Bulletin T.CXLV de l'Académie serbe des sciences et des arts − 2013
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 38

QUADRATURE PROCESSES AND NEW APPLICATIONS

G. V. MILOVANOVIĆ

(Inaugural lecture delivered on May 31, 2013 at the Serbian Academy of Sciences and Arts in Belgrade)

A b s t r a c t. This survey on quadrature processes and their applications is an extended version of my public lecture given in the Serbian Academy of Sciences and Arts under the same title and it represents a continuation of my "2007–lecture" under the title "Quadrature processes – development and new directions" [*Bull. Cl. Sci. Math. Nat. Sci. Math.* **33** (2008), 11–41]). *Beside the basic concept on the constructive theory of orthogonal polynomials on* $\mathbb{R}$*, an account on Gaussian quadratures for nonclassical weights on the real semiaxis and their applications in summation of slowly convergent series, generalized Birkhoff–Young quadratures, nonstandard quadratures of Gaussian type* (*interval quadratures and quadratures based on operator values*)*, as well as on some applications in solving Fredholm integral equations of the second kind in one and two dimensions are presented. The presented results have been obtained in our recent papers.*

AMS Mathematics Subject Classification (2000): 41A55, 33C45, 33C47, 42C05, 65D30, 65D32
Key Words: Quadrature process; Gauss–Christoffel quadrature formula; Orthogonal polynomials; Moments; Three–term recurrence relation; Weight;

Node; Summation of series; Non–standard quadratures; Interval quadratures; Fredholm integral equation of the second kind.

1. *Introduction*

This is a survey on some results on quadrature processes and their applications, which I have obtained together with my collaborators or alone in the last period and it represents an extended version of my public lecture given recently in the Serbian Academy of Sciences and Arts (May 20, 2013). An account of the classical Newton–Cotes rules, Gauss–Christoffel quadratures and quadratures with multiple nodes (cf. [40, 27, 44, 77], [61] –[65]), as well as several their generalizations and extensions (cf. [43, 53]), was given in my previous lecture [45] (delivered at the 7th Meeting of the Department of Mathematics, Physics and Geo Sciences in the Serbian Academy of Sciences and Arts, October 26, 2007), so that this survey is its continuation.

The paper is organized as follows. In Section 2 we give a connection between Gaussian type of quadratures and orthogonal polynomials and describe a role of the fundamental three–term recurence relation for orthogonal polynomials, as well as the basic concept on the constructive theory of orthogonal polynomials on $\mathbb{R}$. Section 3 is devoted to Gauss–Christoffel quadrature formulae for non–classical weight functions on the real semiaxis. Using recent progress in symbolic computation and variable–precision arithmetic we show how to generate coefficients in the three–term recurrence relation directly by using the original Chebyshev method of moments in sufficiently high precision or even in symbolic form. Five interesting types of weight functions on $\mathbb{R}_+$ are investigated. Two methods for summation of slowly convergent series are presented in Section 4. Generalized Birkhoff–Young interpolatory quadrature formulae for weighted integrals of analytic functions in the complex plane are studied in Section 5. Their node polynomials can be interpreted in terms of the type II multiple orthogonal polynomials. Two kinds of nonstandard quadratures – interval quadratures of Gaussian type and Gaussian quadratures based on operator values – are considered in Section 6. Finally, Section 7 is devoted to Fredholm integral equations of the second kind. Beside an important one–dimensional case of the integral equation on the finite interval $\mathcal{D} = A = [-1, 1]$, with respect to the Jacobi weight, we consider also a two–dimensional case on a triangle $\mathcal{D} = \mathrm{T}$, which can be reduced to the square $\mathcal{D} = \mathrm{Q} = A^2$. The proposed methods are very efficient and they are based on the recent progress in polynomial interpolation (cf. [35]).

2. *Gauss–Christoffel quadratures and orthogonal polynomials*

In 1814 C.F. Gauss [14] developed his famous method of numerical integration which dramatically improves the earlier method of Newton and Cotes. This discovery was the most significant event of the 19th century in the field of numerical integration and perhaps in all of numerical analysis. An elegant alternative derivation of these formulas was provided by Jacobi, and a significant generalization to arbitrary measures was given by Christoffel, and therefore, today these formulae with maximal degree of precision are known as the *Gauss–Christoffel quadrature formulae.* Their error term and convergence were proved by Markov and Stieltjes, respectively. It was only in 1928 Uspensky gave the first proof for the convergence of Gaussian formula on unbounded intervals with the classical measures of Laguerre and Hermite. A nice survey of Gauss–Christoffel quadrature formulae was written by Gautschi [15].

In modern terminology, the formulation of this classical theory can be given as follows.

Let $\mathcal{P}$ be the space of real polynomials and $\mathcal{P}_n \subset \mathcal{P}$ the space of polynomials of degree at most n. Suppose $\mathrm{d}\mu(t)$ is a positive measure on $\mathbb{R}$ with finite or unbounded support, for which all moments $\mu_k = \int_{\mathbb{R}} t^k \mathrm{d}\mu(t)$ exist and are finite, and $\mu_0 > 0$. Then, for each $n \in \mathbb{N}$, there exists the n–point Gauss–Christoffel quadrature formula

$$\int_{\mathbb{R}} f(t)\mathrm{d}\mu(t) = \sum_{k=1}^{n} A_k f(\tau_k) + R_n(f), \tag{2.1}$$

which is exact for all algebraic polynomials of degree at most $2n-1$, i.e., $R_n(f) = 0$ for each $f \in \mathcal{P}_{2n-1}$.

The Gauss–Christoffel quadrature formula (2.1) can be characterized as an interpolatory formula for which its *node polynomial* $\pi_n(t) = \prod_{k=1}^{n}(t-\tau_k)$ is orthogonal to $\mathcal{P}_{n-1}$ with respect to the inner product defined by

$$(p,q) = \int_{\mathbb{R}} p(t)q(t)\mathrm{d}\mu(t) \quad (p,q \in \mathcal{P}). \tag{2.2}$$

Therefore, orthogonal polynomials play an important role and they are today the basic tool in this theory. The inner product (2.2) gives rise to a unique system of monic orthogonal polynomials $\pi_k(\,\cdot\,) = \pi_k(\,\cdot\,;\mathrm{d}\mu)$, such that

$$\pi_k(t) \equiv \pi_k(\mathrm{d}\mu;t) = t^k + \text{terms of lower degree}, \quad k = 0,1,\ldots,$$

and

$$(\pi_k, \pi_n) = ||\pi_n||^2 \delta_{kn} = \begin{cases} 0, & n \neq k, \\ ||\pi_n||^2, & n = k. \end{cases}$$

2.1. *Fundamental three–term recurrence relation.* Because of the property $(tp, q) = (p, tq)$, these orthogonal polynomials satisfy the three–term recurrence relation

$$\pi_{k+1}(t) = (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \quad k = 0, 1, 2 \ldots, \tag{2.3}$$

with $\pi_0(t) = 1$ and $\pi_{-1}(t) = 0$, where $(\alpha_k) = (\alpha_k(\mathrm{d}\mu))$ and $(\beta_k) = (\beta_k(\mathrm{d}\mu))$ are sequences of recursion coefficients which depend on the measure $\mathrm{d}\mu$. The coefficient β_0 may be arbitrary, but is conveniently defined by $\beta_0 = \mu_0 = \int_{\mathbb{R}} \mathrm{d}\mu(t)$.

There are many reasons way the coefficients α_k and β_k in the three–term recurrence relation (2.3) are fundamental quantities in the constructive theory of orthogonal polynomials (for details see [17]).

First, α_k and β_k provide a compact way of representing and easily calculating orthogonal polynomials, their derivatives, and their linear combinations, requiring only a linear array of parameters. Also, the same recursion coefficients α_k and β_k appear in the *Jacobi continued fraction associated with the measure* $\mathrm{d}\mu$,

$$F(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{z - t} \sim \frac{\beta_0}{z - \alpha_0 -} \, \frac{\beta_1}{z - \alpha_1 -} \cdots,$$

which is known as the *Stieltjes transform* of the measure $\mathrm{d}\mu$ (for details see [22, p. 15], [35, p. 114]). For the n-th *convergent* of this continued fraction, it is easy to see that

$$\frac{\beta_0}{z - \alpha_0 -} \, \frac{\beta_1}{z - \alpha_1 -} \cdots \frac{\beta_{n-1}}{z - \alpha_{n-1}} = \frac{\sigma_n(z)}{\pi_n(z)}, \tag{2.4}$$

where σ_n are the so–called *associated polynomials*, defined by

$$\sigma_k(z) = \int_{\mathbb{R}} \frac{\pi_k(z) - \pi_k(t)}{z - t} \, \mathrm{d}\mu(t), \quad k \geq 0,$$

as well as that these polynomials satisfy the same fundamental relation (2.3), i.e.,

$$\sigma_{k+1}(z) = (z - \alpha_k)\sigma_k(z) - \beta_k \sigma_{k-1}(z), \quad k \geq 0,$$

with starting values $\sigma_0(z) = 0$, $\sigma_{-1}(z) = -1$.

The function of the second kind,

$$\varrho_k(z) = \int_{\mathbb{R}} \frac{\pi_k(t)}{z-t}\,\mathrm{d}\mu(t), \quad k \geq 0,$$

where z is outside the spectrum of $\mathrm{d}\mu$, also satisfy the same three–term recurrence relation (2.3) and represent its *minimal solution*, normalized by $\varrho_{-1}(z) = 1$, as observed by Gautschi in [16].

Notice that the rational function (2.4) has simple poles at the zeros $z = x_{n,k}$, $k = 1, \ldots, n$, of the polynomial $\pi_n(t)$. By $\lambda_{n,k}$ we denote the corresponding residues, i.e.,

$$\lambda_{n,k} = \lim_{z \to x_{n,k}} (z - x_{n,k}) \frac{\sigma_n(z)}{\pi_n(z)} = \frac{1}{\pi_n'(x_{n,k})} \int_{\mathbb{R}} \frac{\pi_n(t)}{t - x_{n,k}}\,\mathrm{d}\mu(t),$$

so that the continued fraction representation (2.4) gets the following form

$$\frac{\sigma_n(x)}{\pi_n(x)} = \sum_{k=1}^{n} \frac{\lambda_{n,k}}{x - x_{n,k}}\,.$$

The coefficients $\lambda_{n,k}$ are exactly the weight coefficients (Christoffel numbers) in the Gauss–Christoffel quadrature formula (2.1) and they can be expressed by the so–called Christoffel function $\lambda_n(\mathrm{d}\mu; t)$ (cf. [35, Chapters 2 & 5]) in the form

$$A_k = \lambda_n(\mathrm{d}\mu; \tau_k), \quad k = 1, \ldots, n,$$

and zeros of the polynomial $\pi_n(t)$ are the nodes of (2.1), i.e., $\tau_k = x_{n,k}$, $k = 1, \ldots, n$.

