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SOME STATIC ROMAN DOMINATION NUMBERS FOR FLOWER SNARKS

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Abstract: *This paper is devoted to the problem of finding the Roman, restrained Roman and signed total Roman domination number for flower snark graphs. The exact values of Roman and restrained Roman domination number are determined and proved. For signed total Roman domination number tight upper bound is presented.*

Keywords: *Roman domination, restrained Roman domination, signed total Roman domination, flower snarks graphs*

1. INTRODUCTION

The problem of Roman domination on graphs has arisen and was motivated by article of Ian Stewart in (Stewart (1999)), though concept was even earlier suggested by ReVelle (1997). The original problem was formulated in military history.

At the beginning of the 4th century AD, Roman Empire was consolidated and reformed by Emperor Constantine I (306 – 337). The previous century brought much destruction both through barbarian invasions and civil wars and is known as The Crisis of the Third Century. The Empire was economically exhausted and in one period (258 – 274) split in three part all of them boasting the Roman inheritance: central government (in Italy, Africa, Balkans and Central Europe), Gallic Empire (Hispania, Gallia, Germania and Britannia) and Palmyrene Empire (Middle East, Asia Minor and Egypt). After military victories of the central government under Aurelian (270 – 275) and administrative reorganization under Diocletian (284 – 305), Empire was stabilized both militarily and economically. Barbarians were driven out, Empire was united, inflation was put under control by government intervention. Another round of civil war after the death of Constantius I (father of Constantine I) has put on the throne as a sole ruler in 324, Emperor Constantine.

He did some remarkable things: reorganized the economy by introducing golden coin solidus (or denarius), allowing religious tolerance (Milan Edict in 312), built new capital (Constantinople) and finally reorganizing army. In the third century, borders were easily punctuated by barbarians because Roman strategy was based on preemptive strikes on large groups of barbarians on their territory, outside the Emipre. Civil wars and quick succession of short-lived emperors, prevented the strategy of such attacks and empire switched to defensive, at first only on the borders and afterwards defensive-in-depth. Constantine I organized army in order to implement this defensive-in-depth, which placed armed forces not only on the borders but throughout all of the Empire. To achieve this, army was organized in stationary units, stationed mostly in border regions (limitanei) and garrisons on key points and mobile troops (comitatensis) stationed on few key points in the Empire which will act quickly in support to endangered regions. Mobile units were better equipped, trained and payed than stationary units. To accomplish this strategy it was necessary to decide were to establish units for quick response and were to put stationary units. Major problem was payment of the army (which was up to 3/4 of the tax revenues), so optimal disposition was of utmost importance. Constantine I decided to differentiate communities of the Empire in 3 categories: those with mobile and stationary troops, those with only stationary troops and those without troops at all. The condition was that communities without troops must be in neighborhood of communities with mobile troops, so that in case of attack they can be defended. Detailed explanations of strategies applied by Roman Empire through history can be found in (Luttwak (2016)).

This disposition represents very interesting optimization problem. The territory of the Empire is represented by the graph $G = (V, E)$ in which communities are vertices and edges exist between neighboring communities. Formally, problem can be formulated as finding function $f : V \to \{0, 1, 2\}$ such that value $\sum\limits_{v \in V} f(v)$ is minimal, while any vertex v with $f(v) = 0$ must be adjacent to some vertex u with $f(u) = 2$. This function is called *Roman dominating function* (RDF) and was introduced by Cockayne et al. (2004). If we define $w(f) =$

$f(V) = \sum_{v \in V} f(v)$ as a weight of function f, than *Roman domination number*, denoted as $\gamma_R(G)$, is defined as $\gamma_R(G) = \min\{w(f)|f \text{ is a RDF on} G\}$.

Let us present the notion of Roman domination with the following example.

Example. On Figure 1 a simplified territory of the Roman Empire is presented. Find an optimal RDF and γ_R.

