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k-METRIC ANTIDIMENSION OF WHEELS AND GRID GRAPHS

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Abstract: *In this paper we study the k-metric antidimension problem on two special classes of graphs: wheels W_n and grid graphs $G_{m,n}$. We prove that W_n is n-metric antidimensional and find the k-metric antidimension for each k where it exists. For $G_{m,n}$ we find the k-metric antidimension for $k = 1,2$. Additionally, we determine 4-metric antidimension in the case when m and n are both odd.*

Keywords: *k-metric antidimension, Wheel graphs, Grid graphs*

1. INTRODUCTION

The concept of metric dimension of a graph G was introduced independently by Slater (1975) and Harary and Melter (1976). It is based on the notion of resolving set R of vertices which has the property that each vertex is uniquely identified by its metric coordinates with respect to R. The minimal cardinality of resolving sets is called the metric dimension of graph G.

Some interesting applications of the metric dimension include navigation of robots in networks (Khuller et al., 1996), applications to chemistry (Johnson, 1993, 1998) and application in computer graphics (Melter and Tomescu, 1984).

There are several variations of the metric dimension concept:

- Weighted metric dimension (Epstein et al., 2015);
- Resolving dominating sets (Brigham et al., 2003);
- Local metric dimension (Okamoto et al., 2010);
- Independent resolving sets (Chartrand et al., 2003);
- Strong metric dimension (Sebő and Tannier, 2004);
- Minimal doubly resolving sets (Cáceres et al., 2007);
- k-metric dimension (Estrada-Moreno et al., 2015);
- Simultaneous metric dimension (Ramírez-Cruz et al., 2014).

Recently, Trujillo-Rasua and Yero (2016a) introduced the concepts of k-antiresolving set S and k-metric antidimension. Different vertices of $V \setminus S$ now can have the same metric coordinates with respect to S, but no vertex can be identified with probability higher than $1/k$. For a given k, the minimal cardinality of k-antiresolving sets represents the k-metric antidimension of graph G. Zhang and Gao (2017) have proved that the problem of finding the k-metric antidimension is NP-hard in general case.

The concept of k-metric antidimension has been used to define privacy measures aimed at evaluating the resistance of social graphs to active attacks. Trujillo-Rasua and Yero (2016a) define a novel privacy measure, so called (k,l)-anonymity. Mauw et al. (2016) propose the first privacy-preserving transformation method for social graphs that counteracts active attacks.

The k-metric antidimension of special classes of graphs has been studied by several authors. Trujillo-Rasua and Yero (2016a,b) consider the k-metric antidimension of paths, cycles, trees and complete bipartite graphs. They also provided efficient algorithms which can be used to decide whether a tree or a unicyclic graph is 1-metric antidimensional.

Zhang and Gao (2017) analyze the size of k-antiresolving sets in random graphs and in the case of Erdos-Renyi random graphs establish three bounds on the size.

Kratica et al. (2018) study mathematical properties of the k-antiresolving sets and the k-metric antidimension of some generalized Petersen graphs. In this paper the analysis is extended to wheels and grid graphs.

2. PROBLEM DEFINITION

Let $G = (V,E)$ be a simple connected undirected graph. The degree of a vertex of a graph is the number of edges incident to that vertex. The maximum degree of a graph G, denoted by Δ_G, is the maximum degree of its vertices.

Let us denote by $d(u,v)$ the length of the shortest path between vertices u and v. The metric representation $r(v|S)$ of vertex v with respect to an ordered set of vertices $S = \{u_1,...,u_t\}$ is defined as $r(v|S) = (d(v,u_1),...,d(v,u_t))$. Value $d(v,u_i)$ represents the metric coordinate of v with respect to vertex u_i, $i = 1,...,t$. The following definitions introduce the concepts of k-antiresolving set, k-metric antidimension of graph G and the notion of k-metric antidimensional graph.

► Definition 1. (Trujillo-Rasua and Yero (2016a)) Set S is called a k-antiresolving set for G if k is the largest positive integer such that for every vertex $v \in V \setminus S$ there exist at least $k-1$ different vertices $v_1,...,v_{k-1} \in V \setminus S$ with $r(v|S) = r(v_1|S) = ... = r(v_{k-1}|S)$, i.e. v and $v_1,...,v_{k-1}$ have the same metric representation with respect to S.

