}\left(A(t)w(t)\frac{dy}{dt}\right) + \lambda_n w(t)y = 0. \tag{2.3}$$

The coefficients λ_n are also displayed in Table 1.

Similarly, the mth derivative of $Q_n(t)$ satisfies the differential equation

$$\frac{d}{dt}\left(A(t)w_m(t)\frac{dy}{dt}\right) + \lambda_{n,m} w_m(t)y = 0,$$

where

$$\lambda_{n,m} = -(n-m)\left[\frac{1}{2}(n+m-1)A''(0) + B'(0)\right]. \tag{2.4}$$

We note that $\lambda_{n,0} = \lambda_n$.

The characterization of the classical orthogonal polynomials by differential equation (2.2), i.e., (2.3), was proved by Lesky [28], and conjectured by Aczél [1] (see also Bochner [11]). Such a differential equation appears in many mathematical models in atomic physics, electrodynamics and acoustics. As an example we mention the well-known Schrödinger equation.

The classical orthogonal polynomials possess a Rodrigues' type formula (cf. [8,43–45]):

$$Q_n(t) = \frac{C_n}{w(t)} \cdot \frac{\mathrm{d}^n}{\mathrm{d}t^n}[A(t)^n w(t)],$$

where C_n are constants different from zero. Its integral form is

$$Q_n(t) = \frac{C_n}{w(t)} \cdot \frac{n!}{2\pi \mathrm{i}} \oint_\Gamma \frac{A(z)^n w(z)}{(z-t)^{n+1}}\,\mathrm{d}z,$$

where Γ is a closed contour such that $t \in \mathrm{int}\Gamma$.

The constants C_k in the previous formulas can be chosen in different way (for example, $Q_n(t)$ to be monic, orthonormal, etc.). A historical reason leads to

$$C_n = \begin{cases} \frac{(-1)^n}{2^n n!} & \text{for } P_n^{(\alpha,\beta)}(t), \\ 1 & \text{for } L_n^s(t), \\ (-1)^n & \text{for } H_n(t). \end{cases}$$

In addition, the Gegenbauer polynomials $C_n^\lambda(t)$ and the Chebyshev polynomials $T_n(t)$ and $S_n(t)$ need

$$\begin{aligned} C_n^\lambda(t) &= \frac{(2\lambda)_n}{(\lambda+(1/2))_n} P_n^{(\alpha,\alpha)}(t) \quad (\alpha = \lambda - 1/2), \\ T_n(t) &= \frac{n!}{(1/2)_n} P_n^{(-1/2,-1/2)}(t), \\ S_n(t) &= \frac{(n+1)!}{(3/2)_n} P_n^{(1/2,1/2)}(t), \end{aligned}$$

where $(s)_k$ is the standard notation for Pochhammer's symbol

$$(s)_k = s(s+1)\cdots(s+k-1) = \frac{\Gamma(s+k)}{\Gamma(s)} \quad (\Gamma \text{ is the gamma function}).$$

3. Landau and Kolmogoroff type polynomial inequalities

Consider now the inner product (2.1) with the weight $w(t) = \exp(-t^2)$ on $(-\infty, +\infty)$ and put

$$\|f\|^2 = (f,f) = \int_{-\infty}^{+\infty} \mathrm{e}^{-t^2} f(t) g(t)\,\mathrm{d}t. \tag{3.1}$$

Similarly to the well-known inequalities of Landau type [27] and Kolmogoroff type [25] for continuously differentiable functions, as well as their generalizations (see, for example, [16,17,22,25,29,39,41]), it is possible to consider such kind of inequalities for algebraic polynomials of fixed degree.

For sufficiently smooth functions on $\mathbb{R}$, Kolmogoroff [25] established an inequality in the uniform norm

$$\|f^{(k)}\|_\infty \leqslant C_{n,k}\|f^{(n)}\|_\infty^{k/n}\|f\|_\infty^{1-k/n} \quad (0<k<n),$$

with the best possible constant $C_{n,k}$. Similar inequalities in integral norms have been also considered (cf. [34, Chapter I]). For example, Kupcov [35] considered the same inequality in L^2-norm on the positive half-line and proved that it is equivalent to

$$\|f^{(k)}\|_2^2 \leqslant \frac{1}{\gamma_{n,k}}\left(\|f^{(n)}\|_2^2+\|f\|_2^2\right),$$

where

$$M_{n,k}^2=\frac{1}{\gamma_{n,k}}\left\{\left(\frac{n-k}{n}\right)^{k/n}+\left(\frac{k}{n-k}\right)^{1-k/n}\right\}.$$