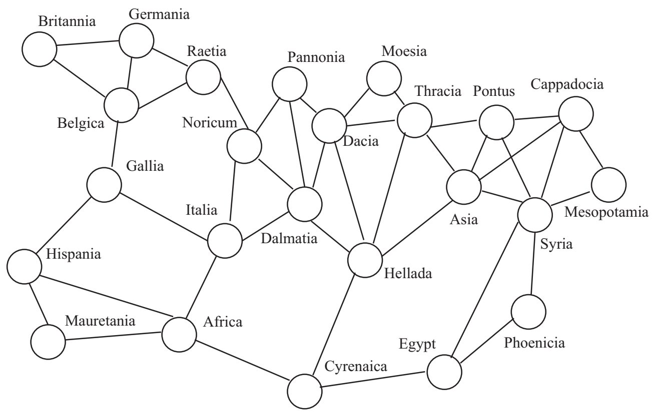

Figure 1 Graph of Roman Empire

The Roman domination number of graph on Figure 1 has value 9. Provinces with 2 units are Belgica, Africa, Dacia and Syria. Province with 1 unit is Noricum. As it can be seen Belgica protects Britannia, Germania, Raetia and Gallia. Africa protects Hispania, Mauretania, Italia and Cyrenaica. Dacia protects Pannonia, Dalmatia, Hellada, Moesia and Thracia. Syria protects Asia, Pontus, Cappadocia, Mesopotamia, Phoenicia and Egypt. The only province that was not protected is Noricum so it must have 1 unit to garrison it and protect it.

The Roman dominating function f partitions the set V in three disjunct sets V_0, V_1 and V_2, where $V_i = \{v \in V | f(v) = i\}$. Numbers $n_i = |V_i|$, $i = 0,1,2$ will represent cardinality of sets V_i. Now, the weight of a Roman dominating function f is equal to $f(V) = \sum_{v \in V} f(v) = 2 \cdot n_2 + 1 \cdot n_1 + 0 \cdot n_0 = 2 \cdot n_2 + n_1$.

From the basic problem of Roman domination, a multitude of similar problems were formulated. In this paper we will consider two of them: Restrained Roman Domination Problem (RRDP) and Signed Total Roman Domination Problem (STRDP).

The Restrained Roman domination problem was introduced by Pushpam and Padmapriea (2015) and the problem of finding the minimal number of troops such that entire Empire would be defended but with changed conditions was compared to RDP. A community is considered to be secured if at least one troop is stationed within. The condition for a community without troops within is now that it is secure if it is adjacent to at least one community with two troops and to at least one community without troops. The appropriate function is denoted as *Restrained Roman Domination Function* (RRDF). Mathematically speaking, condition that a community is secure is that for any vertex $v \in V$ value $f(v)$ is either $f(v) \geq 1$ or $f(v) = 0$ and there exist two vertices u and w adjacent to v such that $f(u) = 0$ and $f(w) = 2$. Appropriate domination number, denoted as $\gamma_{rR}(G) = min\{w(f)|f \text{ is a RRDF on} G\}$.

Similarly, the Signed Total Roman Domination Problem (STRDP), introduced by Volkmann (2016), can be defined as finding function $f : V \to \{-1,1,2\}$ such that (i) for every vertex $v \in V$ such that $f(v) = -1$ there is adjacent vertex u such that $f(u) = 2$ and (ii) if we denote $N(v) = \{u \in V | \{u,v\} \in E\}$ i.e. *open neighborhood* of v, for every $v \in V$ holds $f(N(v)) = \sum_{u \in N(v)} f(u) \geq 1$. Interpretation for this variant of RDP is that vertices with value $f(v) = -1$ represent weak spots in the defense. The appropriate function is denoted as *Signed Total Roman Domination Function* (STRDF) and appropriate domination number, denoted as $\gamma_{stR}(G) = min\{w(f)|f \text{ is a STRDF on} G\}$. The STRDP was introduced by Ahangar et al. (2014).

Example 2. Find optimal RRDF and SRDF and respective domination numbers γ_{rR} and γ_{stR} for graph on Figure 1.