Using procedures of numerical linear algebra, notably the QR or QL algorithm, it is easy to compute the zeros of the orthogonal polynomials $\pi_n(t)$ rapidly and efficiently as eigenvalues of the *Jacobi matrix* of order n associated with the measure $\mathrm{d}\mu$,

$$J_n(\mathrm{d}\mu) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathbf{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}.$$

The first components of the corresponding normalized eigenvectors $\mathbf{v}_k = [v_{k,1} \ \dots \ v_{k,n}]^{\mathrm{T}}$ ($\mathbf{v}_k^{\mathrm{T}}\mathbf{v}_k = 1$) give also immediately the Christoffel numbers $A_k = \lambda_{n,k} = \beta_0 v_{k,1}^2$, $k = 1, \dots, n$, where $\beta_0 = \mu_0 = \int_{\mathbb{R}} \mathrm{d}\mu(t)$ (cf. Golub and Welsch [29]).

Unfortunately, the recursion coefficients are known explicitly only for some narrow classes of orthogonal polynomials. One of the most important classes for which these coefficients are known explicitly are surely the so–called *very classical* orthogonal polynomials (Jacobi, the generalized Laguerre, and Hermite polynomials), which appear frequently in applied analysis and computational sciences. Orthogonal polynomials for which the recursion coefficients are not known we call *strongly non–classical polynomials.* For these, if we know how to compute the first n recursion coefficients α_k and β_k, $k = 0, 1, \dots, n-1$, then we can compute all orthogonal polynomials of degree at most n by a straightforward application of the three–term recurrence relation (2.3), construct the corresponding Gauss–Christoffel quadratures for any number of nodes less than or equal to n, etc.

2.2. *Constructive theory of orthogonal polynomials on* $\mathbb{R}$. In [17] Walter Gautschi starts with an arbitrary positive measure $\mathrm{d}\mu(t)$, which is given explicitly or implicitly via moment information, and considers the actual (numerical) construction of orthogonal polynomials as a basic computational problem: *For a given measure* $\mathrm{d}\mu$ *and for given* $n \in \mathbb{N}$, *generate the first coefficients* $\alpha_k(\mathrm{d}\mu)$ *and* $\beta_k(\mathrm{d}\mu)$, $k = 0, 1, \dots, n-1$. In about two dozen papers, he developed the so–called *constructive theory of orthogonal polynomials on* $\mathbb{R}$, including effective algorithms for numerically generating orthogonal polynomials, a detailed stability analysis of such algorithms, the corresponding software implementation, etc. (cf. [19], [48], [49]). Our collaboration started in that time (precisely in 1983) and the story about it has recently been told by Walter Gautschi [25] on the occasion of my 60th anniversary. I was then in my thirties, so his influence to my scientific work and my further development was of crucial importance; for this I am very grateful to Walter Gautschi!

In the numerical construction of recursion coefficients an important aspect is the sensitivity of the problem with respect to small perturbation in the data (e.g., perturbations in the first $2n$ moments μ_k, $k = 0, 1, \dots, 2n-1$, when we calculate the coefficients for $k \leq n-1$). There is a simple algorithm, due to Chebyshev, which transforms the moments to desired recursion coefficients, $[\mu_k]_{k=0}^{2n-1} \mapsto [\alpha_k, \beta_k]_{k=0}^{n-1}$, but its viability is strictly dependent on the conditioning of this mapping. Usually it is severely ill conditioned so that these calculatios via moments, in finite precision on a computer, are quite ineffective, especially for measures on unbounded supports. The only salvation, in this case, is to either use symbolic computation, which however requires special resources and often is not possible, or else to use the explicit form of the measure. In the latter case, an appropriate discretization of the measure and subsequent approximation of the recursion coefficients is a viable alternative.

There are three basic procedures for generating these recursion coefficients: (1) the *method of* (*modified*) *moments*, (2) the *discretized Stieltjes–Gautschi procedure*, and (3) the *Lanczos algorithm*, and they play the central role in the constructive theory of orthogonal polynomials. The basic references are [17], [20], [22], and [35].

Remark 2.1 In this paper we restrict our attention only to the case of orthogonal algebraic polynomials and quadrature rules with maximal algebraic degree of exactness. Also, one can consider orthogonality and Gaussian type quadrature in some other functional spaces. For example, for quadrature rules with maximal trigonometric degree of exactness and orthogonal systems of trigonometric polynomials (of integer or semi–integer degree of exactness) readers are referred to [69] –[75].

3. *Gauss–Christoffel quadratures for non–classical weights on* $\mathbb{R}_+$

3.1. *Gaussian quadraures on the real semiaxis.* In this section we consider Gauss–Christoffel quadratures (2.1) on real semiaxis $(0,+\infty)$ for absolutely continuous measures, which can be expressed as $\mathrm{d}\mu(t) = w(t)\mathrm{d}t$, where the weight function $t \mapsto w(t)$ is non–negative and measurable in Lebesgue's sense for which all moments exists and $\mu_0 > 0$. Numerical construction of (Gaussian) quadrature parameters (the nodes τ_k and Christoffel numbers A_k, $k = 1, \ldots, n$) in

$$\int_0^{+\infty} f(t)w(t)\mathrm{d}t = \sum_{k=1}^{n} A_k f(\tau_k) + R_n(f),$$

requires knowledge of the first n recursive coefficients α_k and β_k, $k = 0, 1, \ldots, n-1$.

Recent progress in *symbolic computation* and *variable–precision arithmetic* now makes it possible to generate the coefficients α_k and β_k in the three–term recurrence relation (2.3) directly by using the original Chebyshev method of moments in sufficiently high precision or even in symbolic form. Respectively symbolic/variable–precision software for orthogonal polynomials is available: Gautschi's package `SOPQ` in Matlab (cf. [49]) and our Mathematica package `OrthogonalPolynomials` [8], [58]. Thus, all that is required is a procedure for symbolic or numerical calculation of the moments in variable–precision arithmetic. Such an approach enables us to overcome the numerical instability. The Gaussian parameters can be obtained very easy by the stable Golub–Welsch procedure [29], realized in the Mathematica Package `OrthogonalPolynomials` as the function `aGaussianNodesWeights`, which has different calling formats (see [8], [58]).

In the next subsection we consider a few very important cases of the weights on the real semiaxis $\mathbb{R}_+$. Gaussian quadratures with respect to such weights can be used in diverse areas of applied and numerical analysis, as well as in many other areas of applied and computational sciences. For example, they can be applied for computing special functions (Airy functions, modified Bessel functions of imaginary order, parabolic cilinder functions, etc.), by selecting their suitable integral representations (cf. [21], [28]), summation of slowly convergent series (see [26], [41], [42], [47]), integral equations, probability, approximation theory, etc.

3.2. *Variable–precision recurrence coefficients.* In this subsection we give five cases of interesting non–classical weights on $(0,+\infty)$ for which we can calculate the moments in symbolic form and then obtain the recurrence coefficients with an arbitrary precision.

1° *One side exponential weight* $w(t) = t^\gamma \exp(-t^\beta)$ or *the half–range Freud weight function*, with $\gamma > -1$ and $\beta > 0$. The moments are given by

$$\mu_k = \int_0^{+\infty} t^k w(t)\mathrm{d}t = \frac{1}{\beta}\Gamma\left(\frac{k+\gamma+1}{\beta}\right), \quad k \in \mathbb{N}_0.$$

For $\beta = 1$ it reduces to the classical generalized Laguerre case, and for $\gamma = 0$ and $\beta = 2$ to the case with the half–range Hermite weight function.

Gamma function can be evaluated to arbitrary numerical precision in Mathematica (see [85]). To obtain the three–term recursion coefficients using our package `OrthogonalPolynomials`, for example for $\gamma = 1/2$ and $\beta = 4$ and $n \le 40$ with `WorkingPrecision->80`, one only needs to execute the following commands:

```
<< orthogonalPolynomials`
  gamma=1/2; beta=4; mom = Table[Gamma[(k+gamma+1)/beta],{k,0,80}];
  {al,be}=aChebyshevAlgorithm[mom, WorkingPrecision -> 80];
```

Taking the `WorkingPrecision` sufficiently large, for example to be `120`, we get that the maximal relative error in the previous obtained recursion coefficients is 4.85×10^{-39} and conclude that at least 38 decimal digits in the `{al,be}` are exact. It means that we can compute the parameters (nodes and weights) in all n–point Gaussian formulae for $n \leq 40$ with the same precision, because the Golub–Welsch algorithm is well–conditioned.

Recently, quadratures with these exponential weights have been used in [30].

2° *Bose–Einstein's weight* $w(t) = \varepsilon(t) = t/(\mathrm{e}^t - 1)$ on $(0, +\infty)$. The moments are

$$\mu_k(\varepsilon) = \int_0^{+\infty} t^k w(t)\mathrm{d}t = (k+1)!\zeta(k+2), \quad k \in \mathbb{N}_0,$$

where the zeta function can be evaluated to arbitrary numerical precision. Furthermore, for certain special arguments, `Zeta` (in Mathematica) automatically evaluates to exact values. Thus, as in the previous case, a direct application of the Chebyshev method of moments gives recursion coefficients, as well as the parameters of quadratures.

A general problem with the weight function $w(t) = [\varepsilon(t)]^r$, where $r \in \mathbb{N}$, can be also consider in a similar way. In that case, the corresponding moments $\mu_k^{(r)}(\varepsilon)$, $r > 1$, can be obtained recursively by

$$\mu_k^{(r)}(\varepsilon) = \frac{k+r}{r-1}\,\mu_k^{(r-1)}(\varepsilon) - \mu_{k+1}^{(r-1)}(\varepsilon).$$

For example, $\mu_k^{(2)}(\varepsilon) = (k+2)![\zeta(k+2) - \zeta(k+3)]$, $k \in \mathbb{N}_0$ (cf. [26], [23]).

Integrals with this weight frequently appear in solid state physics, e.g., the total energy of thermal vibration of a crystal lattice can be expressed in the form $\int_0^{+\infty} f(t)\varepsilon(t)\,\mathrm{d}t$, where $f(t)$ is related to the phonon density of states. Also, integrals of this type can be used for summation of slowly convergent series (cf. [26]).

3° *Fermi–Dirac weight* $w(t) = \varphi(t) = 1/(\mathrm{e}^t + 1)$ on $(0, +\infty)$. The moments are given by

$$\mu_k(\varphi) = \int_0^{+\infty} \frac{t^k}{\mathrm{e}^t + 1}\,\mathrm{d}t = \begin{cases} \log 2, & k = 0, \\ (1 - 2^{-k})k!\zeta(k+1), & k > 0. \end{cases}$$

Integrals with this weight are encountered in the dynamics of electrons in metals, as well as in summation of the slowly convergent series (see [26]).

Gaussian quadratures with respect to the next two hyperbolic weights can be applied in summation of slowly convergent series (see [41], [42], [47]).

4° *Hyperbolic weights* $w_1(t) = 1/\cosh^2 t$ *and* $w_1(t) = \sinh t/\cosh^2 t$ *on* $(0, +\infty)$. The moments can be calculated exactly as

$$\mu_k^{(1)} = \int_0^{+\infty} t^k w_1(t)\mathrm{d}t = \begin{cases} 1, & k = 0, \\ \log 2, & k = 1, \\ C_k\zeta(k), & k \geq 2, \end{cases} \tag{3.1}$$

where $C_k = (2^{k-1} - 1)k!/4^{k-1}$ (see [58]), and

$$\mu_k^{(2)} = \int_0^{+\infty} t^k w_2(t)\mathrm{d}t = \begin{cases} 1, & k = 0, \\ k\left(\dfrac{\pi}{2}\right)^k |E_{k-1}|, & k\,(\text{odd}) \geq 1, \\ \dfrac{2k}{4^k}\left(\psi^{(k-1)}(1/4) - \psi^{(k-1)}(3/4)\right), & k\,(\text{even}) \geq 2, \end{cases}$$

where E_k are Euler's numbers, defined by the generating function

$$\frac{2}{\mathrm{e}^t + \mathrm{e}^{-t}} = \sum_{k=0}^{+\infty} E_k \frac{t^k}{k!},$$

and $\psi(z)$ is the so–called digamma function, i.e., the logarithmic derivative of the gamma function, given by $\psi(z) = \Gamma'(z)/\Gamma(z)$.