Similar inequalities in the norm (3.1)

$$\|P^{(k)}\|^2 \leqslant A\|P^{(m)}\|^2+B\|P\|^2 \quad (0<k<m\leqslant n), \tag{3.2}$$

for polynomials from $\mathscr{P}_n$, were studied by Varma [48] for $m=2,3,4$. For example, Varma [48] proved the following inequality:

$$\|P'\|^2 \leqslant \frac{1}{2(2n-1)}\|P''\|^2+\frac{2n^2}{2n-1}\|P\|^2, \tag{3.3}$$

for all polynomials $P(t)\in\mathscr{P}_n$, with equality case if and only if $P(t)=cH_n(t)$, where $H_n(t)$ is the Hermite polynomial of degree n and c is an arbitrary real constant.

Bojanov and Varma [13] proved the following result.

Theorem 3.1. *Let* $0<k<m\leqslant n$ *be integers. For every* $P(t)\in\mathscr{P}_n$ *and any* A *such that*

$$A\leqslant \frac{k}{m2^{m-k}}\cdot\frac{1}{(n-k)\cdots(n-m+1)},$$

the inequality

$$\|P^{(k)}\|^2 \leqslant A\|P^{(m)}\|^2+\left\{2^k\binom{n}{k}k!-A2^m\binom{n}{m}m!\right\}\|P\|^2. \tag{3.4}$$

Moreover, by choosing $P(t)=H_n(t)$ *we obtain equality in* (3.4).

The proof was based on the following simple fact.

Lemma 3.2. *Suppose that inequality* (3.2) *holds for* $P(t)=H_k(t)$, $k=0,1,\ldots,n$. *Then it holds for every* $P(t)\in\mathscr{P}_n$.

Recently, Alves and Dimitrov [4] proved a more general result.

Theorem 3.3. *Let $0 < k < m \leqslant n$ be integers and A and B positive constants.*

(i) *If*

$$\frac{A}{B} \leqslant 2^{-m} \frac{(n-m)!}{n!} \frac{k}{m-k},$$

then

$$\|P^{(k)}\|^2 \leqslant \frac{A\|P^{(m)}\|^2 + B\|P\|^2}{A2^{m-k}(n-k)![(n-m)!]^{-1} + B2^{-k}(n-k)!(n!)^{-1}}$$

for every $P(t) \in \mathscr{P}_n$. Moreover, equality is attained if and only if $P(t)$ is a constant multiple of $H_n(t)$.

(ii) *If*

$$\frac{2^{-m}}{(m+1)!} \frac{k}{m-k} \leqslant \frac{A}{B} \leqslant 2^{-m} \frac{k}{m!(m-k)},$$

then

$$\|P^{(k)}\|^2 \leqslant \frac{A\|P^{(m)}\|^2 + B\|P\|^2}{A2^{m-k}(m-k)! + B2^{-k}(m-k)!(m!)^{-1}} \tag{3.5}$$

for every $P(t) \in \mathscr{P}_n$. Moreover, equality is attained if and only if $P(t)$ is a constant multiple of $H_m(t)$.

(iii) *If*

$$\frac{A}{B} > 2^{-m} \frac{k}{m!(m-k)},$$

then

$$\|P^{(k)}\|^2 \leqslant \frac{A\|P^{(m)}\|^2 + B\|P\|^2}{B2^{-k}(m-k-1)![(m-1)!]^{-1}} \tag{3.6}$$

for every $P(t) \in \mathscr{P}_n$. Moreover, equality is attained if and only if $P(t)$ is a constant multiple of $H_{m-1}(t)$.

(iv) *If $A/B = 2^{-m}k/((m-k)m!)$, then the inequalities* (3.5) *and* (3.6) *coincide and they hold for every $P(t) \in \mathscr{P}_n$. In this case equality is attained if and only if $P(t)$ is any linear combination of $H_{m-1}(t)$ and $H_m(t)$.*

Notice that Theorem 3.1 is an immediate consequence of the statement (i) in the previous theorem. In the case $k = 1$ and $m = 2$, Alves and Dimitrov [4] provided a complete characterization of the positive constants A and B, for which the corresponding Landau type polynomial inequalities hold. For example, if $0 < A/B < (4n(n-1))^{-1}$, then

$$\|P'\|^2 \leqslant \frac{A\|P''\|^2 + B\|P\|^2}{2A(n-1) + B(2n)^{-1}},$$

for every $P(t) \in \mathscr{P}_n$. Setting $B = 4n^2 A$ this inequality reduces to Varma's inequality (3.3).