► Definition 2. (Trujillo-Rasua and Yero (2016a)) For fixed k, the k-metric antidimension of graph G, denoted by $adim_k(G)$, is the minimum cardinality amongst all k-antiresolving sets in G. A k-antiresolving set of cardinality $adim_k(G)$ is called a k-antiresolving basis of G.

► Definition 3. (Trujillo-Rasua and Yero (2016a)) Graph G is k-metric antidimensional if k is the largest integer such that G contains a k-antiresolving set.

Now the k-metric antidimension problem can be formulated as follows: for a given k find the k-metric antidimension of graph G if it exists. The following two properties will be used in the proofs of theorems in Section 3 and Section 4.

► Property 1. (Trujillo-Rasua and Yero (2016a)) If G is a connected k-metric antidimensional graph of maximum degree Δ_G, then $1 \leq k \leq \Delta_G$.

Property 2 presents a simple necessary and sufficient condition for a set of vertices to be k-antiresolving. Let $S \subset V$ be a subset of vertices of G and let ρ_S be equivalence relation on G defined by

$$(\forall a,b \in V)\ (a\rho_S b \ \Leftrightarrow\ r(a|S) = r(b|S))$$

and let $S_1,...,S_m$ be the equivalence classes of ρ_S. It is easy to see that the following property is satisfied.

► Property 2. (Kratica et al. (2018)) Let k be a fixed integer, $k \geq 1$. Then S is a k-antiresolving set for G if and only if $\min\limits_{1 \leq i \leq m} |S_i| = k$.

3. WHEELS

Wheel $W_n = (V,E)$ of dimension n is a graph with $n+1$ vertices and $2n$ edges, where central vertex v_0 is connected to all vertices, while other vertices v_i, $i = 1,...n$, are connected as in a cycle. Hence central vertex v_0 has n neighbours, while all other vertices $v_i, i = 1,...,n$, have three neighbours.

► Theorem 1. For $n \geq 6$ graph W_n is n-metric antidimensional and

$$adim_k(W_n) = \begin{cases} 2, \ k = 1 \vee k = 2 \\ 1, \ k = 3 \vee k = n \end{cases}$$

Proof. Step 1. $adim_1(W_n) \leq 2$

Let us consider set $S = \{v_1, v_2\}$. The equivalence classes of ρ_S are given in Table 1. More precisely, the first column of Table 1 contains set S, while in the second and the third column the equivalence classes of relation ρ_S and their cardinalities are given. In the fourth column the corresponding metric representations with respect to S are shown. Since the minimal cardinality of equivalence classes is one, according to Property 2, it follows that $S = \{v_1, v_2\}$ is 1-antiresolving set.

Step 2. $adim_1(W_n) = 2$

Suppose the contrary, that $adim_1(W_n) = 1$. Then there exists an 1-antiresolving set S of cardinality one. We have two cases:

- Case 1. Suppose that $S = \{v_0\}$. From Table 1 it is evident that there exists only one equivalence class $\{v_i | i = 1,...,n\}$ of cardinality n.

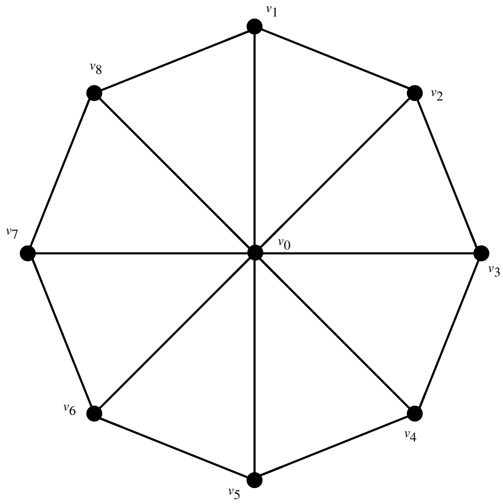

Figure 1 Graph W_8

- Case 2. Suppose that $S = \{v_i\}$, $1 \leq i \leq n$. Without loss of generality we can assume $i = 1$. From Table 1 it can be seen that for $n \geq 6$ the equivalence class with minimal cardinality is $\{v_0, v_2, v_n\}$.

In both cases we have a contradiction with the assumption that there exists an 1-antiresolving set S of cardinality one, so $adim_1(W_n) \geq 2$. According to Step 1, $adim_1(W_n) = 2$.