Mathematica evaluates derivatives $\psi^{(n)}(z)$ to arbitrary numerical precision, using the function `PolyGamma[n,z]`. In our case, executing the following commands:

```
<< orthogonalPolynomials`
   mom=Join[{1},Table[If[OddQ[k],k(Pi/2)^k Abs[EulerE[k-1]],
       2k/4^k(PolyGamma[k-1,1/4]-PolyGamma[k-1,3/4])],{k,1,99}]];
   {al,be}=aChebyshevAlgorithm[mom, WorkingPrecision -> 80];
```

we obtain the first 50 recurrence coefficients with the maximal relative error of 2.51×10^{-43}.

5° *The weight* $w^{(\alpha,\beta)}(t) = \exp(-t^{-\alpha} - t^{\beta})$, $\alpha, \beta > 0$, *on* $(0, +\infty)$. In the case $\alpha = \beta$, the moments are

$$\mu_k^{(\beta,\beta)} = \int_0^{+\infty} t^k w(t)\mathrm{d}t = \frac{2}{\beta} K_{(k+1)/\beta}(2), \quad k \in \mathbb{N}_0,$$

where $K_r(z)$ is the modified Bessel function of the second kind. In the MATHEMATICA package this function is implemented as `BesselK[r,z]`, and its value can be evaluated with an arbitrary precision. As we have recently shown in [38], the calculation of the recursive coefficients is a very sensitive process. For example, if we need the first $n = 100$ coefficients for $\beta = 2$, with relative errors less than $\varepsilon = 10^{-52}$, then it is enough to put

```
<< orthogonalPolynomials`
    beta=2; mom=Table[2/beta BesselK[(k+1)/beta,2], {k,0,199}];
    {al,be}=aChebyshevAlgorithm[mom, WorkingPrecision -> 150];
```

and then, in the worst case, the process causes a loss of about 98 decimal digits!

The case $\alpha \neq \beta$ is more complicated than the previous one for $\alpha = \beta$, especially for symbolic computations. However, in some cases for integer (or rational) values of parameters, the moments can be expressed in terms of the Meijer G–function. In a standard case, the Meijer G–function is defined as (cf. [4, p. 207])

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) \equiv G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1,\ldots,a_n; a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m; b_{m+1},\ldots,b_q \end{matrix}\right)$$

$$= \frac{1}{2\pi \mathrm{i}} \int_L \frac{\prod\limits_{\nu=1}^{m} \Gamma(b_\nu - s) \prod\limits_{\nu=1}^{n} \Gamma(1 - a_\nu + s)}{\prod\limits_{\nu=m+1}^{q} \Gamma(1 - b_\nu + s) \prod\limits_{\nu=n+1}^{p} \Gamma(a_\nu - s)} z^s \,\mathrm{d}s,$$

where an empty product is interpreted as 1, $1 \leq m \leq q$, $1 \leq n \leq p$, and parameters a_ν and b_ν are such that no pole of $\Gamma(b_\nu - s)$, $\nu = 1,\ldots,m$, coincides with any pole of $\Gamma(1 - b_\mu + s)$, $\mu = 1,\ldots,n$. Roughly speaking, the contour L separates the poles of functions $\Gamma(b_1 - s)$, $\ldots$, $\Gamma(b_m - s)$ from the poles of $\Gamma(1 - a_1 + s)$, $\ldots$, $\Gamma(1 - a_n + s)$, and a discussion on three different paths of integration is given in [4, p. 207]. An alternative equivalent definition of the Meijer G–function can be done in terms of inverse Mellin transform (cf. [83, p. 793]). The Meijer G–function is a very general function which reduces to simpler special functions in many common cases. In MATHEMATICA, the Meijer G–function is implemented as

```
MeijerG[{{a1,...,an},{an1,,...,ap}},{{b1,...,bm},{bm1,...,bq}},z]
```

and it is suitable for both symbolic and numerical manipulation and its value can be evaluated with an arbitrary precision. In many special cases, `MeijerG` is automatically converted to other functions.

Following [38] we mention here the corresponding moments $\mu_k^{(\alpha,\beta)}$ expressed in terms of the Meijer G–function for a few specific values of the parameters α and β:

$$\begin{aligned}
\mu_k^{(1,2)} &= \frac{1}{2^{k+2}\sqrt{\pi}}\, G_{2,4}^{3,1}\left(\frac{1}{4}\,\middle|\, \begin{matrix} -\,;- \\ -\frac{k+1}{2}, -\frac{k}{2}, 0; - \end{matrix}\right), \quad k \ge 0;\\
\mu_k^{(2,1)} &= \frac{2^k}{\sqrt{\pi}}\, G_{2,4}^{3,1}\left(\frac{1}{4}\,\middle|\, \begin{matrix} -\,;- \\ 0, \frac{k+1}{2}, \frac{k+2}{2}; - \end{matrix}\right), \quad k \ge 0;\\
\mu_k^{(3,1)} &= \frac{3^{k+1/2}}{2\pi}\, G_{2,5}^{4,1}\left(\frac{1}{27}\,\middle|\, \begin{matrix} -\,;- \\ 0, \frac{k+1}{3}, \frac{k+2}{3}, \frac{k+3}{3}; - \end{matrix}\right), \quad k \ge 0;\\
\mu_k^{(1/2,3/2)} &= \frac{1}{3^{2k+5/2}\pi}\, G_{2,5}^{4,1}\left(\frac{1}{27}\,\middle|\, \begin{matrix} -\,;- \\ -\frac{2k+2}{3}, -\frac{2k+1}{3}, -\frac{2k}{3}, 0; - \end{matrix}\right), \quad k \ge 0.
\end{aligned}$$

A direct application of the Chebyshev method of moments gives the recursive coefficients. This weight function has an application in the weighted polynomial approximation on $\mathbb{R}^+$. In [38] we have also considered some "truncated" Gaussian rules w.r.t. this weight function for $\alpha > 0$ and $\beta > 1$ and proved their stability and convergence with the order of the best polynomial approximation in suitable function spaces.

4. *Summation of slowly convergent series*

For slowly convergent series which are appeared in many problems in mathematics, physics and other sciences, there are several numerical methods based on linear and nonlinear transformations. In general, starting from the sequence of partial sums of the series, these transformations give other sequences with a faster convergence to the same limit, i.e., to the sum of the series. There is a rich literature on this subject (cf. references in the book of Mastroianni and Milovanović [35]).

In this section we give an account on some summation processes for series ($n = \infty$) and finite sums,

$$\sum_{k=1}^{n} (\pm 1)^k f(k), \tag{4.1}$$

with a given function $z \mapsto f(z)$ with certain properties with respect to variable z, based on ideas related to Gauss–Christoffel quadratures. In a general case, the function f can depend on other parameters, e.g., $f(z;x,\ldots)$, so that these summation proceses can be applied also to some classes of functional series, not only to numerical series.

The basic idea in our methods is to transform the sum (4.1) to an integral with respect to some measure $\mathrm{d}\mu$ on $\mathbb{R}_+$, and then to approximate this integral by a finite quadrature sum,

$$\sum_{k=1}^{n}(\pm 1)^k f(k) = \int_{\mathbb{R}_+} g(t)\mathrm{d}\mu(t) \approx \sum_{\nu=1}^{N} A_\nu g(x_\nu), \tag{4.2}$$

where the function g is connected with f in some way, and the weights $A_\nu \equiv A_\nu^{(n)}$ and abscissae $x_\nu \equiv x_\nu^{(n)}$, $\nu = 1,\ldots,N$, are chosen in such a way as to approximate closely the sum (4.1) for a large class of functions with a relatively small number $N \ll n$. In our approach we take a Gaussian quadrature sum as the sum on the right–hand side in (4.2).

In the sequal, we mention only two methods for such kind of transformations: *Laplace transform method* and *Contour integration over a rectangle.*

4.1. *Laplace transform method.* For a fixed $m \in \mathbb{N}_0$, let

$$f(s) = \int_0^{+\infty} t^m \mathrm{e}^{-st} g(t)\mathrm{d}t, \qquad \Re s \geq 1.$$

Then

$$\sum_{k=1}^{n}(\pm 1)^k f(k) = \sum_{k=1}^{n}(\pm 1)^k \int_0^{+\infty} t^m \mathrm{e}^{-kt} g(t)\mathrm{d}t = \int_0^{+\infty}\left(\sum_{k=1}^{n}(\pm \mathrm{e}^{-t})^k\right) t^m g(t)\mathrm{d}t,$$

i.e.,

$$\sum_{k=1}^{n}(\pm 1)^k f(k) = \pm\int_0^{+\infty} \frac{t^m}{\mathrm{e}^t \mp 1}[1-(\pm 1)^n \mathrm{e}^{-nt}]g(t)\mathrm{d}t. \tag{4.3}$$

Thus, the summation of series are now transformed to the integration problem, which is very appropriated for infinite series ($n = \infty$), when the *Bose–Einstein* and *Fermi–Dirac* weight functions,

$$\varepsilon(t) = \frac{t}{\mathrm{e}^t - 1} \quad \text{and} \quad \varphi(t) = \frac{1}{\mathrm{e}^t + 1},$$

respectively, can be employed. These weight are studied in the previous section and several examples with infinity series ($n = \infty$) can be found in [26] and [47].

In the case of finite series, applying Gaussian quadrature with Bose–Einstein's weight $\varepsilon(t)$ to the integral on the right side in (4.3), the convergence of the process (as n increases) slows down considerably. The reason for this is the behavior of the function $t \mapsto h_n(t) = 1 - (\pm 1)^n \mathrm{e}^{-nt}$, which tends to a discontinuous function when $n \to +\infty$. Notice that $h_n(0) = 1 - (\pm 1)^n$ has the values 0 or 2, and $\lim_{n\to+\infty} h_n(t) = 1$.

For a fixed n, the factor $[1 - (\pm 1)^n \mathrm{e}^{-nt}]$ can be included in the corresponding weights, so that

$$\mu_k^{(n)}(\varepsilon) = \int_0^{+\infty} t^k \varepsilon(t)[1 - \mathrm{e}^{-nt}]\mathrm{d}t = \mu_k(\varepsilon) - (k+1)!\zeta(k+2, n+1)$$

and

$$\begin{aligned}\mu_k^{(n)}(\varphi) &= \int_0^{+\infty} t^k \varphi(t)[1 - (-1)^n \mathrm{e}^{-nt}]\mathrm{d}t \\ &= \mu_k(\varphi) + \frac{(-1)^n}{2}\begin{cases} H\Big(\dfrac{n-1}{2}\Big) - H\Big(\dfrac{n}{2}\Big), & k = 0, \\ \dfrac{k!}{2^k}\left[\zeta\Big(k+1, \dfrac{n}{2}+1\Big) - \zeta\Big(k+1, \dfrac{n+1}{2}\Big)\right], & k \ge 1, \end{cases}\end{aligned}$$

where $H(k)$ is the k–th harmonic number and $\zeta(s,a)$ is the generalized Riemann zeta function defined by $\zeta(s,a) = \sum_{\nu=0}^{+\infty}(\nu+a)^{-s}$. In that case, the corresponding Gaussian formulas (generated by these moments) converge rapidly for smooth functions g. However, this approach would be interesting only if someone calculates finite sums a large number of times with the same number of terms. In the next subsection we consider another summation method which is much more applicable for the finite sums.

