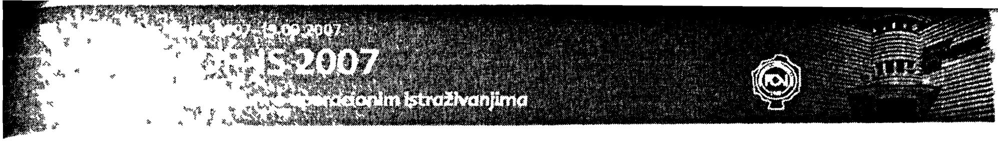

SAVRŠENI GRAFOVI U SMISLU RASTOJANJA

DISTANCE-PERFECT GRAPHS

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Sažetak *U radu se razmatra problem određivanja savršenih grafova po rastojanju u smislu metričke dimenzije. Pokazuje se da su savršeni grafovi u smislu rastojanja ili putevi ili imaju dijametar najviše 3.*

Ključne reči *savršeni grafovi, metrička dimenzija, petersenov graf*

Abstract *In this paper we consider the problem of determining distance-perfect graphs in the sense of the metric dimension. We show that distance-perfect graphs are either paths or have diameter of most 3.*

Kkey words *metric-perfect graphs, metric dimension, petersen graph*

1. INTRODUCTION

The metric dimension problem, introduced by F. Harary in 1976 [4], has been recently widely investigated. It arises in many diverse areas including network discovery and verification, robot navigation, connected joints in graphs, chemistry, etc.

Given a simple connected graph $G = (V, E)$ and $u, v \in V$, $d(u,v)$ denotes the *distance* between u and v in G, i.e. the length of the shortest u-v path. A vertex x of the graph G is said to *resolve* two vertices u and v of G if $d(x,u) \neq d(x,v)$. An ordered vertex set $B = \{x_1, x_2, \ldots, x_k\}$ of G is a *resolving set* of G if every two distinct vertices of G are resolved by some vertex of B. Given a vertex t, the k-tuple $d(t,B) = (d(t,x_1), \ldots, d(t,x_k))$ is called the *vector of metric coordinates* (or the *metric vector*) of t with respect to B. The *metric basis* of G is a resolving set of the minimum cardinality. The *metric dimension* $\beta(G)$ of G is the cardinality of its metric basis.

Example 1. Consider the graph G of Figure 1. The set $W_1 = \{A, B, C\}$ is a resolving set for G since the vectors of metric coordinates for the vertices of G with respect to W_1 are

$d(A,W_1) = (0,1,1)$, $d(B,W_1) = (1,0,1)$, $d(C,W_1) = (1,1,0)$, $d(D,W_1) = (1,2,1)$, $d(E,W_1) = (2,1,1)$.

However W_1 is not the minimum resolving set since $W_2 = \{A, B\}$ is also a resolving set with smaller cardinality, as can be seen from Figure 1. On the other hand, the set $W_3 = \{B\}$ is not a resolving set since $d(A,W_3) = d(C,W_3) = 1$. Using a similar argument it is

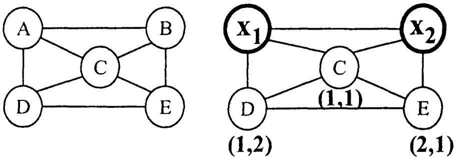

Figure 1. A graph and its metric basis

easy to check that none of singleton nodes forms a resolving set, and hence $\beta(G) = 2$ with the metric basis $W_2 = \{x_1, x_2\}$, $x_1 = A$, $x_2 = B$.

The metric dimension has many interesting properties which are out of the scope of this paper. Interested reader is referred e.g. to [1]. In [5] it was proved that the problem of computing the metric dimension of an arbitrary graph is NP-hard. Nevertheless, for some classes of graphs it is possible to obtain explicit formulas for the metric dimension: path on n vertices has $\beta(P_n) = 1$, cycle on n vertices has $\beta(C_n) = 2$, complete graph on n vertices has $\beta(K_n) = n - 1$. On the other hand, the metric dimensions of some important classes of graphs such as hypercubes and Hamming graphs are still not known. In [6,7] are given results of solving the metric dimension problem as a combinatorial optimization problem for some important theoretical classes of graphs as well as for several classes of graphs from practice.