In an unpublished manuscript from 1987, we ([31]) considered a general extremal problem:

For fixed k, m and λ $(1 \leqslant k < m \leqslant n,\ 0 \leqslant \lambda \leqslant 1)$ determine the best constant $C_n \equiv C_n(k, m, \lambda)$ such that

$$\|P^{(k)}\|^2 \leqslant C_n\left(\lambda\|P\|^2 + (1-\lambda)\|P^{(m)}\|^2\right) \tag{3.7}$$

for each $P(t) \in \mathscr{P}_n$, where $\|f\|^2 = (f, f)$, $(f, g) = \int_{\mathbb{R}} f(t)g(t)\,\mathrm{d}\mu(t)$, and $\mathrm{d}\mu(t)$ is a nonnegative measure on the real line, with compact or infinite support for which all moments $\mu_k = \int_{\mathbb{R}} t^k \mathrm{d}\mu(t)$, $k = 0, 1, \ldots$, exist and are finite, and $\mu_0 > 0$.

At first, we can put $\alpha = \lambda C_n$ and $\beta = (1-\lambda)C_n$, so that inequality (3.7) becomes

$$\|P^{(k)}\|^2 - \beta\|P^{(m)}\|^2 \leqslant \alpha\|P\|^2. \tag{3.8}$$

Let $\{p_k(t)\}$ be a system of orthonormal polynomials with respect to the measure $\mathrm{d}\mu(t)$. Then for an arbitrary polynomial $P(t) \in \mathscr{P}_n$, we have expressions

$$P(t) = \sum_{i=0}^{n} c_i p_i(t) \quad \text{and} \quad P^{(k)}(t) = \sum_{i=k}^{n} c_i p_i^{(k)}(t).$$

Therefore

$$\|P\|^2 = \sum_{i=0}^{n} c_i^2$$

and

$$F = \|P^{(k)}\|^2 - \beta\|P^{(m)}\|^2 = \sum_{i,j=k}^{n} c_i c_j b_{ij}^{(k)} - \beta \sum_{i,j=m}^{n} c_i c_j b_{ij}^{(m)},$$

where

$$b_{ij}^{(s)} = \int_{\mathbb{R}} p_i^{(s)}(t) p_j^{(s)}(t)\mathrm{d}\mu(t) \quad (s > 0).$$

Let $Q = [q_{ij}]_{k \leqslant i,j \leqslant n}$ be the corresponding symmetric matrix of the quadratic form F, i.e.,

$$F = \langle Q\mathbf{c}, \mathbf{c}\rangle = \mathbf{c}^{\mathrm{T}} Q\mathbf{c}, \quad \mathbf{c} = [c_k, c_{k+1}, \ldots, c_n]^{\mathrm{T}}.$$

Then we have

$$\frac{F}{\|P\|^2} \leqslant \frac{\mathbf{c}^{\mathrm{T}} Q\mathbf{c}}{\mathbf{c}^{\mathrm{T}}\mathbf{c}} \leqslant \lambda_{\max},$$

where $\lambda_{\max} \equiv \lambda_{\max}(\beta)$ is the maximal eigenvalue of the matrix Q. Regarding (3.8) we have now $\alpha = \lambda_{\max}(\beta)$, i.e.,

$$\lambda C_n = \lambda_{\max}((1-\lambda)C_n), \tag{3.9}$$

from which we conclude that *the solution of Eq.* (3.9) *gives the best constant* C_n.

Two special cases $\lambda = 1$ and $\lambda = 0$ are well known.

For $\lambda = 1$ the problem (3.7) reduces to the well-known Markov inequality in L^2-norm

$$\|P^{(k)}\|^2 \leqslant C_n \|P\|^2 \quad (P(t) \in \mathscr{P}_n)$$

(see [33, Chapter 6]).

The case $\lambda = 0$, i.e.,

$$\|P^{(k)}\|^2 \leqslant C_n \|P^{(m)}\|^2 \quad (P(t) \in \mathscr{P}_n),$$

can be reduced to the L^2-inequalities of Turán type (cf. [33, Section 6.2.6]).