4.2. *Contour integration over a rectangle.* We consider an alternative summation/integration procedure for the series (4.1), when for $k \ge m$, the function f is analytic in the region

$$\{z \in \mathbb{C} \mid \Re z \ge \alpha,\ m - 1 < \alpha < m\}. \tag{4.4}$$

In fact, we consider the series

$$T_{m,n} = \sum_{k=m}^{n} f(k) \quad \text{and} \quad S_{m,n} = \sum_{k=m}^{n} (-1)^k f(k), \tag{4.5}$$

where $m \in \mathbb{Z}$ and n is a finite number greather than m or $n = +\infty$.

The method requires the indefinite integral F of f chosen so as to satisfy certain decay properties (see [41], [45], [47]). Using contour integration over a rectangle in the complex plane we are able to reduce $T_{m,n}$ and $S_{m,n}$ to a problem of Gaussian quadrature rules on $(0,+\infty)$ with respect to the hyperbolic weight functions considered in Subsection 3.2 (case 4°),

$$w_1(t) = \frac{1}{\cosh^2 t} \quad \text{and} \quad w_2(t) = \frac{\sinh t}{\cosh^2 t}. \tag{4.6}$$

For a holomorphic function f in

$$G = \Big\{ z \in \mathbb{C} : \alpha \le \Re z \le \beta,\ |\Im z| \le \frac{\delta}{\pi} \Big\},$$

where $m-1<\alpha<m$, $n<\beta<n+1$ $(m,n\in\mathbb{Z}, m<n)$, $\delta>0$, $\Gamma=\partial G$, using Cauchy's residue theorem, the series (4.5) can be expressed in the forms

$$T_{m,n} = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \frac{\pi}{\tan \pi z} \mathrm{d}z, \quad S_{m,n} = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z) \frac{\pi}{\sin \pi z} \mathrm{d}z.$$

After integration by parts, these formulas reduce to

$$T_{m,n} = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \Big(\frac{\pi}{\sin \pi z}\Big)^2 F(z) \mathrm{d}z, \quad S_{m,n} = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \Big(\frac{\pi}{\sin \pi z}\Big)^2 \cos \pi z\, F(z) \mathrm{d}z,$$

where F is an integral of f.

Assume the following conditions for the function F:

(C1) *F is a holomorphic function in the region* (4.4);

(C2) $\displaystyle\lim_{|t|\to+\infty} \mathrm{e}^{-c|t|} F\left(x + \mathrm{i}t/\pi\right) = 0$, *uniformly for* $x \ge \alpha$;

(C3) $\displaystyle\lim_{x\to+\infty} \int_{\mathbb{R}} \mathrm{e}^{-c|t|} \left| F\left(x+\mathrm{i}t/\pi\right)\right| \mathrm{d}t = 0$,

where $c=2$ or $c=1$, when we consider $T_{m,n}$ or $S_{n,m}$, respectively.

Setting $\alpha=m-1/2$, $\beta=n+1/2$, and letting $\delta\to+\infty$, the previous integrals over Γ reduce to the integrals with respect to the weight functions (4.6),

$$T_{m,n} = \int_0^{+\infty} w_1(t) \left[\Phi\left(\alpha, t/\pi\right) - \Phi\left(\beta, t/\pi\right)\right] \mathrm{d}t$$

and

$$S_{m,n} = \int_0^{+\infty} w_2(t) \left[(-1)^m \Psi\left(\alpha, t/\pi\right) + (-1)^n \Psi\left(\beta, t/\pi\right)\right] \mathrm{d}t,$$

where

$$\Phi(x,y) = -\frac{1}{2}\left[F(x+\mathrm{i}y) + F(x-\mathrm{i}y)\right], \quad \Psi(x,y) = \frac{1}{2\mathrm{i}}\left[F(x+\mathrm{i}y) - F(x-\mathrm{i}y)\right].$$

Thus, the numerical construction of Gaussian quadratures with respect to the hyperbolic weights w_1 and w_2, given in (4.6), provides appropriate summation processes for the sums $T_{m,n}$ and $S_{m,n}$, respectively.

We mention now a recent result on the generalized Mathieu series and its alternating variant,

$$S_m(r) = \sum_{n\ge 1} \frac{2n}{(n^2+r^2)^{m+1}}, \qquad \widetilde{S}_m(r) = \sum_{n\ge 1} (-1)^{n-1} \frac{2n}{(n^2+r^2)^{m+1}},$$

derived by contour integration using rectangular integration path (see [60]):

Theorem 4.1 *The following integral representation formulae hold true*

$$S_m(r) = \frac{\pi}{m}\int_0^\infty \frac{\displaystyle\sum_{j=0}^{[m/2]} (-1)^j \binom{m}{2j} \left(r^2 - x^2 + \frac{1}{4}\right)^{m-2j} x^{2j}}{\left[\left(x^2 - r^2 + \frac{1}{4}\right)^2 + r^2\right]^m}\, w_1(\pi x)\mathrm{d}x,$$

$$\widetilde{S}_m(r) = \frac{\pi}{m}\int_0^\infty \frac{\displaystyle\sum_{j=0}^{[(m-1)/2]} (-1)^j \binom{m}{2j+1} \left(r^2 - x^2 + \frac{1}{4}\right)^{m-2j-1} x^{2j+1}}{\left[\left(x^2 - r^2 + \frac{1}{4}\right)^2 + r^2\right]^m}\, w_2(\pi x)\mathrm{d}x,$$

where the weight functions w_1 *and* w_2 *are given in* (4.6).

By means of these established integral forms of generalized Mathieu series, we obtain also a new integral expression for the Bessel function of the first kind of half integer order, solving a related Fredholm integral equation of the first kind with nondegenerate kernel.

The series $S_1(r)$ was introduced and studied for the first time by Émile Leonard Mathieu (1835–1890) in his book [34] devoted to the elasticity of solid bodies.

Example 4.1. For the following simple example

$$\sum_{k=1}^{n} \frac{1}{(2k+1)^2} = \frac{1}{8}\pi^2 - 1 - \frac{1}{4}\psi'\Big(n + \frac{3}{2}\Big),$$

we have

$$T_{1,n} = T_{1,m-1} + T_{m,n}, \quad 1 \le m < n.$$

For $m = 1$ the first sum on the right side in the previous formula is empty. Here, $f(z) = (2z+1)^{-2}$, and $F(z) = -(2z+1)^{-1}/2$, the integration constant being zero on account of the condition (C3). Thus,

$$\Phi(x,y) = \Re\frac{1}{2(2z+1)} = \frac{x+1/2}{(2x+1)^2+4y^2}\,.$$

Now, we apply the Gaussian quadrature formulae with respect to the hyperbolic weights w_1 to $T_{m,n}$, so that

$$T_{1,n} \approx T_{1,m-1} + Q^{(N)}_{m,n} = \sum_{k=1}^{m-1}\frac{1}{(2k+1)^2} + \sum_{k=1}^{N} A_k\left[\Phi\left(\alpha,\tau_k/\pi\right) - \Phi\left(\beta,\tau_k/\pi\right)\right],$$

with $\alpha = m - 1/2$ and $\beta = n + 1/2$.

For example, for $n = 10000$, $n = 100000$, and $n = +\infty$, with the 50–point Gaussian quadrature we obtain the values

$$\begin{aligned}
T_{1,10^4} &= 0.23367555263594067943632186977207496889924865,\\
T_{1,10^5} &= 0.23369805016116959818951969508876368316979552,\\
T_{1,+\infty} &= 0.23370055013616982735431137498451889191421243.
\end{aligned}$$

Table 1 shows the relative errors

$$r_N^{(m,n)} = \left|\frac{(T_{1,m-1}+Q^{(N)}_{m,n}) - T_{1,n}}{T_{1,n}}\right| = \left|\frac{Q^{(N)}_{m,n} - T_{m,n}}{T_{1,n}}\right|$$

for $N = 5(5)40$ and $m = 1(1)4$. Numbers in parentheses indicate decimal exponents. Notice that the exact value of the sum $T_{1,n}$ is a rational number and it can be calulated exactly.

As we can see the results can be significantly improved if we apply quadrature process to sums with a bigger m. The rapidly increasing of convergence of the summation process as m increases in due to the poles $\pm \mathrm{i} m\pi$ of $\Phi(m-1/2, t/\pi)$ moving away from the real line. It is interesting to note that a similar approach with the “Laplace transform method” does not lead to acceleration of convergence (cf. [41]).

Example 4.2. As an interesting example we consider

$$T_1(a) = \sum_{k=1}^{+\infty}\frac{1}{\sqrt{k}(k+a)}. \tag{4.7}$$

Table 1: Relative errors $r_N^{(m,n)}$ in Gaussian approximation of the finite sum $T_{1,n}$, $n = 10000$, for $N = 5(5)40$ and $m = 1(1)4$

N	$m = 1$	$m = 2$	$m = 3$	$m = 4$
5	7.78(−6)	1.78(−8)	6.76(−10	2.69(−11)
10	1.13(−8)	1.24(−12)	5.24(−15)	2.21(−17)
15	7.44(−12)	2.63(−16)	3.56(−19)	2.36(−24)
20	8.48(−13)	2.65(−18)	3.57(−22)	2.14(−25)
25	2.57(−14)	1.88(−20)	4.55(−25)	7.46(−29)
30	8.88(−16)	5.73(−23)	4.89(−28)	6.33(−32)
35	3.95(−17)	1.33(−24)	3.85(−30)	1.45(−34)
40	2.05(−18)	3.05(−26)	3.13(−32)	4.56(−37)

This series with $a = 1$ appeared in a study of spirals and defines the well–known Theodorus constant (see [10]). The first 1 000 000 terms of the series $T_1(1)$ give the result $1.8580\ldots$, i.e., $T_1 \approx 1.86$ (only 3–digit accuracy).

Using the *method of Laplace transform*, Gautschi (see [18, Example 5.1]) calculated (4.7) for $a = .5$, 1, 2, 4, 8, 16, and 32. As a increases, the convergence of the Gauss quadrature formula slows down considerably. For example, when $a = 8$, the corresponding quadrature with $N = 40$ nodes gives a result with the relative error 2.6(−8).