2. DISTANCE-PERFECT GRAPHS

Let G be a graph on n vertices with the metric dimension $\beta(G) = k$ and the metric basis $B=\{x_1, x_2, \ldots, x_k\}$. For any $x \in V(G)$: $d(x) = d(x, B) = (d_1, d_2, \ldots, d_k)$ is the metric vector of x, where $d_i = d(x, x_i)$. Since B is a resolving set we have $x \neq y \Rightarrow d(x) \neq d(y)$.

Let D be the diameter of graph G, i.e. $D = \max_{u,v \in V(G)} d(u,v)$.

For $x \notin B$ all k coordinates of $d(x)$ belong to the set $\{1, 2, \ldots, D\}$. Since there are at most D^k such vertices, we have $n \leq k + D^k$. If $n = k + D^k$, the graph is called *distance-perfect*.

Examples. Distance-perfect graphs exist. Trivial examples are:

1) Path P_n on n vertices. Then $D = n-1$, $\beta(P_n) = 1$ (any end vertex provides a metric basis). Hence, $n = 1 + (n-1)^1$ and therefore P_n is distance-perfect.

2) Complete graph K_n on n vertices. Then $D = 1$, $\beta(K_n) = n-1$. We have $n = n-1 + 1^{n-1}$ and therefore K_n is the distance-perfect.

One nontrivial example is wheel on 6 vertices (Figure 2). Set of vertices $B=\{x_1,x_2\}$ represents a metric basis. We have k=2 and D=2 and therefore, $n = 6 = 2 + 2^2 = k + D^k$. Metric vectors for $x \notin B$ are all 4 possible distance vectors and they are given in Figure 2.

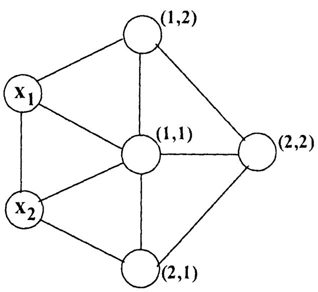

Figure 2. Wheel on 6 vertices

In this paper we will concentrate on the following research problem.

Problem 1. Characterize (or find all, or describe some properties of) distance-perfect graphs. One can consider special cases with restricted n, D, k or/and assuming that graphs have some special properties.

2.1. Distance-perfect graphs of diameter 2

If D=2, then for distance-perfect graphs $n = k + 2^k$. If $d(x) = (1,1, \ldots, 1)$, then vertex x is called the *top* of the graph.

Proposition. If the top of a distance-perfect graph of diameter 2 is not adjacent to a vertex (outside the metric basis), the graph remains distance-perfect when adding an edge between these two vertices.

Proof. The top is, of course, adjacent to all vertices from the metric basis. However, adding an edge between the top and a vertex outside the basis does not influence the distance between any two vertices of the graph.

Definition. A distance-perfect graph G of diameter 2 is called complete if the top is adjacent to all vertices of G.

A complete distance-perfect graph G of diameter 2 can be represented as a cone $G = H \nabla K_1$, where K_1 represents the top of G and ∇ the join of graphs.

Definition. The subgraph H of $G = H \nabla K_1$ is called an almost complete distance-perfect graph of diameter 2.

Definition. The subgraph of a graph G induced by a metric basis B of G is called a foundation of G.

Problem 2. Determine all regular almost complete distance-perfect graphs of diameter 2 with the given foundation.

Example 2. Let a foundation be the graph K_2. Then we have k=2 and since D=2 it follows $n = 2 + 2^2 - 1 = 5$. Let $B=\{x_1, x_2\}$, where x_1 and x_2 are adjacent. The remaining three vertices have distance vectors (1,2), (2,1), (2,2). It is easy to check that the only possibility for G is given in Figure 3. We get a regular graph of degree 2, i.e. the pentagon C_5. The corresponding complete distance-perfect graph is the wheel $C_5 \nabla K_1$, already given in Figure 2.