RRD number is 9. Number of troops in the provinces is the same as for RD. Any province has one neighbor without unit which can be checked directly, for example province of Hellada is a neighbor to Cyrenaica.

STRD number is 9. Provinces with 2 units are Germania, Hispania, Dalmatia, Dacia, Hellada, Egypt and Syria. Provinces with 1 unit are Belgica, Gallia, Mauretania, Raetia, Asia and Phoenicia. All other provinces are weak spots. Number of troops in the neighborhood of any province is greater or equal to 1 which can be ascertained by direct probe. For example, number of troops in the neighborhood of Thracia is 1, since Dacia and Hellada have 2 troops, while Moesia, Asia and Pontus are weak spots (value is -1). The fact that every province which is weak spot has a neighbor with 2 units can be also easily checked.

Except in the military history, domination in graph have found various applications in different kinds of networks, like social networks, biological networks, distributed networks etc. as in (Behtoei et al. (2014)). Specifically, some facility location problems can be interpreted as Roman domination (Chambers et al. (2009)). Instead of interpreting $f(v)$ as the number of units placed at v, it can be viewed as a cost function. Units with cost 2 may be able to serve neighboring locations, while units with cost 1 can serve only their own location. For example, in a communication network, wireless hubs are more expensive but can serve neighboring locations, while wired hubs are low-range but are cheaper.

The Flower Snarks are specific class of regular graphs and are shown on Figure 2. The Flower snark graphs were introduced by Isaacs (1975) as an example of a cubic bridgeless graph family that is not 3-edge-colorable. The degree of vertices in Flower snarks are 3. The set of vertices is $V = \{a_i, b_i, c_i, d_i, |i = 0, \ldots, r-1\}$. The set of edges can be generated through neighborhood of characteristic vertices. So, noting that indices are by modulo r, we have

$$N(a_i) = \{b_i, c_i, d_i\}, N(b_i) = \{b_{i-1}, a_i, b_{i+1}\}, i = 0, \ldots, r-1$$

and

$$N(c_i) = \{c_{i-1}, a_i, c_{i+1}\}, N(d_i) = \{d_{i-1}, a_i, d_{i+1}\}, i = 1, \ldots, r-2$$

while

$$N(c_0) = \{d_{r-1}, a_0, c_1\}, N(c_{r-1}) = \{c_{r-2}, a_{r-1}, d_0\}, N(d_0) = \{c_{r-1}, a_0, d_1\}, N(d_{r-1}) = \{d_{r-2}, a_{r-1}, c_0\},$$