In a special case for $a = 1$, Gautschi [18] (see also [24]) proved that

$$T_1(1) = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \frac{D(\sqrt{t})}{\sqrt{t}}\, w(t)\mathrm{d}t, \tag{4.8}$$

where D is Dawson's integral $D(x) = \mathrm{e}^{-x^2} \int_0^x \mathrm{e}^{t^2} \mathrm{d}t$ and w is the corresponding weight function,

$$w(t) = t^{-1/2}\varepsilon(t) = \frac{t^{1/2}}{\mathrm{e}^t - 1}.$$

In the construction of Gaussian quadratures with respect to this weight, the moments are

$$\mu_k = \int_0^{+\infty} \frac{t^{k+1/2}}{\mathrm{e}^t - 1}\mathrm{d}t = \Gamma\Big(k + \frac{3}{2}\Big)\zeta\Big(k + \frac{3}{2}\Big), \quad k = 0, 1, \ldots,$$

where the gamma function and the Riemann zeta function are computable by variable–precision calculation. Using the Chebyshev algorithm in sufficiently high precision, Gautschi [24] obtained Gaussian quadratures and applied to (4.8) for $N = 5(10)75$.

Now, we directly apply the *method of contour integration over the rectangle* to (4.7), i.e.,

$$T_1(a) = \sum_{k=1}^{m-1} \frac{1}{\sqrt{k}(k+a)} + T_m(a), \qquad T_m(a) = \sum_{k=m}^{+\infty} \frac{1}{\sqrt{k}(k+a)}, \tag{4.9}$$

and then use Gaussian quadrature formula with respect to the weight $w_1(t) = 1/\cosh^2 t$ on $\mathbb{R}_+$ to calculate $T_m(a)$.

In order to construct Gaussian rules for $N \le 100$ we need recursion coefficients α_k and β_k for $k \le N - 1 = 99$, i.e., the moments (3.1) for $k \le 2N - 1 = 199$. Taking the `WorkingPrecision` to be 160, we obtain the first hundred recursion coefficients α_k and β_k, with the relative errors less than 1.86×10^{-78}.

For the series (4.9) we have

$$f(z) = \frac{1}{\sqrt{z}(z+a)} \quad \text{and} \quad F(z) = \frac{2}{\sqrt{a}} \left(\arctan \sqrt{\frac{z}{a}} - \frac{\pi}{2} \right),$$

where the integration constant is taken so that $F(\infty) = 0$. Thus,

$$T_1(a) \approx Q_m^{(N)}(a) = \sum_{k=1}^{m-1} \frac{1}{\sqrt{k}(k+a)} + \sum_{k=1}^{N} A_k \Phi(m - 1/2, \tau_k/\pi),$$

with $\Phi(x, y) = -\frac{1}{2}\left[F(x + \mathrm{i}y) + F(x - \mathrm{i}y)\right]$, where τ_k and A_k are nodes and Christoffel numbers of the N–point Gaussian rule.

Table 2: Gaussian approximation $Q_m^{(N)}(1)$ for $m = 10$

N	$Q_{10}^{(N)}(1)$
5	1.86002507922119$\underline{1}$6
15	1.8600250792211903071806959157171$\underline{7}$4
25	1.86002507922119030718069591571714332466652414$\underline{4}$3
35	1.86002507922119030718069591571714332466652412152345 1$\underline{5}$3
45	1.860025079221190307180695915717143324666524121523451493049199$\underline{2}$
55	1.86002507922119030718069591571714332466652412152345149304919950359 83$\underline{8}$

The relative errors in the previous approximate formuala for $T_1(1)$ are presented in Fig. 1 for $N = 5(5)100$ and different values of m. For example, the Gaussian

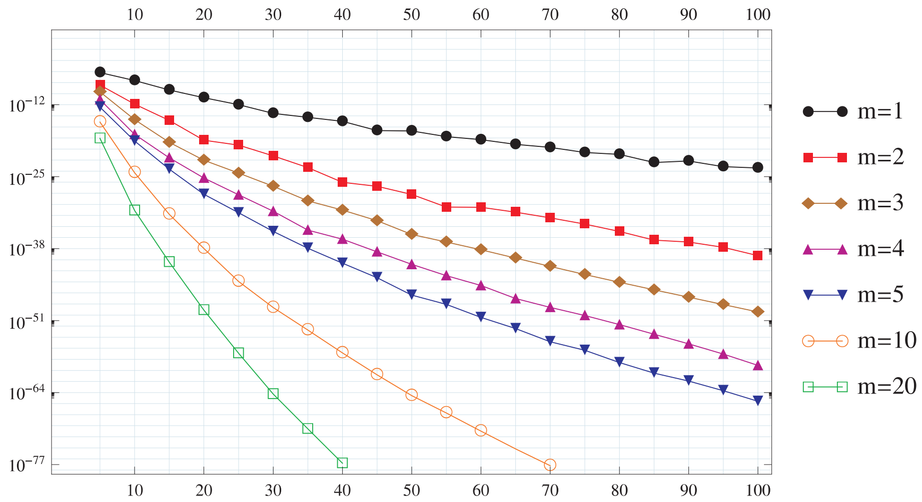

Figure 1: The relative errors in calculating $T_1(1)$ for $N = 5(5)100$ and different values of m

approximation $Q_m^{(N)}(a)$ for $a = 1$, $m = 10$, and $N = 5(10)55$ are presented in Table 2. The first digit in error is underlined.

As we can see, the method is very efficient. Numerical results also show that the convergence is slightly faster if the parameter a is larger. For example, if $a = 1000$, then taking $m = 20$ and $N = 5(5)25$, the corresponding relative errors in Gaussian approximations are $2.32(-20)$, $1.06(-33)$, $6.01(-43)$, $1.18(-51)$, $1.89(-59)$, respectively.

5. *Generalized Birkhoff–Young quadratures*

Recently we have introduced the generalized Birkhoff–Young interpolatory quadrature formula for weighted integrals of analytic functions in the unit disk $\Omega = \{z : |z| \le 1\}$ (cf. [46]),

$$I(f) := \int_{-1}^{1} f(z)w(z)\mathrm{d}z = Q_N(f) + R_N(f), \tag{5.1}$$

where $w : (-1, 1) \to \mathbb{R}^+$ is an *even* nonnegative weight function, for which all moments $\mu_k = \int_{-1}^{1} z^k w(z)\mathrm{d}z$, $k = 0, 1, \ldots$, exist and $\mu_0 = \int_{-1}^{1} w(z)\mathrm{d}z > 0$.

For a given fixed integer $m \ge 1$ and for each $N \in \mathbb{N}$, we put $N = 2mn + \nu$, where $n = [N/(2m)]$ and $\nu \in \{0, 1, \ldots, 2m-1\}$, and define the node polynomial as

$$\omega_N(z) = z^\nu p_{n,\nu}(z^{2m}) = z^\nu \prod_{k=1}^{n} (z^{2m} - r_k), \quad 0 < r_1 < \cdots < r_n < 1. \tag{5.2}$$

Then the corresponding interpolatory quadrature rule $Q_N(f)$ has the form

$$Q_N(f) = \sum_{j=0}^{\nu-1} C_j f^{(j)}(0) + \sum_{k=1}^{n} \sum_{j=1}^{m} A_{k,j} \left[f\big(x_k \mathrm{e}^{\mathrm{i}\theta_j}\big) + f\big(-x_k \mathrm{e}^{\mathrm{i}\theta_j}\big) \right], \tag{5.3}$$

where

$$x_k = \sqrt[2m]{r_k}, \quad k = 1, \ldots, n; \quad \theta_j = \frac{(j-1)\pi}{m}, \quad j = 1, \ldots, m.$$

For $\nu = 0$, the first sum in $Q_N(f)$ is empty. $R_N(f)$ in (5.1) is the corresponding remainder.

The polynomial $p_{n,\nu}$ in (5.2) can be interpreted in terms of the type II multiple orthogonal polynomials (cf. [1], [87], [66], [67]) and we can prove the following result [46]:

Theorem 5.1 *Let m be a fixed positive integer and w be a nonnegative even weight function w on $(-1, 1)$, for which all moments $\mu_k = \int_{-1}^{1} z^k w(z)\mathrm{d}z$, $k \geq 0$, exist and $\mu_0 > 0$. For any $N \in \mathbb{N}$ there exists a unique interpolatory quadrature $Q_N(f)$, with a maximal degree of exactness*

$$d_{\max} = 2(m+1)n + \begin{cases} \nu - 1, & \nu \text{ even}, \\ \nu, & \nu \text{ odd}, \end{cases}$$

if and only if the polynomial $p_{n,\nu}(t)$ is the type II multiple orthogonal polynomial, with respect to the weight functions $w_j(t)$,

$$\int_0^1 t^\ell p_{n,\nu}(t) w_j(t)\mathrm{d}t = 0, \quad \ell = 0, 1 \ldots, \left[\frac{n-j}{m}\right],$$

where $w_j(t) = t^{(s+2j)/(2m)-1} w(t^{1/(2m)})$, $j = 1, \ldots, m$.

In the case $m = 1$, the node polynomial $\omega_{2n+\nu}(z) = z^\nu p_{n,\nu}(z^2)$, with $\nu = 0$ or $\nu = 1$, is a monic polynomial of degree $2n+\nu$, which is orthogonal to $\mathcal{P}_{2n+\nu-1}$ with respect to the even weight function w on $(-1, 1)$, so that the quadrature formula (5.1), with (5.3), is in fact a standard Gaussian formula on $(-1, 1)$ with $2n + \nu$ nodes. The polynomial sequences

$$p_{n,0}(t) = \omega_{2n}(\sqrt{t}) \quad \text{and} \quad p_{n,1}(t) = \frac{\omega_{2n+1}(\sqrt{t})}{\sqrt{t}}$$

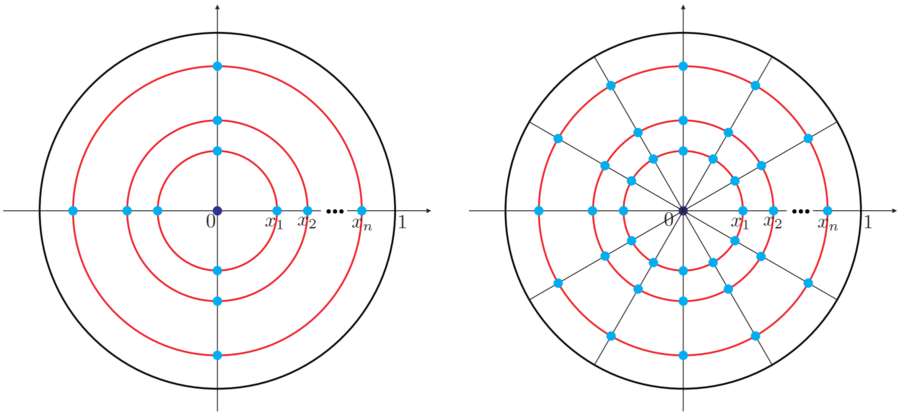

Figure 1: Distribution of nodes for $m = 2$ (left) and $m = 6$ (right)

are orthogonal on $(0, 1)$ with respect to the weight functions $w(\sqrt{t})/\sqrt{t}$ and $w(\sqrt{t})\sqrt{t}$, respectively (see [35, Theorem 2.2.11]). Notice that the origin is appeared as a quadrature node only when $\nu = 1$.

Distributions of nodes for $m = 2$ and $m = 6$ are presented in Figure 1.