Step 3. $adim_2(W_n) = 2$

Let $S = \{v_0, v_i\}$, $1 \leq i \leq n$. Without loss of generality we can assume $i = 1$. Since $\{v_2, v_n\}$ is the equivalence class of minimal cardinality (see Table 1), according to Property 2 it follows that $S = \{v_0, v_1\}$ is a 2-antiresolving set. As it can be seen from Step 2, singleton sets S have equivalence classes with cardinality of at least three, so such sets can not be 2-antiresolving sets. Therefore, $S = \{v_0, v_1\}$ is a 2-antiresolving basis of W_n and $adim_2(W_n) = 2$.

Step 4. $adim_3(W_n) = 1$

Let us consider set $S = \{v_1\}$. Since the equivalence class with the minimal cardinality is $\{v_0, v_2, v_n\}$, according to Property 2 it follows that $S = \{v_1\}$ is a 3-antiresolving set. As $|S| = 1$, $S = \{v_1\}$ is a 3-antiresolving basis of W_n, so $adim_3(W_n) = 1$.

Step 5. $adim_n(W_n) = 1$

Let $S = \{v_0\}$. There is only one equivalence class $\{v_i | i = 1, ..., n\}$ of cardinality n, so it is obvious that $S = \{v_0\}$ is an n-antiresolving set of W_n. Since $|S| = 1$, it is also an n-antiresolving basis of W_n. ◀

Table 1: Equivalence classes of ρ_S on W_n

S	*Equivalence class*	*Card.*	*M. rep.*
$\{v_1, v_2\}$	$\{v_0\}$	1	(1,1)
	$\{v_n\}$	1	(1,2)
	$\{v_3\}$	1	(2,1)
	$\{v_i \| i = 4, ..., n-1\}$	$n-4$	(2,2)
$\{v_0, v_1\}$	$\{v_2, v_n\}$	2	(1,1)
	$\{v_i \| i = 3, ..., n-1\}$	$n-3$	(1,2)
$\{v_1\}$	$\{v_0, v_2, v_n\}$	3	(1)
	$\{v_i \| i = 3, ..., n-1\}$	$n-3$	(2)
$\{v_0\}$	$\{v_i \| i = 1, ..., n\}$	n	(1)

Values $adim_k(W_n)$ for $n \in \{3, 4, 5\}$ and the corresponding k-antiresolving bases are obtained by total enumeration and presented in Table 2.

► Theorem 2. For $l \in \{4, ..., n-1\}$ there does not exist $S \subset V$ such that S is an l-antiresolving set.

Proof. From the proof of Theorem 1 it follows that for $n \geq 6$, $S = \{v_0\}$ is an n-antiresolving set of W_n and $S = \{v_i\}$, $1 \leq i \leq n$, is a 3-antiresolving set of W_n. Also, $S = \{v_0, v_i\}$, $1 \leq i \leq n$, is a 2-antiresolving set. Let us consider all other possibilities for S.

Table 2: $adim_k(W_n)$ for $n \in \{3,4,5\}$

n	k	*Basis*	$adim_k(W_n)$
3	1	$\{v_1, v_2, v_3\}$	3
	2	$\{v_1, v_2\}$	2
	3	$\{v_1\}$	1
4	1	$\{v_1\}$	1
	2	$\{v_0, v_2, v_4\}$	3
	3	$\{v_1, v_3\}$	2
	4	$\{v_0\}$	1
5	1	$\{v_1, v_2\}$	2
	2	$\{v_1\}$	1
	3, 4	*does not exist*	*undefined*
	5	$\{v_0\}$	1

Case 1. $|S| \geq 2$, $v_0 \notin S$.

- If $S = \{v_1, ..., v_n\}$ then there is only one equivalence class $\{v_0\}$ with respect to S and hence S is an 1-antiresolving set.
- If $S \subset \{v_1, ..., v_n\}$ then there exists $v_i \in S$ such that $v_{i'} \notin S$ or $v_{i''} \notin S$, where $v_{i'}$ and $v_{i''}$ represent the previous and the next vertex in the cycle, respectively. Formally, $i' = \begin{cases} i-1, \; 2 \leq i \leq n \\ n, \; i = 1 \end{cases}$ and $i'' = \begin{cases} i+1, \; 1 \leq i \leq n-1 \\ 1, \; i = n \end{cases}$. Then v_0 and either $v_{i'}$ or $v_{i''}$, or both, are the only vertices from $V \setminus S$ which have 1 as a coordinate with respect to v_i. It follows that the equivalence class of ρ_S with minimal cardinality, has cardinality less or equal to three, i.e. S is an l-antiresolving set, where $l \leq 3$.

