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270 Minutes on Categorial Proof Theory

Abstract. The aim of these notes is to provide an introduction of basic categorial notions to a reader interested in logic and proof theory. The first part is devoted to justification of these notions through a cut elimination procedure. In the second part a classification of formulae up to isomorphism, and an example of coherence are given.

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Contents

1. About these notes

This text is based on a series of talks delivered by the author at the Algebra and Logic Seminar of the Mathematical Faculty in Belgrade. Zoran Petrović, the chairman of the seminar, invited the author to prepare a short course on categorial proof theory. Once a week, during three weeks of spring 2014, a 90 minutes talk was organized. Hence the title "270 Minutes...".

The aim of these notes is to provide an introduction of basic categorial notions to a reader interested in logic and proof theory. After this introduction, two important topics of categorial proof theory, namely the classification of formulae up to isomorphism and coherence, are given by simple examples.

Categorial proof theory as a field of general proof theory was established during the 1960s. We refer to [2, Preface] for some historical notes on this topic. We can't go into details here because our intention is to make a text readable in four and a half hour. However, two names should be mentioned. The work of Jim Lambek and Bill Lawvere made a substantial influence to the field.

In order to make it self-contained, the definitions of relevant categorial and other notions are given in frame boxes at appropriate places throughout this text. Some proofs are not finished—this is left as an exercise for the reader.

2. The three propositional languages

We deal with fragments of propositional logic throughout this text. Our considerations will vary from one such fragment to another keeping the context as simple as possible. We introduce three propositional languages whose *alphabet* consists of an infinite set of propositional letters p, q, $r, \ldots$, two binary propositional connectives $\wedge$, $\rightarrow$, one nullary connective $\top$ and two auxiliary symbols (and). However, it is not the case that all the symbols occur in every language. The *words*, are the finite sequences of symbols from the alphabet. A *language* is a set of words called *formulae*.

The *formulae* are defined inductively as follows.

(1) Propositional letters and $\top$ are formulae.

(2) If A and B are formulae, then $(A \wedge B)$ and $(A \rightarrow B)$ are formulae.

(3) Nothing else is a formula.

We omit the outermost parentheses in formulae taking them for granted. The *implicational language* consists of all the formulae in which $\top$ and $\wedge$ do not occur. The *conjunction language* consists of all the formulae in which $\top$ and $\rightarrow$ do not occur. The *mixed language* consists of all the formulae.

The mixed language has its *strict* variant which we introduce for the sake of simplicity of the forthcoming notions. In this variant, we consider $\wedge$ to be associative and $\top$ to be the unit for $\wedge$. The *formulae* of this language are defined inductively as follows.

(1) Propositional letters and $\top$ are formulae called *atomic*.

(2) If A and B are formulae, then $\langle A\rangle \rightarrow \langle B\rangle$ is a formula (called *implication*), where $\langle X\rangle$ is (X), when X is not atomic, or X, otherwise.

(3) If A and B are formulae different from $\top$, then $\|A\| \wedge \|B\|$ is a formula, where $\|X\|$ is (X), when X is an implication, or X, otherwise.

(4) Nothing else is a formula.

For example, $(p \wedge q \wedge r) \rightarrow p$ is a formula of the strict mixed language.

In Section 4, where we use this language, we write just $A \rightarrow B$ instead of the more complicated, but correct $\langle A\rangle \rightarrow \langle B\rangle$. Also, we write just $A \wedge B$, which is A when B is $\top$, or which is B when A is $\top$, or which is $\|A\| \wedge \|B\|$. This is just in order not to overburden the notation.

Formal systems, equational systems

A *formal system* is given by:

1. *Alphabet*, a set of *symbols*.

2. *Language*, a set of words in this alphabet. The elements of the language are called *formulae*.

3. *Axioms*, a set of formulae.

4. *Rules of inference*, a set of relations on the language of arities greater or equal to 2, where $\rho(A_1, \ldots, A_n, A)$ means that the formula A is a direct consequence of the formulae $A_1, \ldots, A_n$. The rules of inference given by schematic line rules are called *inference figures*.

A *derivation* in a formal system is a finite tree with formulae in its vertices. The formulae in the leaves, which are not axioms, are called *hypotheses*, and the *derived formula* is in the root. Every branching of this tree corresponds to one of the rules of inference. For example, if ρ is a rule of inference and we have $\rho(A_1, A_2, A_3, A)$, then a derivation may contain the following branching:

$$\begin{array}{ccc} A_1 & A_2 & A_3 \\ \diagdown & | & \diagup \\ & A & \end{array}$$

A *proof* is a derivation without hypothesis. A formula derived without hypothesis is a *theorem* of the system.

An *equational system* is a formal system whose alphabet contains a special symbol usually denoted by $=$. The formulae of an equational system are of the form $u = v$, where u and v are words in the alphabet, called *terms*. The axioms and the rules of inference are such that the relation $\equiv$, defined on the set of terms by

$$u \equiv v \quad \text{when} \quad u = v \text{ is a theorem of the system,}$$

is an equivalence relation congruent with the operations on terms corresponding to the operation symbols of the alphabet.