In the simplest case with the Hermite measure $\mathrm{d}\mu(t) = \mathrm{e}^{-t^2}\,\mathrm{d}t$ on $(-\infty, +\infty)$, the matrix Q is diagonal

$$Q = \operatorname{diag}\left(b_{k,k}^{(k)}, \ldots, b_{m-1,m-1}^{(k)}, b_{m,m}^{(k)} - \beta b_{m,m}^{(m)}, \ldots, b_{n,n}^{(k)} - \beta b_{n,n}^{(m)}\right),$$

where $b_{i,j}^{(s)} = 2^s s! \binom{i}{s}$.

Let $\varphi_i(\beta) = 2^k k! \binom{i}{k} - \beta 2^m m! \binom{i}{m}$, $m \leqslant i \leqslant n$. Then

$$\lambda_{\max}(\beta) = \max\left\{2^k k! \binom{m-1}{m}, \max_{m \leqslant i \leqslant n} \varphi_i(\beta)\right\}. \tag{3.10}$$

In the case $k = 1$, $m = 2$, (3.10) reduces to

$$\lambda_{\max}(\beta) = \max\left\{2^k k! \binom{m-1}{m}, \max_{2 \leqslant i \leqslant n} (2i - 4i(i-1)\beta)\right\}.$$

We see that for $\beta = (1-\lambda)C_n \leqslant \frac{1}{2}(2n-1)^{-1}$,

$$\lambda_{\max}(\beta) = 2n[1 - 2(n-1)\beta],$$

and then

$$C_n = \frac{2n}{\lambda + 4n(n-1)(1-\lambda)} \quad \left(\frac{4n^2}{1+4n^2} \leqslant \lambda \leqslant 1\right).$$

For example, if $\lambda = 4n^2/(1+4n^2)$ we get Varma's inequality (3.3). For $\lambda = 1$ we have the classical Schmidt's result [37].

The following characterization of the classical orthogonal polynomials was given by Agarwal and Milovanović [2].

Theorem 3.4. *For all* $P(t) \in \mathscr{P}_n$ *the inequality*

$$(2\lambda_n + B'(0))\|\sqrt{A}P'\|^2 \leqslant \|AP''\|^2 + \lambda_n^2 \|P\|^2 \tag{3.11}$$

holds, with equality if only if $P(t) = cQ_n(t)$, *where* $Q_n(t)$ *is the classical orthogonal polynomial of degree n orthogonal to all polynomials of degree* $\leqslant n-1$ *with respect to the weight function* $t \mapsto w(t)$ *on* (a,b), *and c is an arbitrary real constant. The* $\lambda_n, A(t)$ *and* $B(t)$ *are given in Table* 1.

In order to prove (3.11) we use the differential equation (2.2). So, we have

$$\begin{aligned}\|L[P]\|^2 = {}& \|AP''\|^2 + \|BP'\|^2 + \lambda_n^2\|P\|^2 + 2(AP'',BP') + 2\lambda_n(AP'',P) \\ & + 2\lambda_n(BP',P).\end{aligned}$$

A simple application of integration by parts gives

$$2(AP'',BP') = -B'(0)\|\sqrt{A}P'\|^2 - \|BP'\|^2$$

and

$$\|\sqrt{A}P'\|^2 = -(AP'',P) - (BP',P).$$

Then we find

$$\|L[P]\|^2 = \|AP''\|^2 - B'(0)\|\sqrt{A}P'\|^2 + \lambda_n^2\|P\|^2 - 2\lambda_n\|\sqrt{A}P'\|^2.$$

Since $\|L[P_n]\| \geqslant 0$, we obtain (3.11).

It is easy to see that the equality case is given by $P_n(t) = cQ_n(t)$. Namely, the polynomial solution of Eq. (2.2) is only $cQ_n(t)$, where c is a constant.

We mention now some special cases.

First, for $w(t) = e^{-t^2}$ on $(-\infty,+\infty)$, the inequality (3.11) reduces to Varma's inequality (3.3).

In the generalized Laguerre case, the inequality (3.11) becomes

$$\|\sqrt{t}P'\|^2 \leqslant \frac{n^2}{2n-1}\|P\|^2 + \frac{1}{2n-1}\|tP''\|^2,$$

where $w(t) = t^s e^{-t}$ $(s > -1)$ on $(0,+\infty)$.