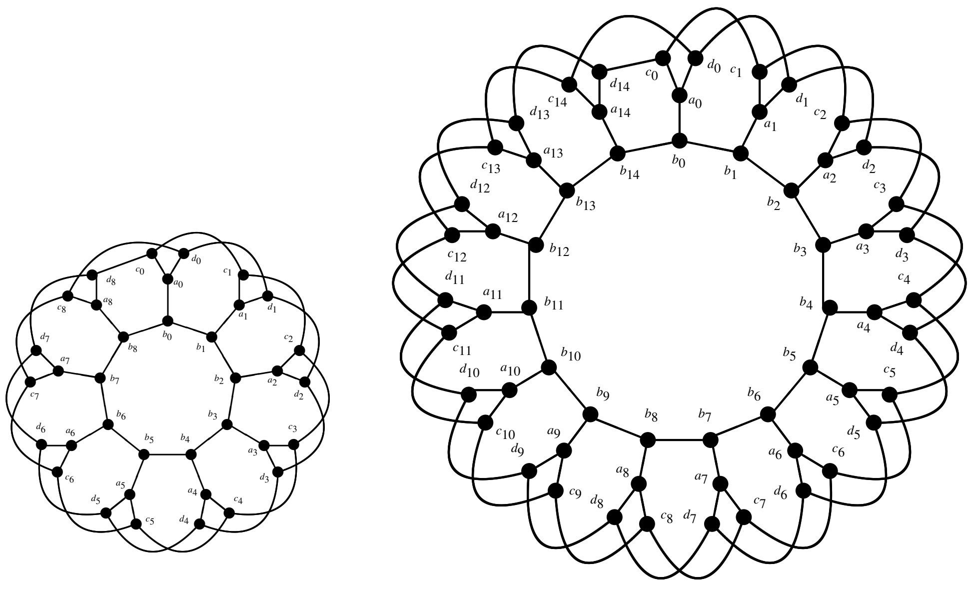

Figure 2 Flower snarks J_9 and J_{15}

Let $G = (V, E)$ be a graph, where order of G is $n = |V|$ and size of G is $m = |E(G)|$.

Let δ and Δ be the minimum and the maximum degree of vertices in G, respectively.

A set $S \subseteq V$ is the *dominating set* if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of the dominating set in G.

2. PREVIOUS WORK

2.1. Roman domination problem

There is extensive literature on the domination set problems in previous decades, and RDP as one of these problems was also intensively studied.

As mentioned above, the definition of RDF is given in (Cockayne et al. (2004)) and some basic properties of these functions were studied. The authors also found $\gamma_R(G)$ for some classes of graphs. In their article they established following relationship between domination and Roman domination numbers for arbitrary graph G:

$$\gamma(G) \le \gamma_R(G) \le 2 \cdot \gamma(G).$$

They characterized the graphs for which $\gamma_R(G) = \gamma(G) + k$ for $k \le 2$. Xing et al. (2006) extended this result to arbitrary k. Similar result for trees is given by Song and Wang (2006) with $\gamma_R(T) = \gamma(T) + 3$.

Of special interest for this paper are graphs with $\gamma_R(G) = 2\gamma(G)$ which are called Roman graphs, because we will prove that flower snark graphs are Roman graphs. Some well-known classes of graphs such as $P_{3k}, P_{3k+2}, C_{3k}, C_{3k+2}$ for $k \ge 1$ and $K_{m,n}$ for $\min(n,m) \ne 2$ were proved to be Roman graphs by Xueliang et al. (2009). They also proved that some regular graphs are Roman graphs: some subclasses of circulant and generalized Petersen graphs, Cartesian product of graphs $C_{5m} \Box C_{5n}$, where $m \ge 1, n \ge 1$. Characterization of Roman trees was presented by Henning (2002).

The proof that RDP is NP-hard in a general case was given by Dreyer (2000). In (Shang and Hu (2007)) it was proven that Roman domination problem in unit disk graphs is also NP-hard. Nascimento and Sampaio (2015) proved that the RDP is NP-hard even for the subgraphs of grid graphs and bipartite planar graphs with $\Delta = 4$. Nevertheless, for some classes of graphs solution could be find in polynomial time. Roman domination number for interval graphs could be calculated by linear-time algorithms (Liedloff et al. (2005)). The polynomial-time algorithm for AT-free graphs was presented in the same article.

It is of interest to find lower and upper bounds, both in general case and for some specific classes of graphs.

The following lower bound, established by Cockayne et al. (2004), is very useful thanks to its simplicity:

► Proposition 1. For any graph G, $\gamma_R(G) \ge \frac{2|V(G)|}{1+\Delta(G)}$.

In (Chambers et al. (2009)) it was proved that $\gamma_R(G) \le 4n/5$ if $\delta(G) \ge 1$ and $\gamma_R(G) \le 8n/11$ if $\delta(G) \ge 2$. Similarly, if $\delta \ge 3$ than $\gamma_R(G) \le 2n/3$ (Liu and Chang (2012b)).

Some lower and upper bounds using the diameter and the girth were proposed by Mobaraky and Sheikholeslami (2008). These bounds were improved by Bermudo et al. (2014) for $\delta(G) \ge 2$

$$\gamma_R(G) \le n - \left(\left\lfloor \frac{Diam(G)}{3} \right\rfloor + 1 \right) (\delta(G) - 1)$$

In the same paper two useful upper bounds were proposed:

► Proposition 2. Let G be a graph of order n. Then $\gamma_R(G) \le \left\lfloor \frac{2n\delta(G)}{3\delta(G)-1} \right\rfloor$.