Case 2. $|S| \geq 3$, $v_0 \in S$.
As $v_0 \in S$ and $S \subset V$, it follows that there exists $v_i \in S$ such that $v_{i'} \notin S$ or $v_{i''} \notin S$, where $v_{i'}$ and $v_{i''}$ again represent the previous and the next vertex in the cycle, respectively. Now $v_{i'}$ or $v_{i''}$, or both, are the only vertices from $V \setminus S$ which have 1 as a coordinate with respect to v_i. Similarly as in Case 1 it follows that S is an l-antiresolving set where $l \leq 2$. ◀

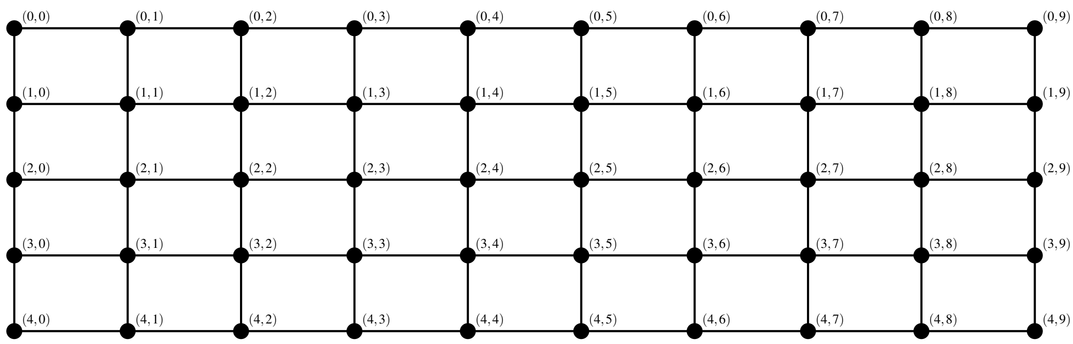

Figure 2 Graph $G_{5,10}$

4. GRID GRAPHS

Grid graph $G_{m,n} = (V,E)$ of dimension $m \cdot n,\ m,n > 1$, can be considered as a graph with set of vertices $V = \{(i,j) | 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$, where vertices are organized in m rows with n vertices in each row. Two vertices are adjacent if they belong to the same row and to adjacent columns, or to the same column and to adjacent rows. Formally, vertices (i,j) and (i',j') are adjacent if $i = i'$ and $|j - j'| = 1$ or $|i - i'| = 1$ and $j = j'$, for $0 \leq i,i' \leq m-1$ and $0 \leq j,j' \leq n-1$. The interior vertices of the grid graph have 4 neighbors, the vertices along the sides have 3 neighbors and only four of them (in the corners of the rectangle) have 2 neighbors.

It is obvious that a grid graph $G_{m,n}$ can be viewed as a Cartesian product of paths P_m and P_n, i.e. $G_{m,n} \cong P_m \square P_n$. Moreover, grid graphs of the dimensions $m \cdot n$ and $n \cdot m$ are isomorphic ($G_{m,n} \cong G_{n,m}$), so without loss of generality we can suppose that $m \leq n$.

Since $G_{1,n} \cong P_n$ and the k-metric antidimension of paths is considered in (Trujillo-Rasua and Yero, 2016a), we consider work only cases when $m,n \geq 2$.

► Theorem 3. For $m,n \geq 2$ it follows:

- $adim_1(G_{m,n}) = 1$;
- $adim_2(G_{m,n}) = \begin{cases} 2, & m,n \text{ both even} \\ 1, & otherwise \end{cases}$;
- $adim_4(G_{m,n}) = 1$ for m, n both odd.

Proof. Step 1. $adim_1(G_{m,n}) = 1$
Let us consider set $S = \{(0,0)\}$. The equivalence classes of ρ_S are given in Table 3 (item 1). Since the equivalence class with minimal cardinality is $\{(m-1,n-1)\}$, according to Property 2, it follows that $S = \{(0,0)\}$ is an 1-antiresolving set. As $|S| = 1$, $S = \{(0,0)\}$ is an 1-antiresolving basis of $G_{m,n}$, so $adim_1(G_{m,n}) = 1$.