The first quadrature rule of this type was appeared in 1950 by Birkhoff and Young [5]. They proposed a quadrature formula of the form

$$\int_{z_0-h}^{z_0+h} f(z)\mathrm{d}z \approx \frac{h}{15}\Big\{24f(z_0) + 4\big[f(z_0+h) + f(z_0-h)\big] - \big[f(z_0+\mathrm{i}h) + f(z_0-\mathrm{i}h)\big]\Big\},$$

with the error term $R_5^{BY}(f)$, for numerical integration over a line segment in the complex plane, where $f(z)$ is a complex analytic function in $\big\{z : |z - z_0| \le r\big\}$ and $|h| \le r$. This five point quadrature formula is exact for all algebraic polynomials of degree at most five and its remainder $R_5^{BY}(f)$ can be estimated by (see [88] and [11, p. 136])

$$|R_5^{BY}(f)| \le \frac{|h|^7}{1890} \max_{z\in S} |f^{(6)}(z)|,$$

where S denotes the square with vertices $z_0 + \mathrm{i}^k h$, $k = 0, 1, 2, 3$. By a reduction of the line segment $[z_0 - h, z_0 + h]$ to $[-1, 1]$, this five–point rule reduces to

$$\int_{-1}^{1} f(z)\mathrm{d}z = \frac{8}{5}f(0) + \frac{4}{15}\big[f(1) + f(-1)\big] - \frac{1}{15}\big[f(\mathrm{i}) + f(-\mathrm{i})\big] + R_5(f). \tag{5.4}$$

In 1978 Tošić [86] obtained a significant improvement of (5.4) in the form

$$\int_{-1}^{1} f(z)\mathrm{d}z \quad = \quad \frac{16}{15}f(0) + \frac{1}{6}\left(\frac{7}{5} + \sqrt{\frac{7}{3}}\right)\big[f(r) + f(-r)\big]$$

$$+\frac{1}{6}\left(\frac{7}{5}-\sqrt{\frac{7}{3}}\right)\big[f(\mathrm{i}r)+f(-\mathrm{i}r)\big]+R_5^T(f),$$

where $r=\sqrt[4]{3/7}$ and

$$R_5^T(f)=\frac{1}{793800}f^{(8)}(0)+\frac{1}{61122600}f^{(10)}(0)+\cdots\ .$$

More than three decades ago, in a joint paper with Đorđević [59] we extended this formula to the following nine point rule,

$$\begin{aligned}\int_{-1}^{1}f(z)\mathrm{d}z \;\;=\;\; & Af(0)+C_{11}\big[f(r_1)+f(-r_1)\big]+C_{12}\big[f(\mathrm{i}r_1)+f(-\mathrm{i}r_1)\big]\\ & +C_{21}\big[f(r_2)+f(-r_2)\big]+C_{22}\big[f(\mathrm{i}r_2)+f(-\mathrm{i}r_2)\big]+R_9(f;r_1,r_2),\end{aligned}$$

where $0<r_1<r_2<1$, and proved that for

$$r_1=r_1^*=\sqrt[4]{\frac{63-4\sqrt{114}}{143}}\quad\text{and}\quad r_2=r_2^*=\sqrt[4]{\frac{63+4\sqrt{114}}{143}},$$

this quadrature rule has the algebraic degree of precision $p=13$, with the error–term

$$R_9(f;r_1^*,r_2^*)=\frac{1}{28122661066500}f^{(14)}(0)+\cdots\ \approx 3.56\cdot 10^{-14}f^{(14)}(0).$$

Evidently, it is a special case, which can be obtained from Theorem 5.1 for $N=9$ and $m=2$. In that case, $n=2$ and $\nu=1$, so that $d_{\max}=2(m+1)n+s=13$, and

$$p_{2,1}(z)=z^2-\frac{126}{143}z+\frac{15}{143}.$$

A special case with the Chebyshev weight of the first kind $w(z)=1/\sqrt{1-z^2}$ was considered recently in [68].

6. *Nonstandard quadratures of Gaussian type*

If the information data $\{f(\tau_k)\}_{k=1}^n$ in the standard quadrature

$$\int_{\mathbb{R}}f(t)\mathrm{d}\mu(t)=\sum_{k=1}^{n}w_kf(\tau_k)+R_n(f). \tag{6.1}$$

are replaced by $\{(\mathcal{A}^{h_k}f)(\tau_k)\}_{k=1}^n$, where $\mathcal{A}^h$ is an extension of some linear operator $\mathcal{A}^h:\mathcal{P}\to\mathcal{P}$, $h\geq 0$, we get a non–standard quadrature formula

$$\int_{\mathbb{R}}f(t)\mathrm{d}\mu(t)=\sum_{k=1}^{n}w_k(\mathcal{A}^{h_k}f)(\tau_k)+R_n(f). \tag{6.2}$$

Notice that we use the same notation for the linear operator defined on the space of all algebraic polynomials and for its extension to the certain class of integrable functions X $(f \in X)$. As a typical example for such operators is the *average operator*

$$(\mathcal{A}^h f)(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)\mathrm{d}t, \qquad h > 0. \tag{6.3}$$

In the case $h = 0$ this operator is interpreted as the *identity operator* $\mathcal{A}^0 = \mathcal{I}$, so that, for continuous f, its value at x is $f(x)$, i.e.,

$$(\mathcal{A}^0 f)(x) = \lim_{h \to 0} (\mathcal{A}^h f)(x) = (\mathcal{I} f)(x) = f(x).$$

Quadratures (6.2) with the average operator (6.3) are known as the *interval quadrature formulae* and they studied by several authors (cf. Omladič, Pahor, and Suhadolc [81], Pitnauer and Reimer [82], Kuz'mina [31], Sharipov [84], Babenko [3], Motornyi [78]).

In many applications, especially in experimental physics and engineering, it is not possible to accurately measure the values $f(\tau_\nu)$, $\nu = 1, \ldots, n$, only their mean values, so that instead of the standard quadrature (6.1) one can use only an interval quadrature with the average operator (6.3) (see Fig. 6.1 (left)).

Standard quadraures can be interpreted as quadratures with an operator defined by

$$(\mathcal{A}^h f)(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(x)\mathrm{d}t = f(x), \tag{6.4}$$

so that $(\mathcal{A}^{h_k} f)(\tau_k) = f(\tau_k)$, $k = 1, \ldots, n$ (see Fig. 6.1 (right)).

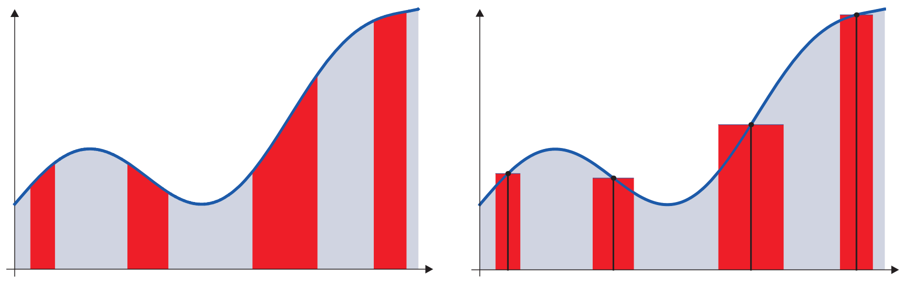

Fig. 6.1. The information data: (left) in an interval quadrature with the average operator; (right) in the standard quadrature interpreted as a nonstandard one with the operator (6.4)

Instead of (6.3) it is possible to consider also a *weighted average operator* in the form

$$(\mathcal{A}_w^h f)(x) = \frac{(\mathcal{A}^h fw)(x)}{(\mathcal{A}^h w)(x)} = \frac{\displaystyle\int_{x-h}^{x+h} f(t)w(t)\mathrm{d}t}{\displaystyle\int_{x-h}^{x+h} w(t)\mathrm{d}t}, \tag{6.5}$$

or even simpler as

$$(\mathcal{B}_w^h f)(x) = (\mathcal{A}^h fw)(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)w(t)\mathrm{d}t, \tag{6.6}$$

where $h > 0$ and w is a given weight function on a finite interval $[a,b]$, i.e., a nonnegative Lebesgue integrable function, such that for each subinterval $(\alpha,\beta) \subseteq [a,b]$, $\alpha < \beta$, we have $\int_\alpha^\beta w(t)\mathrm{d}t > 0$.

6.1. *Gaussian interval quadrature formulae.* In this subsection, for a given weight function $w : I \to \mathbb{R}^+$ and for $\mathbf{h} = (h_1,\ldots,h_n)$, $h_\nu \geq 0$, $\nu = 1,\ldots,n$, we use the weighted average operator (6.5) and define the following interval quadrature formula of Gaussian type on I as

$$\int_I f(x)w(x)\mathrm{d}x = \sum_{\nu=1}^n \frac{\sigma_\nu}{w(I_\nu)}\int_{I_\nu} f(x)w(x)\mathrm{d}x + R_n(f), \tag{6.7}$$

which is exact for all algebraic polynomial of degree at most $2n-1$, i.e., $R_n(\mathcal{P}_{2n-1}) = 0$, where $I_\nu = (x_\nu - h_\nu, x_\nu + h_\nu)$, $\nu = 1,\ldots,n$, are nonoverlapping intervals, whose union is the proper subset of I. Quantities $w(I_\nu)$, $\nu = 1,\ldots,n$, are given by

$$w(I_\nu) = \int_{I_\nu} w(x)\mathrm{d}x, \quad \nu = 1,\ldots,n.$$

The midpoints x_ν of the intervals I_ν, $\nu = 1,\ldots,n$, are called the *nodes* of the interval quadrature rule (6.7), and the quantities σ_ν, $\nu = 1,\ldots,n$, are called the *weights*.

Notice that for a continuous function f in (6.7), we have

$$\lim_{h_\nu\to 0^+} \frac{1}{w(I_\nu)}\int_{I_\nu} f(x)w(x)\mathrm{d}x = f(x_\nu), \quad \nu = 1,\ldots,n,$$

so that the Gaussian interval quadrature rule, for $\mathbf{h} = 0$, reduces to the standard Gaussian quadrature rule.

Remark 6.1 If we take the weighted average operator in the form (6.6), then (6.7) becomes

$$\int_I f(x)w(x)\mathrm{d}x = \sum_{\nu=1}^n \frac{w_\nu}{2h_\nu}\int_{I_\nu} f(x)w(x)\mathrm{d}x + R_n(f), \tag{6.8}$$

where $w_\nu = 2h_\nu\sigma_\nu/w(I_\nu)$, $\nu = 1,\ldots,n$.

For a finite interval $I = [a, b]$, in 2001 Bojanov and Petrov [6] proved that the Gaussian interval quadrature rule (6.8) exists, with positive weight coefficients w_ν, $\nu = 1, \dots, n$, as well as that for $h_\nu = h$, $1 \le \nu \le n$, this Gaussian interval quadrature formula is unique. In that case, for each $f \in C^{2n}[a, b]$ there exists a point $\xi \in (a, b)$ such that

$$R_n(f) = \frac{f^{(2n)}(\xi)}{(2n)!} \int_a^b Q_{2n}(x) w(x) \mathrm{d}x,$$

where $Q_{2n}(x)$ is a unique monic polynomial of degree $2n$ such that

$$\int_{x_\nu - h}^{x_\nu + h} Q_{2n}(x) w(x) \mathrm{d}x = \int_{x_\nu - h}^{x_\nu + h} x Q_{2n}(x) w(x) \mathrm{d}x = 0, \quad \nu = 1, \dots, n.$$

Moreover, in [7] Bojanov and Petrov proved the uniqueness of (6.8) for the Legendre weight ($w(x) = 1$) for any set of lengths $h_\nu \ge 0$, $\nu = 1, \dots, n$, satisfying the condition that I_ν are nonoverlapping intervals, whose union is the proper subset of $I = [a, b]$. In concluding remarks they noted that the extention of the uniqueness result to weighted quadratures for any fixed weight function w could be very difficult problem.