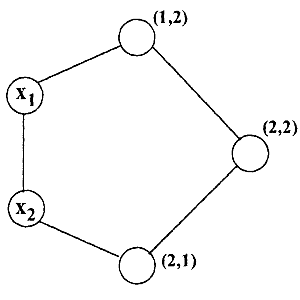

Figure 3. Pentagon C_5

Example 3. Figure 4 contains the well-known Petersen graph P with a metric basis $\{x_1, x_2, x_3\}$ and the metric dimension k=3. The Petersen graph has diameter D=2 and vertices outside the basis have all possible metric vectors except (1,1,1). This means that the cone $P \nabla K_1$ is a distance-perfect graph having $n = k + D^k = 3+2^3 = 11$ vertices.

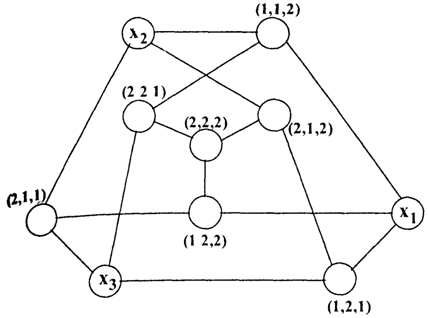

Figure 4. Petersen graph P

These two examples of regular and almost complete distance-perfect graphs are very suggestive and one might look for generalizations However, it seems that it is difficult to find other examples

The pentagon C_5 and the Petersen graph P from Examples 2 and 3 are known to be examples of the so called Moore graphs

A *Moore graph* is a graph of diameter D and girth (the length of the shortest cycle) $2D+1$ It appears that at most four Moore graphs exist [3] In addition to C_5 and P, there is the Hoffman-Singleton graph on 50 vertices and possibly a graph on 3250 vertices and all they have diameter D=2 Since equations $k + 2^k = 51$ and $k + 2^k = 3251$ do not have integer solutions, C_5 and P are the only regular almost complete distance-perfect Moore graphs

A graph is called *strongly regular* with parameters r e f if it is regular of degree r, any two adjacent vertices have exactly e common neighbours and any two non-adjacent vertices have exactly f common neighbours It is known that strongly regular graphs have diameter 2
Since C_5 and P are also strongly regular graphs, another possibility is to try to find other almost complete distance-perfect strongly regular graphs It seems that also in this direction it is unlikely that examples will be found For example, there are no strongly regular graphs of degree 4 on 20 vertices

The existence of strongly regular graphs with given parameters is investigated using eigenvalues of the adjacency matrix (see, for example, [2], p 195) However, the following problem might be interesting

Problem 3 Construct an algorithm to find a metric basis of a strongly regular graph

2.2. Distance-perfect graphs of diameter 3

The paper [1] contains a distance-perfect graph of diameter D=3, metric dimension k=2 and $n = 2+3^2 = 11$ vertices, although the paper does not consider the concept of distance-perfect graphs This graph is reproduced here in Figure 5

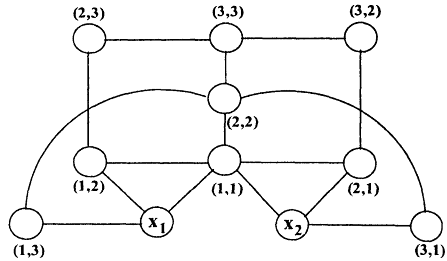

Figure 5 A distance-perfect graph of diameter 3

It is also shown in [1], that the equality $n = k + D^k$ can not hold for $k \geq 2$ and $D \geq 4$ Hence, distance-perfect graphs of diameter $D > 3$ have $k = 1$ and are reduced to paths P_n

3. CONCLUSIONS

We have introduced the concept of distance-perfect graphs and have shown that such graphs are either paths or have diameter at most 3 One might hope that all distance-perfect graphs can be found in further research We have posed several research problems concerning distance-perfect graphs

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