Step 2. If both m and n are even, $adim_2(G_{m,n}) = 2$
Let us define set $S = \{(0,0),(m-1,n-1)\}$. The equivalence classes of ρ_S are given in Table 3 (item 2). Since the equivalence classes with minimal cardinality are $\{(0,1),(1,0)\}$ and $\{(m-2,n-1),(m-1,n-2)\}$, according to Property 2, it follows that $S = \{(0,0),(m-1,n-1)\}$ is a 2-antiresolving set. Next we will prove that there does not exists a 2-antiresolving set S of $G_{m,n}$ of cardinality one. Suppose the contrary, that there exists a 2-antiresolving set of cardinality one, $S = \{(i',j')\}$. Let us define $i'' = \begin{cases} 0, & i' \geq m/2 \\ m-1, & i' < m/2 \end{cases}$ and $j'' = \begin{cases} 0, & j' \geq n/2 \\ n-1, & j' < n/2 \end{cases}$. Vertex (i'',j'') is the unique most distant vertex from vertex (i',j'), so $\{(i'',j'')\}$ is an equivalence class of ρ_S of cardinality one, which is in contradiction with the assumption that $S = \{(i',j')\}$ is a 2-antiresolving set of $G_{m,n}$. Hence $adim_2(G_{m,n}) = 2$.

Step 3. If either m or n is odd, $adim_2(G_{m,n}) = 1$
Case 1. Both m and n are odd.
Let $p = \lfloor m/2 \rfloor$ and $q = \lfloor n/2 \rfloor$, i.e. $m = 2p+1$ and $n = 2q+1$. Let us define set $S = \{(p-1,q)\}$. Since both m and n are odd and $m,n \geq 2$ it follows that $m,n \geq 3$. As $m = 2p+1 \geq 3$ and $n = 2q+1 \geq 3$ then $p,q \geq 1$. Equivalence classes of ρ_S are given in Table 3 (item 3). For each r, $2 \leq r \leq p+q$ there exists vertex (i',j') such that $j' < q$ and $d((p-1,q),(i',j')) = r$. Due to symmetry of $G_{m,n}$, $d((p-1,q),(i',2q-j')) = r$, and hence each equivalence class ρ_S has cardinality at least two. Since equivalence class $\{(m-1,0),(m-1,n-1)\}$ has cardinality two it follows that $S = \{(p-1,q)\}$ is a 2-antiresolving set. As $|S| = 1$, $S = \{(p-1,q)\}$ is a 2-antiresolving basis of $G_{m,n}$, so $adim_2(G_{m,n}) = 1$.
Case 2. m is even and n is odd.
Let $p = \lfloor m/2 \rfloor$ and $q = \lfloor n/2 \rfloor$, i.e. $m = 2p$ and $n = 2q+1$. Let us define set $S = \{(p,q)\}$. Since n is odd and $n \geq 2$ it follows that $n \geq 3$. As $n = 2q+1 \geq 3$ then $q \geq 1$. The equivalence classes of ρ_S are given in Table 3 (item 4). Similarly as in Case 1, for each r, $2 \leq r \leq p+q-1$ there exists vertex (i',j') such that $j' < q$ and $d((p,q),(i',j')) = r$. Due to symmetry of $G_{m,n}$, $d((p,q),(i',2q-j')) = r$, and hence each equivalence class ρ_S has cardinality at least two. Since equivalence class $\{(0,0),(0,n-1)\}$ has cardinality two it follows that $S = \{(p,q)\}$ is a 2-antiresolving set. As $|S| = 1$, $S = \{(p,q)\}$ is a 2-antiresolving basis of $G_{m,n}$, so $adim_2(G_{m,n}) = 1$.
Case 3. m is odd and n is even.
Having in mind the fact that $G_{m,n} \cong G_{n,m}$, this case is reduced to Case 2.