► Proposition 3. If G is a graph of order n and size m, then $\gamma_R(G) \le \min \left\{ \left\lfloor \frac{3\Delta(G)n-2m}{3\Delta(G)-1} \right\rfloor, \left\lfloor \frac{(3\Delta(G)+4)n-2m}{3\Delta(G)+4} \right\rfloor \right\}$.

Moreover, if G is a C_5-free graph, then $\gamma_R(G) \le \left\lfloor \frac{(3\Delta(G)+2)n-2m}{3\Delta(G)+2} \right\rfloor$.

Another upper bound $\gamma_R(G) \le 2\left(1 - \frac{2^{1/\delta(G)}\delta(G)}{(1+\delta(G))^{1+1/\delta(G)}}\right) n$, including the proof that it is asymptotically best possible is given by Zverovich and Poghosyan (2011).

One of the most used estimates of the RD number is given by Cockayne et al. (2004) and concerns relationship between RD number and domination number of the same graph, namely, $\gamma(G) \le \gamma_R(G) \le 2\gamma(G)$. Combining works by Favaron et al. (2009) and Bermudo and Fernau (2012), Bermudo et al. (2014) derived:

► Proposition 4. If G is a graph of order $n \ge 3$, then $n - \gamma(G)(\Delta(G) - 1) \le \gamma_R(G) \le n - \frac{\gamma(G)}{2}$.

Another lower bound using total domination number, $\gamma_t(G) \le \gamma_R(G)$ is presented by Hedetniemi et al. (2013).

There are some specific bounds for connected graphs given in (Liu and Chang (2012a); Favaron et al. (2009)) respectively, $\gamma_R(G) \le \max\{\lceil 2n/3 \rceil, \lceil 23n/34 \rceil\}$ and $\gamma_R(G) + \gamma(G)/2 \le n$ where $n \ge 3$ is order of G. Finally, let G be a nontrivial, connected graph with maximum degree Δ. Then $\gamma_R(G) \ge \frac{\Delta+1}{\Delta}\gamma(G)$. The proof is given by Chellali et al. (2016).

The differential of a vertex set S is defined as $\partial(S) = |B(S)| - |S|$, where $B(S)$ is the set of vertices in $V \setminus S$ that have a neighbor in the vertex set S, and the differential of a graph is defined as $\partial(G) = \max\{\partial(S) | S \subseteq V\}$. A relation between the Roman domination number and the differential of a graph is studied by Bermudo et al. (2014).

For the Roman domination problem several integer linear programming (ILP) formulations was proposed. The first formulation was introduced by ReVelle and Rosing (2000). Another ILP formulations for Roman domination were proposed by Burger et al. (2013). These formulations were improved using a fewer number of constraints by Ivanović (2016).

Two approximation schemes, one 5-approximation algorithm of linear time and a one of polynomial-time were discussed by Shang and Hu (2007).

Since RDP could not be easily solved in the general case, of interest was to study it for different classes of graphs:

- interval graphs, cographs, asteroidal triple-free graphs and graphs with a d-octopus by Liedloff et al. (2005);
- corona graphs by Yero et al. (2013);
- grid graphs by Currò (2014);
- Generalized Sierpiński graphs by Ramezani et al. (2016);
- Generalized Petersen Graphs $GP(n;2)$ by Wang et al. (2011) and $GP(n;3)$ and $GP(n;4)$ by Zhiqiang Zhang and Xu (2014);
- cardinal product of paths and cycles in Klobučar and Puljić (2014, 2015);
- strongly chordal graphs by Liu and Chang (2013);
- digraphs by Sheikholeslami and Volkmann (2011);
- complementary prisms by Al Hashim (2017).

and others.

