SYM-OP-IS 2008
XXXV Simpozijum o operacionim istraživanjima
Saobraćajni fakultet Univerziteta u Beogradu

THE METRIC DIMENSION OF STRONGLY REGULAR GRAPHS

JOZEF KRATICA, DRAGOŠ CVETKOVIĆ
Mathematical Institute SANU, Kneza Mihaila 36, Belgrade,
jkratica@mi.sanu.ac.yu, ecvetkod@etf.bg.ac.yu

MIRJANA ČANGALOVIĆ, VERA KOVAČEVIĆ-VUJČIĆ
Faculty of Organizatonal Sciences, Jove Ilića 154, Belgrade, {canga | verakov}@fon.bg.ac.yu

JELENA KOJIĆ
Faculty of Mathematics, Studentski Trg 16, Belgrade, k_jelena@yubc.net

---

Abstract: *This paper presents metric dimensions of strongly regular graphs up to 45 vertices. Total number of such graphs is over 43000. The results are obtained by CPLEX solver using the compact integer programming formulation of the metric dimension problem. The metric dimension of any two members of the class of strongly regular graphs, with given parameters, differs by at most one.*

Keywords: *Metric Dimension, Strongly Regular Graphs, Integer Programming.*

1. INTRODUCTION

This paper studies the metric dimension of strongly regular graphs. We have computed metric dimension of all strongly regular graphs with at most 45 vertices. These data represent a basis for further theoretical investigations.

The concept of the metric dimension of a graph was introduced in seventies by P.J. Slater [1] and F. Harary [2] and since then has been investigated by many authors.

Given a simple connected graph $G=(V,E)$ and $u,v \in V$, $d(u,v)$ denotes the *distance* between u and v in G, i.e. the length of the shortest u-v path. A vertex x of the graph G is said to *resolve* two vertices u and v of G if $d(x,u) \neq d(x,v)$. An ordered vertex set $B = \{x_1, x_2, ..., x_k\}$ of G is a *resolving set* of G if every two distinct vertices of G are resolved by some vertex of B. Given a vertex t, the k-tuple $d(t,B)=(d(t,x_1),...,d(t,x_k))$ is called the *vector of metric coordinates* (or the *metric vector*) of t with respect to B. The *metric basis* of G is a resolving set of the minimum cardinality. The *metric dimension* $\beta(G)$ of G is the cardinality of its metric basis.

*Example 1***.** Consider the graph G of Figure 1. The set $W_1=\{A,B,C\}$ is a resolving set for G since the vectors of metric coordinates for the vertices of G with respect to W_1 are:

$r(A, W_1)=(0,1,1)$; $r(B, W_1)=(1,0,2)$; $r(C, W_1)=(1,2,0)$;
$r(D, W_1)=(2,1,1)$; $r(E, W_1)=(1,2,1)$; $r(F, W_1)=(2,1,2)$.

However, W_1 is not the minimum resolving set since $W_2=\{A,C\}$ is also a resolving set with smaller cardinality, and with the following vectors of metric coordinates:
$r(A, W_2)=(0,1)$; $r(B, W_2)=(1,2)$; $r(C, W_2)=(1,0)$;

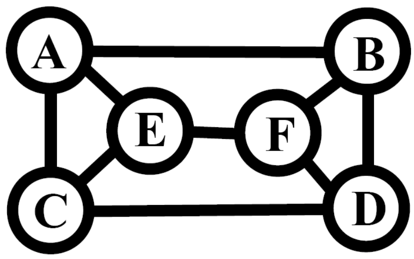

Figure 1. A graph from Example 1

$r(D, W_2)=(2,1)$; $r(E, W_2)=(1,1)$; $r(F, W_2)=(2,2)$.
On the other hand, the set $W_2=\{A\}$ is not a resolving set since $r(B, W_3)= r(C, W_3)=1$. Using a similar argument it is easy to check that none of singleton nodes forms a resolving set, and hence $\beta(G)=2$ with the metric basis $W_2=\{A,C\}$.