In the Jacobi case we get the inequality

$$\begin{aligned}\big((2n-1)(\alpha+\beta) + 2(n^2+n-1)\big)\|\sqrt{1-t^2}P'\|^2 \leqslant {}& n^2(n+\alpha+\beta+1)^2\|P\|^2 \\ & + \|(1-t^2)P''\|^2,\end{aligned}$$

where $w(t) = (1-t)^\alpha(1+t)^\beta (\alpha,\beta > -1)$ on $(-1,1)$.

In the simplest case, when $\alpha = \beta = 0$ (Legendre case), we obtain

$$\|\sqrt{1-t^2}P'\|^2 \leqslant \frac{n^2(n+1)^2}{2(n^2+n-1)}\|P\|^2 + \frac{1}{2(n^2+n-1)}\|(1-t^2)P''\|^2.$$

In Chebyshev case ($\alpha = \beta = -1/2$), we get

$$\|\sqrt{1-t^2}P'\|^2 \leqslant \frac{n^4}{2n^2-1}\|P\|^2 + \frac{1}{2n^2-1}\|(1-t^2)P''\|^2,$$

where $w(t) = 1/\sqrt{1-t^2}$ on $(-1,1)$. Similarly, for $\alpha = \beta = 1/2$ we have

$$\|\sqrt{1-t^2}P'\|^2 \leqslant \frac{n^2(n+2)^2}{2n^2+4n-3}\|P\|^2 + \frac{1}{2n^2+4n-3}\|(1-t^2)P''\|^2,$$

where $w(t) = \sqrt{1-t^2}$ on $(-1,1)$.

4. Weighted polynomial inequalities in L^2-norm and classical orthogonal polynomials

Guessab and Milovanović [20] considered a weighted L^2-analogue of the Bernstein's inequality [9,10], which can be stated in the following form:

$$\|\sqrt{1-t^2}P'(t)\|_\infty \leqslant n\|P\|_\infty. \tag{4.1}$$

Let w be the weight of the classical orthogonal polynomials ($w \in CW$) and $A(t)$ be given as in Table 1. Using the norm $\|f\|_w^2 = (f,f)$, where the inner product (f,g) is defined by (2.1), Guessab and Milovanović [20] solved the following problem connected with the Bernstein's inequality (4.1):

Determine the best constant $C_{n,m}(w)$ $(1 \leqslant m \leqslant n)$ such that the inequality

$$\|A^{m/2}P^{(m)}\|_w \leqslant C_{n,m}(w)\|P\|_w \tag{4.2}$$

holds for all $P(t) \in \mathscr{P}_n$.

Theorem 4.1. *For all $P(t) \in \mathscr{P}_n$ the inequality* (4.2) *holds, with the best constant*

$$C_{n,m}(w) = \sqrt{\lambda_{n,0}\lambda_{n,1}\cdots\lambda_{n,m-1}},$$

where $\lambda_{m,m}$ is given by (2.4). *The equality is attained in* (4.2) *if and only if $P(t)$ is a constant multiple of the classical polynomial $Q_n(t)$ orthogonal with respect to the weight function $w \in CW$.*

Suppose that $P(t) \in \mathscr{P}_n$ and take the corresponding expansion in orthogonal polynomials $\{Q_k(t)\}$, $P(t) = \sum_{v=0}^{n} a_v Q_v(t)$. The main rule in proving this theorem plays the linear functional

$$L_v[P] = \frac{\mathrm{d}}{\mathrm{d}t}\left(A(t)w(t)\frac{\mathrm{d}P(t)}{\mathrm{d}t}\right) + \lambda_v w(t)P(t),$$

where λ_v is defined as in Table 1. Since $L_v[Q_v] \equiv 0$, we get

$$L_n[P] = \sum_{v=0}^{n}(\lambda_n - \lambda_v)a_v w(t)Q_v(t).$$

Using the inner product (2.1), we obtain

$$(w^{-1}L_n[P], P) = \sum_{v=0}^{n}(\lambda_n - \lambda_v)a_v^2\|Q_v\|_w^2. \tag{4.3}$$