2.2. Restrained Roman domination

The problem of Restrained Roman domination is NP-hard which was proved in general case by Rad and Krzywkowski (2015). The Restrained Roman domination on graph $G = (V,E)$ is closely related to the Restrained domination problem which is to find a set of restrained domination of minimal cardinality, where the set of restrained domination $S \subseteq V$ is a set of vertices with neighbors both in S and in $V \setminus S$. The minimal cardinality of restrained domination set is denoted as γ_{str}.

Some of the properties of RRDF are given by Pushpam and Padmapriea (2015) and can be summarized as

► Proposition 5.
- For any graph G $\gamma_{str}(G) \leq \gamma_{rR} \leq 2\gamma_{str}(G)$;
- if G is a graph of order n and has a vertex od degree $n-1$ and $\delta(G) > 1$, then $\gamma_{rR} = 2(\gamma_{str}(G) = 1)$;
- if a graph G has C_3, then $\gamma_{rR} < n$;

Some bounds for the RRD number for connected graphs are given by Rad and Krzywkowski (2015).

► Proposition 6.
- For every connected graph G of order n, $\gamma_{rR}(G) < n+1-\lfloor(diam(G)-2)/3\rfloor$;
- for every connected graph G of order n and circumference $g(G)$ holds $\gamma_{rR}(G) < n+1-\lfloor(g(G)-2)/3\rfloor$;
- for every connected graph G of order n and size m the inequality $\gamma_{rR}(G) \leq 2m-n+2$ holds, and inequality is strict if and only if G is a tree with $\gamma_{rR} = n$.
- for every graph G of order n and if $\delta > 0$ and $n < \delta(\delta-1)/(\ln\delta - \ln 2 + 1)$ than holds

$$\gamma_{rR} \leq n\left(\frac{2\ln(1+\delta)-\ln 4+2}{\delta+1}\right)$$

In (Pushpam and Padmapriea (2015)) are given values of γ_{rR} for some classes of graphs:

► Proposition 7.
- For $n \geq 4$, $\gamma_{rR}(P_n) = \frac{2n+3+r}{3}, \quad n \equiv r(\bmod 3)$;
- $\gamma_{rR}(C_n) = \begin{cases} \frac{2n+3+r}{3}, & n \equiv r(\bmod 3),\ r \in \{1,2\} \\ \frac{2n}{3}, & n \equiv 0(\bmod 3) \end{cases}$;
- $\gamma_{rR}(K_n) = 2$;
- $\gamma_{rR}(K_{m,n}) = 4$.

Several relationships between γ_{rR} and γ_{rst} were proved in (Pushpam and Padmapriea (2015)). Especially, they characterized graphs where $\gamma rR = \gamma_{rst} + k$, $k \in \{1,2\}$ and some other bound on γ_{rR} for trees and bipartite graphs.

2.3. Signed Total Roman domination

Some useful bounds on STRD number are given by Volkmann (2016).

► Proposition 8. Let $f = (V_{-1}, V_1, V_2)$ be a STRDF in a graph G of order n. Let $\delta = \delta(G) \geq 1$ and $\Delta = \Delta(G)$. Then the following holds

- $(2\Delta-1)|V_2| + (\Delta-1)|V_1| \geq (\delta+1)|V_{-1}|$;
- $(2\Delta+\delta)|V_2| + (\Delta+\delta)|V_1| \geq (\delta+1)n$;
- $(\Delta+\delta)w(f) \geq (\delta+2-\Delta)n + (\delta-\Delta)|V_2|$;

- $w(f) \geq (\delta + 2 - 2\Delta)n/(2\Delta + \delta) + |V_2|$.