The metric dimension has many interesting properties which are out of the scope of this paper. Interested reader is referred e.g. to [3]. In [4] it was proved that the problem of computing the metric dimension of an arbitrary graph is NP-hard. Nevertheless, for some classes of graphs it is possible to obtain explicit formulas for the metric dimension: path on n vertices has $\beta(P_n) = 1$, cycle on n vertices has $\beta(C_n) = 2$, complete graph on n vertices has $\beta(K_n) = n - 1$. On the other hand, the metric dimensions of some important classes of graphs such as hypercubes and Hamming graphs are still not known. In [5,6] are given results of solving the metric dimension problem as a combinatorial optimization problem for some important theoretical classes of graphs as well as for several classes of graphs from practice.

Strongly regular graphs were introduced in sixties by R.C. Bose [7].

A graph is called *strongly regular* with parameters r, e, f if it is regular of degree r, any two adjacent vertices have exactly e common neighbors and any two non-adjacent vertices have exactly f common neighbors. One can show that the number n of vertices of a strongly regular graph is determined by its parameters. Numbers n, r, e, f are called parameters of a strongly regular graph and are denoted by the quadruple n-r-e-f.

For many sets of parameters strongly regular graphs do not exist. In the case of existence the number of strongly regular graphs with given parameters varies from 1 to enormous numbers. The existence and the construction of strongly regular graphs is an important subject in investigations in graphs theory in last three decades.

It is known that the diameter of strongly regular graphs is always equal to 2. Also, strongly regular graphs always have 3 distinct eigenvalues. The spectrum can be calculated from parameters and vice versa (see, for example, [8], p. 195):

Let $s = \sqrt{(e-f)^2 - 4\cdot(f-r)}$; then the eigenvalues are $\lambda_1 = r$, $\lambda_{2,3} = (e - f \pm s)/2$. If $s \notin \mathrm{N}$, the multiplicities are 1, $(n-1)/2$, $(n-1)/2$, respectively. In the case $s \in \mathrm{N}$, the multiplicities are 1, $r((r-1+f-e)(s+f-e)-2f)/(2fs)$, $n-1-r((r-1+f-e)(s+f-e)-2f)/(2fs)$, respectively.

The complement of a strongly regular graph is also a strongly regular graph. A disconnected graph whose components are complete graphs with the same number of vertices is strongly regular. Such strongly regular graphs and their complements are called *trivial*. We shall consider only non-trivial strongly regular graphs.

<u>*Example 2*</u>. Figure 2 contains one of the strongly regular graphs, the well-known Petersen graph P. The set $\{x_1, x_2, x_3\}$ is a metric basis and we have $\beta(P)=3$.

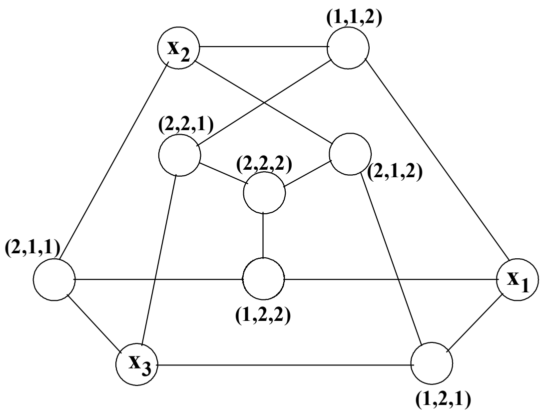

Figure 2. Petersen graph P

The Petersen graph is a strongly regular graph with parameters $n=10$, $r=3$, $e=0$, $f=1$. It has the eigenvalues $3, 1, -2$ with multiplicities $1, 5, 4$ respectively. The Petersen graph has many remarkable properties and is perhaps mostly considered graph in theoretical considerations. A whole book [9] is devoted to the Petersen graph.

All strongly regular graphs up to 45 vertices and many others are known. There exist electronic catalogues of strongly regular graphs. Our source is the catalogue from the address

http://www.maths.gla.ac.uk/~es/srgraphs.html.

We felt that it would be interesting to find a link between the two interesting graph theory concepts, metric dimension and strongly regular graphs. Therefore we have undertaken a computational study of metric dimensions of strongly regular graphs using standard software for integer programming.