Step 4. If both m and n are odd, $adim_4(G_{m,n}) = 1$
Let $p = \lfloor m/2 \rfloor$ and $q = \lfloor n/2 \rfloor$, i.e. $m = 2p+1$ and $n = 2q+1$. Let us define set $S = \{(p,q)\}$. Since both m and n are odd and $m,n \geq 2$ it follows that $m,n \geq 3$. As $m = 2p+1 \geq 3$ and $n = 2q+1 \geq 3$ then $p,q \geq 1$. The equivalence classes of ρ_S are given in Table 3 (item 5). For each r, $2 \leq r \leq p+q-1$ there exists vertex (i',j') such that $i' < p$, $j' < q$ and $d((p,q),(i',j')) = r$. Due to symmetries of $G_{m,n}$ we have $d((p,q),(i',2q-j')) = r$, $d((p,q),(2p-i',j')) = r$ and $d((p,q),(2p-i',2q-j')) = r$ and hence each equivalence class of ρ_S has cardinality at least four. Since equivalence class $\{(0,0),(0,n-1),(m-1,0),(m-1,n-1)\}$ has cardinality four, it follows that $S = \{(p,q)\}$ is a 4-antiresolving set. As $|S| = 1$, $S = \{(p,q)\}$ is a 4-antiresolving basis of $G_{m,n}$, so $adim_4(G_{m,n}) = 1$.

◀

5. CONCLUSION

In this article the k-metric antidimension problem is considered on wheels and grid graphs. Exact values of the k-metric antidimension of wheels W_n are obtained for $k \in \{1,2,3,n\}$ and it is proved that the k-metric antidimension does not exists for $4 \leq k \leq n-1$. In the case of grid graphs $G_{m,n}$ the exact values of the k-metric antidimension are obtained for $k \in \{1,2\}$ for arbitrary m, n and for $k = 4$ when m and n are both odd.

In future reseach it would be interesting to identify the cases when k-metric antidimension of $G_{m,n}$ does not exist. Also the k-metric antidimension problem could be considered on some other challenging classes of graphs.

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Table 3: Equivalence classes of ρ_S on $G_{m,n}$

S	*Equivalence class*	*Card.*	*M. rep.*	*Condidtion*
$\{(0,0)\}$	$\{(0,1),(1,0)\}$	2	(1)	
	$\{(i,j)\|i+j=r, i=0,...,r\}$	$r+1$	(r)	$r=2,...,m-1$
	$\{(i,j)\|i+j=r, i=0,...,m-1\}$	m	(r)	$r=m,...,n-1$
	$\{(i,j)\|i+j=r, i=r-n+1,...,m-1\}$	$m+n-r-1$	(r)	$r=n,...,m+n-3$
	$\{(m-1,n-1)\}$	1	$(m+n-2)$	
$\{(0,0),(m-1,n-1)\}$	$\{(0,1),(1,0)\}$	2	$(1,m+n-3)$	
	$\{(i,j)\|i+j=r, i=0,...,r\}$	$r+1$	$(r,m+n-2-r)$	$r=2,...,m-1$
	$\{(i,j)\|i+j=r, i=0,...,m-1\}$	m	$(r,m+n-2-r)$	$r=m,...,n-1$
	$\{(i,j)\|i+j=r, i=r-n+1,...,m-1\}$	$m+n-r-1$	$(r,m+n-2-r)$	$r=n,...,m+n-3$
	$\{(m-2,n-1),(m-1,n-2)\}$	2	$(m+n-3,1)$	
$\{(p-1,q)\}$	$\{(p-2,q),(p,q),(p-1,q-1),(p-1,q+1)\}$	4	(1)	$m=2p+1$ and $n=2q+1$
	$\{(p-i-1,q-j),(p-i-1,q+j),(p+i-1,q-j),(p+i-1,q+j)\|i+j=r, 0\le i\le m-1, 0\le j\le n-1\}$	≥ 2	(r)	$r=2,...,p+q$
	$\{(m-1,0),(m-1,n-1)\}$	2	(p+q+1)	
$\{(p,q)\}$	$\{(p-1,q),(p+1,q),(p,q-1),(p,q+1)\}$	4	(1)	$m=2p$ and $n=2q+1$
	$\{(p-i,q-j),(p-i,q+j),(p+i,q-j),(p+i,q+j)\|i+j=r, 0\le i\le m-1, 0\le j\le n-1\}$	≥ 2	(r)	$r=2,...,p+q-1$
	$\{(0,0),(0,n-1)\}$	2	(p+q)	
$\{(p,q)\}$	$\{(p-1,q),(p+1,q),(p,q-1),(p,q+1)\}$	4	(1)	$m=2p+1$ and $n=2q+1$
	$\{(p-i,q-j),(p-i,q+j),(p+i,q-j),(p+i,q+j)\|i+j=r, 0\le i\le m-1, 0\le j\le n-1\}$	≥ 4	(r)	$r=2,...,p+q-1$
	$\{(0,0),(0,n-1),(m-1,0),(m-1,n-1)\}$	4	(p+q)	