In the same paper exact values of γ_{stR} for some classes of graphs are proved:

- if $n \geq 3$ then $\gamma_{stR}(K_{1,n-1}) = \gamma_{stR}(K_n) = 3$;
- if $n \geq 1, \gamma_{stR}(K_{n,n}) = 2$, unless $n = 3$ in which case $\gamma_{stR}(K_{3,3}) = 4$;
- if C_n be a cycle of order $n \geq 3$, then $\gamma_{stR}(C_n) = n/2$ if $n \equiv 0(\mod 4)$, $\gamma_{stR}(C_n) = (n+3)/2$ if $n \equiv 1,3(\mod 4)$ and $\gamma_{stR}(C_n) = (n+6)/2$ if $n \equiv 2(\mod 4)$;
- if P_n be a path of order $n \geq 3$, then $\gamma_{stR}(P_n) = n/2$ if $n \equiv 0(\mod 4)$, $\gamma_{stR}(P_n) = \lceil (n+3)/2 \rceil$. otherwise.

Also, there are presented and proved some bounds

- let G be a graph of order n, $\delta \geq 1$. If $\delta < \Delta$, then $\gamma_{stR}(G) \geq \left\lceil \frac{(2\delta+3-2\Delta)n}{2\Delta+\delta} \right\rceil$;
- let G be a graph of order n, $\delta \geq 1$. Then a) $\gamma_{stR}(G) \leq n$, b) if $\delta \geq 3$ then $\gamma_{stR}(G) \leq n-1$;
- let G be a graph of order $n \geq 3$ and $\delta \geq 1$, then $\gamma_{stR}(G) \geq \frac{3}{2}(1+\sqrt{2n+1}) - n$;
- if G is connected graph of order $n \geq 3$ and size m, then $\gamma_{stR}(G) \geq \frac{11n-12m}{4}$;
- if G is bipartite graph of order $n \geq 3$ with $\delta(G) \geq 1$, then $\gamma_{stR}(G) \geq 3\sqrt{n} - n$;
- if G is graph of order $n \geq 3$ with $\delta(G) \geq 1$, then $\gamma_{stR}(G) \geq \max\{\Delta(G)+1-n, \delta(G)+4-n\}$;
- if T is a tree of order n and $\Delta(G) \geq 2$, then $\gamma_{stR}(T) \geq \Delta(G)+4-n$;
- if G is a graph of order n with $\delta(G) \geq 1$, then $\gamma_{stR}(G) \geq \left(1 + \left\lfloor \frac{diam(G)}{3} \right\rfloor\right)(\delta(G)+1) - n$;
- if G is an r-regular graph of order n such $r \geq 1$ and $n-r-1 \geq 1$ then $\gamma_{stR}(G) + \gamma_{stR}(\bar{G}) \geq \frac{4n}{n-1}$, and if n is even then $\gamma_{stR}(G) + \gamma_{stR}(\bar{G}) \geq 4(n-1)/(n-2)$ where $\bar{G}$ is complement of G.

3. THE NEW RESULTS

► Theorem 1. $\gamma_R(J_r) = 2r$

Proof. Step 1. $\gamma_R(J_r) \leq 2r$
Let us define the function $f : V_{J_r} \to \{0,1,2\}$ as partition (V_0, V_1, V_2), such that $V_2 = \{a_i | i = 0, ..., r-1\}$, $V_0 = V_{J_r} \setminus V_2$ and $V_1 = \emptyset$. Since $\bigcup_{i=0}^{r-1} \mathbb{N}[a_i] = \bigcup_{i=0}^{r-1} \{a_i, b_i, c_i, d_i\} = V_{J_r}$ implying that each vertex from V_0 has at least one a-vertex as its neighbor. Since all a-vertices are in V_2, then each vertex from V_0 has at least one neighbor from V_2, so f is Roman domination function with value $f(V_{J_r}) = 2r$ so $\gamma_R(J_r) \leq 2r$.