2. MATHEMATICAL PROGRAMMING FORMULATION

In this section we present a 0-1 integer linear programming formulation for the metric dimension problem from [10].

Given a simple connected undirected graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, $|E| = m$, it is easy to determine the symmetric shortest path distance matrix $D = [d_{uv}]_{nxn}$, where $d_{uv} = d(u,v)$, using any shortest path algorithm.

Let $B \subseteq V$ and let

$$A_{(u,v),j} = \begin{cases} 1, d_{uj} \neq d_{vj}, \\ 0, d_{uj} = d_{vj} \end{cases}, \qquad y_j = \begin{cases} 1, j \in B, \\ 0, j \in V \setminus B. \end{cases}$$

Then, the metric dimension problem can be formulated as:

$$\min \sum_{j=1}^{n} y_j \tag{1}$$

subject to

$$\sum_{j=1}^{n} A_{(u,v),j} \cdot y_j \geq 1 \text{, for all } 1 \leq u < v \leq n, \tag{2}$$

$$y_j \in \{0,1\} \text{, for all } 1 \leq j \leq n. \tag{3}$$

It is easy to see that each feasible solution of (1)–(3) defines a resolving set B of G, and vice versa.

Proposition 1. Set B is a resolving set of G if and only if constraints (2) are satisfied.

Proof. (⇒) Suppose that B is a resolving set. Then for each $u,v\in V$, $u \neq v$ (without loss of generality we may assume $u < v$) there exist $j\in B$ (i.e. y_j=1) such that $d_{uj} \neq d_{vj}$, which implies that $A_{(u,v),j}$=1. Since y_j=1, it follows that $A_{(u,v),j}\cdot y_j = 1$ and consequently $\sum_{j=1}^{n} A_{(u,v),j} \cdot y_j \geq 1$ is satisfied.

(⇐) If constraints (2) are satisfied then for each $1\leq u<v\leq n$ there exist $j\in \{1,...,n\}$ such that $A_{(u,v),j}\cdot y_j \geq 1$, which implies y_j=1 and $d_{uj} \neq d_{vj}$. It follows that the set B defined by $\{ j \mid y_j = 1 \}$ is a resolving set of G. □

Note that, formulation (1)-(3) is linear and has n variables and $n(n$-1)/2 constraints.

3. EXPERIMENTAL RESULTS

This section summarizes computational results for the metric dimension of all 43759 strongly regular graphs with up 45 vertices. The mathematical programming formulations were automatically generated and solved by CPLEX. The tests were carried out on an Intel 2.5 GHz with 4GB RAM. All problems were solved to optimality.

The results are presented in Table 1, which is organized by follows:

- the first column contains the instance parameters;
- the second column contains the three distinct eigenvalues where the superscripts denote multiplicities of eigenvalues;
- the third column contains the metric dimension;
- the fourth column contains *the number of graphs with the corresponding metric dimension \ the total number of graphs in the given class.*

As can be seen from Table 1, strongly regular graphs from the same class (with given parameters) all have the same metric dimension, except in the case of classes 29-14-6-7, 40-12-2-4 and 45-12-3-3. In these three cases the metric dimension of the graphs in the same class differs for at most one.

Trivial strongly regular graphs are not included since the source data base did not, of course, contain them. For example, the circuit on 4 vertices is a trivial strongly regular graph since it is the complement of the disconnected graph consisting of two complete graphs with two vertices.

4. CONCLUSION

This paper presents the metric dimensions of all strongly regular graphs with up to 45 vertices. The results were obtained using CPLEX on compact mathematical programming formulation of the metric dimension problem. All 43759 problems were solved to optimality in reasonable time. These results may give a new insight into the areas of strongly regular graphs and the metric dimension.