Using properties of the topological degree of non–linear mappings, Milovanović and Cvetković [50] proved that Gaussian interval quadrature formula (6.8) is unique for the Jacobi weight function

$$w(x) = (1 - x)^\alpha (1 + x)^\beta, \quad \alpha, \beta > -1, \tag{6.9}$$

on $I = [-1, 1]$, and also proposed an appropriate algorithm for numerical construction of such kind of quadratures. For the special case of the Chebyshev weight of the first kind $w(x) = 1/\sqrt{1 - x^2}$ and the special set of lengths an analytic solution was obtained in [50]. An alternative and much simpler algorithm for the numerical construction of the interval Gaussian quadratures with respect to the Jacobi weight (6.9) has been derived in [51].

The corresponding interval quadrature rules of Gauss–Lobatto type with respect to the Jacobi weight function (6.9),

$$\begin{aligned} \int_I f(x) w(x) \mathrm{d}x \quad &= \quad \frac{\sigma_0}{w(I_0)} \int_{I_0} f(x) w(x) \mathrm{d}x + \sum_{\nu=1}^{n} \frac{\sigma_\nu}{w(I_\nu)} \int_{I_\nu} f(x) w(x) \mathrm{d}x \\ &\quad + \frac{\sigma_{n+1}}{w(I_{n+1})} \int_{I_{n+1}} f(x) w(x) \mathrm{d}x + R_{n+2}^L(f), \end{aligned}$$

where $I_0 = (-1, -1 + h_0)$, $I_\nu = (x_\nu - h_\nu, x_\nu + h_\nu)$, $\nu = 1, \dots, n$, and $I_{n+1} = (1 - h_{n+1}, 1)$, have been considered in [54], as well as special cases with respect to the Chebyshev weight of the first kind. Similar results for the Gauss–Radau quadrature rule have been also proved in [54], as well as the corresponding algorithms for numerical construction of these type of quadratures.

The case of interval quadratures of the Gaussian type on unbounded intervals has been for the first time treated in [52], where the existence and uniqueness of the Gaussian interval quadrature formula with respect to the generalized Laguerre weight function $w(x) = x^\alpha \mathrm{e}^{-x}$ on $\mathbb{R}^+$ have been presented, including an algorithm for the numerical construction of such a formula. The corresponding problem with the Hermite weight $w(x) = \mathrm{e}^{-x^2}$ on $\mathbb{R}$ has been studied in [55]. Thus, in this way we have completed results for all classical weight functions.

Recently, we have considered interval quadrature formulas of Gaussian type with respect to the nonclassical exponential weight functions of the form $w(x) = \mathrm{e}^{-Q(x)}$ on unbounded intervals $I = \mathbb{R}$ or $I = \mathbb{R}^+$ (see [9]), where Q is supposed to be continuous function on I and given such that all algebraic polynomials are integrable with respect to the weight w.

Theorem 6.1 *Given* $\mathbf{h} = (h_1, \dots, h_n) \in \mathbb{R}^n$, $h_\nu \geq 0$, $\nu = 1, \dots, n$, *there exists the unique Gaussian interval quadrature rule* (6.7) *with respect to the weight function* w *on* I, *where intervals* $I_\nu = (x_\nu - h_\nu, x_\nu + h_\nu)$, $\nu = 1, \dots, n$, *are nonoverlapping. In the case of the weight* w *supported on* $I = \mathbb{R}^+$, *in addition, we have* $x_1 - h_1 > 0$.

The results obtained in [9] can be applied also to corresponding quadratures over the finite interval $(-1, 1)$.

6.2. *Gaussian quadratures based on operator values.* Another approach in quadrature formulae of Gaussian type for intervals of the same length with the average operator (6.3) appeared in 1992 in Omladič's paper [80]. The middle points of the intervals are zeros of some kind of orthogonal polynomials. More precisely, Omladič proved that the nodes x_ν, $\nu = 1, \dots, n$, of his quadratures are zeros of the *average Legendre polynomials* $p_n^h(x) \equiv p_n(x)$, which satisfy the three–term recurrence relation

$$p_{n+1}(x) = x\, p_n(x) - \frac{n^2(1 - n^2h^2)}{4n^2 - 1} p_{n-1}(x), \quad n \geq 1.$$

In order to generalize this approach we put $H = H_\delta = [0, \delta)$, $\delta > 0$, and consider families of linear operators $\mathcal{A}^h$, $h \in H$, acting on the space of all algebraic polynomials $\mathcal{P}$, such that the degrees of polynomials are preserved, i.e.,

$$\deg(\mathcal{A}^h p) = \deg(p), \tag{6.10}$$

and

$$\lim_{h \to 0^+} (\mathcal{A}^h p)(x) = p(x), \quad x \in \mathbb{C}, \tag{6.11}$$

for any $p \in \mathcal{P}$ and each $h \in H$. By definition we put $\deg(0) = -1$, so that degree preserving property also means that the zero polynomial is the image only of the zero polynomial.

For a given family of linear operators $\mathcal{A}^h$, $h \in H$, we consider the non–standard interpolatory quadrature of Gaussian type

$$\int_{\mathbb{R}} f(x)\mathrm{d}\mu(x) = \sum_{k=1}^{n} w_k(\mathcal{A}^h f)(x_k) + R_n(f), \tag{6.12}$$

which is exact for all algebraic polynomials of degree at most $2n-1$.

In [56] we have proved the following result:

Theorem 6.2 *Let $\mathcal{A}^h$, $h \in H$, be a family of linear operators satisfying the conditions* (6.10) *and* (6.11) *and* $\mathrm{d}\mu$ *be a finite positive Borel measure on the real line with its support* $\operatorname{supp}(\mathrm{d}\mu) \subset \mathbb{R}$. *For any $n \in \mathbb{N}$ there exists $\varepsilon > 0$, such that for every $h \in H_\varepsilon = [0, \varepsilon)$ there exists the unique interpolatory quadrature formula* (6.12) *of Gaussian type, with nodes $x_k \in \mathrm{Co}(\operatorname{supp}(\mathrm{d}\mu))$ and positive weights $w_k > 0$, $k = 1, \ldots, n$.*

Also, we have proposed a stable numerical algorithm for constructing such quadrature formulae. In particular, for some special classes of linear operators of the form

$$(\mathcal{A}^h p)(x) = \frac{1}{2h}\int_{x-h}^{x+h} p(t)\,\mathrm{d}t,$$

$$(\mathcal{A}^h p)(x) = \sum_{k=-m}^{m} a_k p(x+kh) \quad \text{or} \quad (\mathcal{A}^h p)(x) = \sum_{k=-m}^{m-1} a_k p\left(x + (k+\frac{1}{2})h\right),$$

and

$$(\mathcal{A}^h p)(x) = \sum_{k=0}^{m} \frac{b_k h^k}{k!}\mathcal{D}^k p(x),$$

where m is a fixed natural number and $\mathcal{D}^k = \mathrm{d}^k/\mathrm{d}x^k$, $k \in \mathbb{N}_0$, we obtain interesting explicit results connected with theory of orthogonal polynomials. Details can be found in [56].

Finally, following a starting idea from [33], for finite positive Borel measures supported on the real line we have considered a new type of quadrature rule with maximal algebraic degree of exactness (see Milovanović and Cvetković [57]), which involves function derivatives. We have proved the existence of such quadrature rules and described their basic properties. An additional motivation for this type of quadrature comes also from its applications to initial–value problems for ordinary differential equations.

7. *Fredholm integral equations of the second kind*

Integral equations appear in many fields including continuum and quantum mechanics, kinetic theory of gases, optimization and optimal control systems, communication theory, potential theory, geophysics, electricity and magnetism, biology and population genetics, mathematical economics, queueing theory, etc. Most of the boundary value problems involving differential equations can be converted into problems in integral equations, but also there are certain problems which can be formulated only in terms of integral equations.

We are interested only in the *Fredholm integral equations of the second kind* (FK2),

$$f(\mathbf{y}) + \mu \int_{\mathcal{D}} k(\mathbf{x}, \mathbf{y}) f(\mathbf{x}) w(\mathbf{x}) \mathrm{d}\mathbf{x} = g(\mathbf{y}), \quad \mathbf{y} \in \mathcal{D}, \tag{7.1}$$

where the kernel $k(\mathbf{x}, \mathbf{y})$, the weight w, and g are known functions, $\mu \in \mathbb{R}$ is a parameter, and f is a unknown function.

A computational approach to the solution of integral equations is an essential branch of numerical analysis. There are many numerical methods for solving integral equations (cf. [2], [32], [35, pp. 362–385]). Numerical methods for linear integral equations lead to algebraic systems of linear equations and sometimes the conditional number of the corresponding matrices are large. The solution of an integral equation can be done in a polynomial form, as a peacewise polynomial, spline, etc.

In the sequel we consider an important one–dimensional case of the integral equation (7.1) on $\mathcal{D} = A = [-1, 1]$,

$$f(y) + \mu \int_{A} k(x, y) f(x) w(x) \mathrm{d}x = g(y), \quad y \in A, \tag{7.2}$$

with respect to the Jacobi weight, as well as a two–dimensional case on a triangle $\mathcal{D} = \mathrm{T}$, which can be reduced to the square $\mathcal{D} = \mathrm{Q} = A^2$. The proposed methods are very efficient and they are based on the recent progress in polynomial interpolation (cf. [35]).

7.1. *One–dimensional case on* $[-1, 1]$. We consider the Fredholm integral equations of the second kind FK2 (7.2) on $A = [-1, 1]$, when

$$w(x) = v^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha, \beta > -1,$$

and give a numerical method for their solving in the spaces of continuous functions equipped with certain uniform weighted norms. Letting

$$(Kf)(y) = \mu \int_{-1}^{1} k(x, y) f(x) w(x) \mathrm{d}x, \tag{7.3}$$

the equation (7.2) can be written in the operator form

$$(I + K)f = g, \tag{7.4}$$

where I denotes the identity operator. Assuming the continuity of the kernel $k(x,y)$ we use Nyström methods and we can prove the stability and convergence, as well as the well–conditioning of the corresponding matrices. The last property is derived only from the continuity of the kernel and not from its special form.

Taking another Jacobi weight, $v^{\gamma,\delta}(x)=(1-x)^{\gamma}(1+x)^{\delta}$, with $\gamma,\delta\geq 0$, we define the space of functions as

$$C_{v^{\gamma,\delta}}=\Big\{f\in C^0((-1,1)):\lim_{x\to\pm1}(fv^{\gamma,\delta})(x)=0\Big\},$$

equipped with the norm $\|f\|_{C_{v^{\gamma,\delta}}}=\|fv^{\gamma,\delta}\|_\infty$. Moreover, we denote by

$$E_n(f)_{v^{\gamma,\delta}}=\inf_{P_n\in\mathcal{P}_n}\|(f-P_n)v^{\gamma,\delta}\|_\infty$$

the error of best weighted approximation of a function f in $C_{v^{\gamma,\delta}}$ by means of polynomials of degree at most n.