Step 2. $\gamma_R(J_r) \geq 2r$
It is easy to see that J_r is a regular graph of degree 3, with $4r$ vertices. Then, by Proposition 1 it holds $\gamma_R(J_r) \geq \lceil \frac{2 \cdot 4 \cdot r}{1+3} \rceil = 2r$. ◄

► Proposition 9. $\gamma_{rR}(J_r) = 2r$

Proof. The function f defined in the proof of Theorem 1 has additional property that each vertex from V_0 has at least neighbor from V_0:

- $b_i, b_{i+1} \in V_0$ and $b_{i+1} \in \mathbb{N}[b_i]$;
- If $i \leq r-2$ then $c_i, c_{i+1} \in V_0$ and $c_{i+1} \in \mathbb{N}[c_i]$;
- If $i = r-1$ then $c_{r-1}, d_0 \in V_0$ and $c_{r-1} \in \mathbb{N}[d_0]$;
- If $i \leq r-2$ then $d_i, d_{i+1} \in V_0$ and $d_{i+1} \in \mathbb{N}[d_i]$;
- If $i = r-1$ then $d_{r-1}, c_0 \in V_0$ and $d_{r-1} \in \mathbb{N}[c_0]$.

Having in mind the proof of Theorem 1, the function f is restrained Roman dominating function of J_r, so $\gamma_{rR}(J_r) \leq 2r$. On the other hand, for any graph G holds $\gamma_{rR}(G) \geq \gamma_R(G)$, implying $\gamma_{rR}(J_r) \geq \gamma_R(J_r) = 2r$. Therefore, $\gamma_{rR}(J_r) = 2r$. ◄

► Theorem 2. $\gamma_{stR}(J_r) \leq 3r$

Proof. Let us define the function $f : V_G \to \{-1,1,2\}$ as in (1):

$$f(v) = \begin{cases} -1, \ v = a_i \\ 1, \ v = c_i \vee v = d_i \\ 2, \ v = b_i \end{cases} \qquad i = 0, \ldots, r-1 \tag{1}$$

Firstly, each vertex with value -1 has (at least) one neighbor with value 2, since $f(a_i) = -1$, $f(b_i) = 2$ and $b_i \in N(a_i)$.

Secondly, the sum of function values in the open neighborhood of each vertex is at least 1, since:

- $f(N(a_i)) = f(b_i) + f(c_i) + f(d_i) = 2+1+1 = 4 \geq 1$;
- $f(N(b_i)) = f(a_i) + f(b_{i-1}) + f(b_{i+1}) = -1+2+2 = 3 \geq 1$;
- If $1 \leq i \leq r-2$, $f(N(c_i)) = f(a_i) + f(c_{i-1}) + f(c_{i+1}) = -1+1+1 = 1 \geq 1$;
- If $i = 0$, $f(N(c_0)) = f(a_0) + f(d_{r-1}) + f(c_1) = -1+1+1 = 1 \geq 1$;
- If $i = r-1$, $f(N(c_{r-1})) = f(a_{r-1}) + f(c_{r-2}) + f(d_0) = -1+1+1 = 1 \geq 1$;
- If $1 \leq i \leq r-2$, $f(N(d_i)) = f(a_i) + f(d_{i-1}) + f(d_{i+1}) = -1+1+1 = 1 \geq 1$;
- If $i = 0$, $f(N(d_0)) = f(a_0) + f(c_{r-1}) + f(d_1) = -1+1+1 = 1 \geq 1$;
- If $i = r-1$, $f(N(d_{r-1})) = f(a_{r-1}) + f(d_{r-2}) + f(c_0) = -1+1+1 = 1 \geq 1$.

Finally, $f(V_{J_r}) = f\left(\bigcup_{i=0}^{r-1}\{a_i, b_i, c_i, d_i\}\right) = \sum_{i=0}^{r-1}(f(a_i)+f(b_i)+f(c_i)+f(d_i)) = r\cdot(-1+2+1+1) = 3r$.
Therefore f is the signed total Roman domination function of flower snarks graphs with value $3r$, so $\gamma_{stR}(J_r) \leq 3r$. ◀

4. CONCLUSION

In this paper we have found and proved the exact value for the Roman and restrained Roman domination number of flower snark graphs. The upper bound of the signed total Roman domination number is given, along with the appropriate signed total Roman domination sets.

In the future work the problem of finding these Roman domination numbers for other challenging classes of graphs could be considered. Another direction of future research would be to determine other Roman domination numbers for flower snark graphs.

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