Table 1. Metric dimension of strongly regular graphs

parameters n-r-e-f	*Spectrum*	*β(G)*	*number of graphs*
5-2-0-1	$2, 0.618^{2}, -1.618^{2}$	2	1 \ 1
9-4-1-2	$4, 1^{4}, -2^{4}$	3	1 \ 1
10-3-0-1	$3, 1^{5}, -2^{4}$	3	1 \ 1
13-6-2-3	$6, 1.303^{6}, -2.303^{6}$	4	1 \ 1
15-6-1-3	$6, 1^{9}, -3^{5}$	4	1 \ 1
16-6-2-2	$6, 2^{6}, -2^{9}$	4	2 \ 2
16-5-0-2	$5, 1^{10}, -3^{5}$	4	1 \ 1
17-8-3-4	$8, 1.562^{8}, -2.562^{8}$	4	1 \ 1
21-10-3-6	$10, 1^{14}, -4^{6}$	5	1 \ 1
25-12-5-6	$12, 2^{12}, -3^{12}$	5	15 \ 15
25-8-3-2	$8, 3^{8}, -2^{16}$	6	1 \ 1
26-10-3-4	$10, 2^{13}, -3^{12}$	5	10 \ 10
27-10-1-5	$10, 1^{20}, -5^{6}$	5	1 \ 1
28-12-6-4	$12, 4^{7}, -2^{20}$	6	4 \ 4
29-14-6-7	$14, 2.193^{14}, -3.193^{14}$	5	40 \ 41
		6	1 \ 41
35-16-6-8	$16, 2^{20}, -4^{14}$	6	3854\3854
35-18-9-9	$18, 3^{14}, -3^{20}$	6	227\227
36-10-4-2	$10, 4^{10}, -2^{25}$	7	1 \ 1
36-14-7-4	$14, 5^{8}, -2^{27}$	6	1 \ 1
36-14-4-6	$14, 2^{21}, -4^{14}$	6	180\180
36-15-6-6	$15, 3^{15}, -3^{20}$	6	32548\32548
37-18-8-9	$18, 2.541^{18}, -3.541^{18}$	5	6760\6760
40-12-2-4	$12, 2^{24}, -4^{15}$	7	27 \ 28
		8	1 \ 28
45-12-3-3	$12, 3^{20}, -3^{24}$	7	57 \ 78
		8	21 \ 78

Further research could be directed to computing the metric dimension of strongly regular graphs with more than 45 vertices and to theoretical analysis of the obtained results. Metric dimensions of trivial strongly regular graphs should be determined as well.

REFERENCES

[1] Slater, P.J., “Leaves of trees”, *Congr. Numerantium*, 14 (1975) 549-59.

[2] Harary, F., Melter, R.A., „On the metric dimension of a graph“, *Ars Combinatoria*, 2 (1976) 191–95.

[3] Chartrand, G., Eroh, L., Johnson, M.A., Oellermann, O.R., „Resolvability in graphs and the metric dimension of a graph“, *Discrete Applied Mathematics*, 105 (2000) 99-113.

[4] Khuller, S., Raghavachari, B., Rosenfeld, A., „Landmarks in graphs“, *Discrete Applied Mathematics*, 70 (1996) 217-29.

[5] Kratica, J., Kovačević-Vujčić, V., Čangalović, M., „Computing metric dimension of hypercubes by genetic algorithms“, *Proceedings of the SYMOPIS-2006*, 221-24, Banja Koviljača, 03.-06.10.2006.

[6] Kratica, J., Kovačević-Vujčić, V., Čangalović, M., „Computing the metric dimension of graphs by genetic algorithms“, *Computational Optimization and Applications*; in press. DOI:10.1007/s10589-007-9154-5.

[7] Bose R.C., "Strongly regular graphs, partial geometries and partially balanced designs", *Pacific J. Math,*, 18 (1963) 389-419.

[8] Cvetkovic, D., Doob, M., Sachs, H., *Spectra of graphs – theory and applications*, Academic Press, New York, 1980.

[9] Holton, D.A., Sheehan, J., *The Petersen graph,* Cambridge University Press, Cambridge, 1993.

[10] Currie, J.D., Oellerman, O.R., "The metric dimension and metric independence of a graph," *Journal of Combinatorial Mathematics and Combinatorial Computing*, 39 (2001) 157-67.