Using the n–point Gaussian quadrature with respect to the Jacobi weight $v^{\alpha,\beta}(x)$ we approximate the operator K from (7.3) by the operator K_n defined as

$$(K_nf)(y)=\mu\sum_{k=1}^{n}\lambda_k(v^{\alpha,\beta})k(x_k,y)f(x_k),$$

where x_k, $k=1,\dots,n$, are the zeros of the (orthonormal) Jacobi polynomial $p_n(v^{\alpha,\beta})$ and $\lambda_k(v^{\alpha,\beta})$, $k=1,\dots,n$, are the corresponding Christoffel numbers. In that way, we solve the following approximating equations

$$(I+K_n)f_n=g,\quad n=1,2,\dots. \tag{7.5}$$

Multiplying both sides of (7.5) by $v^{\gamma,\delta}$ and collocating it at the zeros x_i, $i=1,\dots,n$, we obtain the system of linear equations

$$\sum_{k=1}^{n}\left[\delta_{i,k}+\mu\frac{v^{\gamma,\delta}(x_i)}{v^{\gamma,\delta}(x_k)}k(x_k,x_i)\lambda_k(v^{\alpha,\beta})\right]a_k=g(x_i)v^{\gamma,\delta}(x_i),\quad i=1,\dots,n, \tag{7.6}$$

where $a_k=f_n(x_k)v^{\gamma,\delta}(x_k)$, $k=1,\dots,n$, are the unknowns. If for a sufficiently large n (say $n>n_0$), the system (7.6) admits the unique solution $(a_1^*,\dots,a_n^*)$, then we construct the Nyström interpolant

$$f_n^*(y)=g(y)-\mu\sum_{k=1}^{n}k(x_k,y)\frac{\lambda_k(v^{\alpha,\beta})}{v^{\gamma,\delta}(x_k)}a_k^*. \tag{7.7}$$

Under restriction of parameters, $0\leq\gamma<1-\alpha$ and $0\leq\delta<1-\beta$, and properties of the kernel,

$$\lim_n\sup_{|y|\leq1}v^{\gamma,\delta}(y)E_n(k_y)=0\quad\text{and}\quad\lim_n\sup_{|x|\leq1}E_n(k_x)_{v^{\gamma,\delta}}=0, \tag{7.8}$$

we can prove (cf. [37]) that $K : C_{v^{\gamma,\delta}} \to C_{v^{\gamma,\delta}}$ is a compact operator,

$$\sup_n \|K_n\|_{C_{v^{\gamma,\delta}} \to C_{v^{\gamma,\delta}}} \le A < +\infty, \qquad \lim_N \sup_n \sup_{\|fv^{\gamma,\delta}\|_\infty = 1} E_N(K_n f)_{v^{\gamma,\delta}} = 0,$$

$$(\forall f \in C_{v^{\gamma,\delta}}) \qquad \lim_n \|(K - K_n)f\|_{C_{v^{\gamma,\delta}}} = 0,$$

and

$$\lim_n \|(K - K_n)K_n\|_{C_{v^{\gamma,\delta}} \to C_{v^{\gamma,\delta}}} = 0.$$

Theorem 7.1 *Under restriction of parameters,* $0 \le \gamma < 1 - \alpha$ *and* $0 \le \delta < 1 - \beta$*, the conditions* (7.8)*,* $\mathrm{Ker}(I + K) = \{0\}$ *in* $C_{v^{\gamma,\delta}}$ *and* $g \in C_{v^{\gamma,\delta}}$*, the system of equations* (7.6) *has a unique solution for any* $n > n_0$ *and the sequence* $\{f_n^*\}_n$*, defined by* (7.7)*, converges in* $C_{v^{\gamma,\delta}}$ *to the exact solution* f^* *of* (7.4)*, with the following error estimate*

$$\|f^* - f_n^*\|_{C_{v^{\gamma,\delta}}} \le$$
$$C \left\{ \|f\|_{C_{v^{\gamma,\delta}}} \sup_{|y| \le 1} v^{\gamma,\delta}(y) E_{n-1}(k_y) + \sup_{|y|\le 1} v^{\gamma,\delta}(y) \|k_y\|_\infty E_{n-1}(f)_{v^{\gamma,\delta}} \right\},$$

where the constant C *is independent of* n *and* f^**. Moreover, if* V_n *is the matrix of the system of equations* (7.6)*, then* $\mathrm{cond}(V_n) \le \mathrm{cond}(I + K_n) < \mathrm{const}$*.*

The error estimate can be also given in the Sobolev–type space

$$W^r(v^{\gamma,\delta}) = \left\{ f \in C_{v^{\gamma,\delta}} \; : \; f^{(r-1)} \in AC((-1,1)) \text{ and } \|f^{(r)}\varphi^r v^{\gamma,\delta}\|_\infty < +\infty \right\},$$

with $r \ge 1$ and equipped with the norm

$$\|f\|_{W^r(v^{\gamma,\delta})} := \|fv^{\gamma,\delta}\|_\infty + \|f^{(r)}\varphi^r v^{\gamma,\delta}\|_\infty,$$

where $\varphi(x) = \sqrt{1 - x^2}$ and $AC((-1,1))$ denotes the space of all functions which are absolutely continuous in every compact set of the interval $(-1,1)$. For brevity we will set $W^r(v^{0,0}) = W^r$.

Namely, under the assumptions on $\alpha, \beta, \gamma, \delta$, replacing (7.8) by

$$\sup_{|y|\le 1} v^{\gamma,\delta}(y)\|k_y\|_{W^r} < +\infty \quad \text{and} \quad \sup_{|x| \le 1} \|k_x\|_{W^r(v^{\gamma,\delta})} < +\infty$$

and assuming that $g \in W^r(v^{\gamma,\delta})$, the estimate from Theorem 7.1 becomes

$$\|(f^* - f_n^*)v^{\gamma,\delta}\|_\infty = \mathcal{O}(n^{-r}).$$

The proofs, examples and other details can be found in [37]. Similar results can be done for Fredholm integral equations of the second kind with respect to the weight function $w(x) = x^\alpha \mathrm{e}^{-x^\beta}$, $\alpha > -1$, $\beta > 1/2$, over $A = [0, +\infty)$ (see [36], [37]).

7.2. *Fredholm equations in two variables on a triangle.* Recently in [79], Occorsio and Russo have extended the previous results to the numerical solution of two–dimensional Fredholm integral equations by Nyström and collocation methods based on the zeros of Jacobi orthogonal polynomials.

In this subsection we consider the approximation of the solution of the corresponding Fredholm integral equation of the second kind in two variables on a triangle with vertices at the points $(0,0)$, $(0,1)$, $(1,0)$, i.e.,

$$\mathrm{T} = \Big\{(x_1, x_2) : 0 \le x_1 + x_2 \le 1, \quad x_1 \in [0,1]\Big\}.$$

Thus, we consider the Fredholm equation (7.1), over $\mathcal{D} = \mathrm{T}$, where $\mathbf{y} = (y_1, y_2)$, $\mathbf{x} = (x_1, x_2)$, $\mathrm{d}\mathbf{x} = \mathrm{d}x_1\mathrm{d}x_2$, k and g are given functions defined on T, f is the unknown function, and the weight function w is given by

$$w(x_1, x_2) = x_1^{p-1}x_2^{q-1}(x_1 + x_2)^a(1 - x_1 - x_2)^b, \;\; p, q > 0, \; p + q + a > 0, \; b > -1.$$

There are several applications of this type of integral equations in problems arising in fracture mechanics, aerodynamics, two dimensional electromagnetic scattering, etc.

By using a suitable transformation, we can obtain an integral equation on the square $\mathrm{Q} = [-1,1] \times [-1,1]$, where the corresponding weight appearing into the integral is the product of a pair of Jacobi weights. Namely, for any $\mathbf{x} = (x_1, x_2) \in \mathrm{T}$ and $\mathbf{u} = (u_1, u_2) \in \mathrm{Q}$, we introduce the following transformation between the triangle T and the square Q,

$$x_1 = \frac{1}{4}(1 + u_1)(1 + u_2), \qquad x_2 = \frac{1}{4}(1 + u_1)(1 - u_2), \tag{7.9}$$

with the Jacobian,

$$J(x_1, x_2) = \frac{\partial(x_1, x_2)}{\partial(u_1, u_2)} = -\frac{1}{8}(1 + u_1), \quad |J(x_1, x_2)| = \frac{1}{8}v^{0,1}(u_1).$$

Notice that many other maps $\mathrm{T} \leftrightarrow \mathrm{Q}$ are possible, such as, for instance (see [12], [13]),

$$x_1 = \frac{1}{4}(1 + u_1)(1 - u_2), \qquad x_2 = \frac{1}{2}(1 + u_2).$$

However, we use the map (7.9) since, as we will see, it allows us to obtain an integral equation on the square Q, with a weight function product of two Jacobi weights.

Thus, the two–variables Fredholm integral equation (7.1) over $\mathcal{D} = \mathrm{T}$, by using the map (7.9), transforms to the integral equation

$$F(\mathbf{v}) + \nu \int_{\mathrm{Q}} h(\mathbf{u}, \mathbf{v}) F(\mathbf{u}) \Omega(\mathbf{u}) d\mathbf{u} = G(\mathbf{v}), \quad \mathbf{v} \in \mathrm{Q},$$

with a new parameter $\nu = \mu/2^{2p+2q+a+b-1}$, where

$$\begin{aligned} F(\mathbf{v}) &= F(v_1,v_2) = f\left(\frac{(1+v_1)(1+v_2)}{4}, \frac{(1+v_1)(1-v_2)}{4}\right), \\ G(\mathbf{v}) &= G(v_1,v_2) = g\left(\frac{(1+v_1)(1+v_2)}{4}, \frac{(1+v_1)(1-v_2)}{4}\right), \\ \Omega(\mathbf{u}) &= \Omega(u_1,u_2) = w\left(\frac{(1+u_1)(1+u_2)}{4}, \frac{(1+u_1)(1-u_2)}{4}\right), \end{aligned}$$

and the kernel $h(\mathbf{u},\mathbf{v}) = h(u_1,u_2,v_1,v_2) = k(x_1,x_2,y_1,y_2)$ given by

$$h(\mathbf{u},\mathbf{v}) = \\ k\left(\frac{(1+u_1)(1+u_2)}{4}, \frac{(1+u_1)(1-u_2)}{4}, \frac{(1+v_1)(1+v_2)}{4}, \frac{(1+v_1)(1-v_2)}{4}\right).$$

Here, the weight $\Omega(\mathbf{u})$ reduces to a product of two Jacobi weights,

$$\Omega(\mathbf{u}) = v^{b,p+q+a-1}(u_1)v^{q-1,p-1}(u_2),$$

where $v^{\gamma,\delta}(t) = (1-t)^{\gamma}(1+t)^{\delta}$.

By this way, possible singularities of the kernel k on the boundary of T are moved to corresponding singularities along the border of the square and “absorbed” into the Jacobi weights. In [39] we proposed a global approximation of the solution of the integral equation (7.1) over $\mathcal{D} = \mathrm{T}$ by means of a Nyström method based on the tensor product of two univariate Gaussian rules. For such a method we proved the stability and convergence, as well as the error estimates in weighted uniform spaces.

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Mathematical Institute
Serbian Academy of Sciences and Arts
Kneza Mihaila 36
11000 Beograd
Serbia
`gvm@mi.sanu.ac.rs`