On integration by parts, we find that

$$(w^{-1}L_n[P],P) = -\|\sqrt{A}P'\|_w^2 + \lambda_n\|P\|_w^2.$$

Since $\lambda_v \leqslant \lambda_n$ for $v \leqslant n$, from the last equality and (4.3) we conclude that the inequality

$$\|\sqrt{A}P'\|_w \leqslant \sqrt{\lambda_n}\|P\|_w$$

holds. Thus, Theorem 4.1 is true for $m = 1$. Equality case follows from the fact that $(w^{-1}L_n[P],P) = \sum_{v=0}^{n}(\lambda_n - \lambda_v)a_v^2\|Q_v\|_w^2 = 0$ if and only if $a_v = 0$ for $v = 0,1,\ldots,n-1$ and a_n is an arbitrary real constant. Therefore, $P(t) = a_nQ_n(t)$.

Using the corresponding differential equation for kth derivative of $Q_n(t)$, we get the following inequality:

$$\|A^{k/2}P^{(k)}\|_w \leqslant \sqrt{\lambda_{n,k-1}}\|A^{(k-1)/2}P^{(k-1)}\|_w \quad (P(t) \in \mathscr{P}_n)$$

with equality if and only if $P(t) = a_nQ_n(t)$.

Finally, iterating this inequality for $k = 1,\ldots,m$, we finish the proof.

In some special cases we have (see [20]):

(1) Let $w(t) = (1-t)^\alpha(1+t)^\beta$ $(\alpha,\beta > -1)$ on $(-1,1)$ (Jacobi case). Then

$$\|(1-t^2)^{m/2}P^{(m)}\|_w \leqslant \sqrt{\frac{n!\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\Gamma(n+\alpha+\beta+1)}}\|P\|_w, \tag{4.4}$$

with equality if and only if $P(t) = cP_n^{(\alpha,\beta)}(t)$.

Daugavet and Rafal'son [12] and Konjagin [26] considered the extremal problems of the form

$$\|P^{(m)}\|_{p,v} \leqslant A_{n,m}(r,\mu;p,v)\|P\|_{r,\mu} \quad (P(t) \in \mathscr{P}_n),$$

where

$$\|f\|_{r,v} = \begin{cases} \left(\int_{-1}^{1} |f(t)(1-t^2)^\mu|^r \,\mathrm{d}t\right)^{1/r}, & 0 \leqslant r < +\infty, \\ \operatorname*{ess\,sup}_{-1\leqslant t\leqslant 1}|f(t)|(1-t^2)^\mu, & r = +\infty. \end{cases}$$

The case when $p = r \geqslant 1$, $\mu = v = 0$, and $m = 1$, was considered by Hille et al. [23]. The exact constant $A_{n,m}(r,\mu;p,v)$ is known in a few cases, for example, $A_{n,1}(+\infty,0;1,0) = 2n$ and

$$A_{n,m}(2,\mu;2,\mu+m/2) = \sqrt{\frac{n!\Gamma(n+4\mu+m+1)}{(n-m)!\Gamma(n+4\mu+1)}}.$$

The last case, in fact, is the previous result (4.4) with the Gegenbauer weight $(\alpha = \beta = 2\mu)$.

(2) Let $w(t) = t^s\mathrm{e}^{-t}$ $(s > -1)$ on $(0,+\infty)$ (generalized Laguerre case). Then

$$\|t^{m/2}P^{(m)}\|_w \leqslant \sqrt{n!/(n-m)!}\|P\|_w,$$

with equality if and only if $P(t) = cL_n^s(t)$.

(3) The Hermite case with the weight $w(t) = \mathrm{e}^{-t^2}$ on $(-\infty, +\infty)$ is the simplest. Then the best constant is $C_{n,m}(w) = 2^{m/2}\sqrt{n!/(n-m)!}$.
This result can be found in Ph.D. Thesis of Shampine [38] (see also [14,30]).
The case $m = 1$ was investigated by Schmidt [37] and Turán [46].
Recently, Guessab [18] obtained sharp Markov–Bernstein inequalities in L^2 norms that are weighted with classical weights.

Theorem 4.2. *Let $P(t) \in \mathscr{P}_n$ and $w \in CW$. Then*

$$\|w^{-1/2}(V(t)P)'\|_w^2 + \|\sqrt{A(t)C(t)}P\|_w^2 \leqslant \beta_n\|P\|_w^2, \tag{4.5}$$

where $A(t)$ and λ_n are given in Table 1, *and $V(t) = \sqrt{A(t)w(t)}$,*

$$C(t) = \begin{cases} \dfrac{1}{4}\left(\dfrac{\alpha^2-1}{(1-t)^2} + \dfrac{\beta^2-1}{(1+t)^2}\right), & \textit{Jacobi case}, \\ \dfrac{1}{4}\left(\dfrac{s^2-1}{t^2} + 1\right), & \textit{generalized Laguerre case}, \\ t^2, & \textit{Hermite case}, \end{cases}$$

and

$$\beta_n = \lambda_n + \begin{cases} \dfrac{1}{2}(\alpha+1)(\beta+1), & \textit{Jacobi case}, \\ \dfrac{1}{2}(s+1), & \textit{generalized Laguerre case}, \\ 1, & \textit{Hermite case}. \end{cases}$$

The equality is attained in (4.5) *if and only if $P(t)$ is a constant multiple of the classical polynomial $Q_n(t)$ orthogonal with respect to the weight function $t \mapsto w(t)$.*

This elegant result was established by using the second-order Sturm–Liouville type differential equations satisfied by the classical orthogonal polynomials.

Using the method from [20], Guessab [19] has investigated the extremal problem

$$\max_{P(t)\in\mathscr{P}_n^1} \left\|\left(\sqrt{A}/w_m\right)(w_m P^{(m)})'\right\|_{w_m},$$

where $w \in CW$, $w_m = A^m w$, $\mathscr{P}_n^1 = \left\{P \in \mathscr{P}_n \middle| \|P\|_{w_m} \leqslant 1\right\}$, and

$$\|f\|_{w_m} = \left(\int_a^b w_m(t)|f(t)|^2\,\mathrm{d}t\right)^{1/2}.$$

Theorem 4.3. *Let $P(t) \in \mathscr{P}_n^1$ and $w \in CW$. Then*

$$\left\|\left(\sqrt{A}/w_m\right)\left(w_m P^{(m)}\right)'\right\|_{w_m} \leqslant \sqrt{\lambda_{n,0}\lambda_{n,1}\cdots\lambda_{n,m-1}\beta_{n,m}}, \tag{4.6}$$

where $\lambda_{n,\nu}$ *is given in Theorem* 4.1, $\beta_{n,m} = \lambda_{n,m} + B'(0) + (k-1)A''(0)$, *and* $A(t)$ *and* $B(t)$ *are given in Table* 1.

The equality is attained in (4.6) *if and only if* $P(t)$ *is a constant multiple of the classical polynomial* $Q_n(t)$ *orthogonal with respect to the weight function* $t \mapsto w(t)$.

At the end we mention the following extremal problem of Markov's type

$$C_{n,m}(w) = \sup_{P(t)\in\mathscr{P}_n} \frac{\|\mathscr{D}_m P\|_w}{\|A^{m/2}P\|_w} \quad (m \geqslant 1)$$

for the differential operator $\mathscr{D}_m$ defined by

$$\mathscr{D}_m P = \frac{\mathrm{d}^m}{\mathrm{d}t^m}[A^m P] \quad (P(t) \in \mathscr{P}_n).$$

The best constant $C_{n,m}(w)$ was found in the following three cases (see [21]):

1. $w(t) = 1$ on $[-1,1]$ (the Legendre weight):

   $$C_{n,m}(w) = \sqrt{\frac{(n+2m)!}{n!}},$$

   with the extremal polynomial $P^*(t) = \gamma C_n^{m+1/2}(t)$, where $C_n^{\mu}(t)$ is the Gegenbauer polynomial of degree n;
2. $w(t) = \mathrm{e}^{-t}$ on $[0,+\infty)$ (the Laguerre weight):

   $$C_{n,m}(w) = \sqrt{\frac{(n+m)!}{n!}},$$

   with the extremal polynomial $P^*(t) = \gamma L_n^m(t)$, where $L_n^m(t)$ is the generalized Laguerre polynomial of degree n;
3. $w(t) = \mathrm{e}^{-t^2}$ on $(-\infty,+\infty)$ (the Hermite weight):

   $$C_{n,m}(\mathrm{d}\lambda) = 2^{m/2}\sqrt{n!/(n-m)!},$$

   with the extremal polynomial $P^*(t) = \gamma H_n(t)$, where $H_n(t)$ is the Hermite polynomial of degree n.

   Some extremal problems for differential operators were also investigated by Stein [42] and Džafarov [15].

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