3. Natural deduction and sequent systems

In this section, we concentrate on the implicational language only. A *Hilbert-style system* H, for implicational fragment of intuitionistic logic, is a formal system in this language, given by the following axiom schemata

$$A \to (B \to A)$$

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$$

and modus ponens (MP) as the only inference figure.

$$\frac{A \to B \quad A}{B}$$

EXERCISE. *Show that* $(p \to q) \to ((q \to r) \to (p \to r))$ *is a theorem of* H.

This is not an easy task. It would not be a surprise if one starts with proving a metatheorem like the deduction theorem. Then, since r is easily deducible from the hypotheses $p \to q$, $q \to r$ and p, the deduction theorem guarantees the existence of a direct proof, leaving it implicit. Gentzen's idea was to built in the system all the deduction power and leave metatheorems for more serious results.

A *natural deduction system*, for the corresponding fragment of logic, is a formal system in the implicational language without axioms and with the following two inference figures. (The notation $[A]$ in the first figure below means that, after applying this rule of inference, an arbitrary number of hypotheses of the form A above the formula B are no longer active in the derivation.)

$$\dfrac{\begin{array}{c}[A]\\ B\end{array}}{A \to B} \qquad \dfrac{A \to B \quad A}{B}$$

Simply-typed λ calculus

This is an equational system whose alphabet consists of symbols λ, ., $\to$, (,), :, =, and two disjoint infinite sets, the set of *variables* x, y, $z, \dots$ and the set of propositional letters (usually called *atomic types*).
There is a function, called *typing*, which maps the set of variables to the set of implicational formulae. For every implicational formula, there are infinitely many variables mapped to this formula.
The *terms* are words in this alphabet, defined inductively as follows:

(1) If x is a variable whose type is A, then $x\colon A$ is a term.

(2) If $M\colon A \to B$ and $N\colon A$ are terms, then $(MN)\colon B$ is a term.

(3) If $x\colon A$ and $M\colon B$ are terms, then $(\lambda_x.M)\colon A \to B$ is a term.

(4) Nothing else is a term.

If x occurs in the scope of $\lambda_x.$ in a term, then this occurrence of x is *bound*; otherwise, it is *free* in this term. The notation $x \notin FV(M)$ means that every occurrence of x in M is bound. For a term M, a variable x and a term N with the same type as x, we define M^x_N as the result of uniformly substituting the term N for every free occurrence of x in M. By renaming of bound variables, one avoids free variables of N to become bound after substitution.
The language consists of equalities of the form $X = Y$, where X and Y are terms with the same type. The axiom schemata are

$$X = X, \qquad \lambda_x.M = \lambda_y.M^x_y,\ y \notin FV(M),$$

$$(\lambda_x.M)N = M^x_N, \qquad M = \lambda_x.Mx,\ x \notin FV(M),$$

and we have the following inference figures

$$\dfrac{X = Y}{Y = X} \qquad \dfrac{X = Y \quad Y = Z}{X = Z}$$

$$\dfrac{X = Y \quad Z = T}{XZ = YT} \qquad \dfrac{X = Y}{\lambda_z.X = \lambda_z.Y}$$

The Curry-Howard correspondence assigns to a natural deduction derivation a simply-typed λ term, following the clauses:

$$\dfrac{\begin{array}{c}[x : A]\\ M : B\end{array}}{\lambda_x.M : A \to B} \qquad \dfrac{M : A \to B \quad N : A}{MN : B}$$

Hence, a proof, i.e., a derivation without hypotheses, of the formula $(p \to q) \to ((q \to r) \to (p \to r))$ reads

$$\dfrac{\dfrac{\dfrac{\dfrac{[y : q \to r] \qquad \dfrac{[x : p \to q] \quad [z : p]}{xz : q}}{y(xz) : r}}{\lambda_z . y(xz) : p \to r}}{\lambda_{yz} . y(xz) : (q \to r) \to (p \to r)}}{\lambda_{xyz} . y(xz) : (p \to q) \to ((q \to r) \to (p \to r))}$$

The Curry-Howard correspondence acts at three different levels. The formulae correspond to types, the derivations correspond to terms and the normalization of derivations (see [6, II.3]) corresponds to $\beta\eta$-reduction in simply-typed λ-calculus. (Roughly speaking, the last two axiom schemata for simply-typed λ calculus, directed from left to right, underly *$\beta\eta$-reduction*.)

This correspondence is very important from the point of view of general proof theory. It may serve to formalize the notion of equality of derivations based on normalization. However, this correspondence is not in the main stream of these notes. It serves just as an alternative approach to what we intend to do with sequent systems and categories.

Another way to formalize natural deduction is to introduce sequents, i.e., words of the form $\Gamma \vdash A$, where Γ is a set of formulae staying for the hypotheses of a derivation of a formula A. Such a formal system, which is appropriate for our fragment of logic, is given by the axiom scheme $\{A\} \vdash A$ and the inference figures

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \to B} \qquad \frac{\Gamma \vdash B}{\Gamma - \{A\} \vdash A \to B} \qquad \frac{\Gamma \vdash A \to B \quad \Delta \vdash A}{\Gamma \cup \Delta \vdash B}.$$

This motivates the introduction of sequent systems.

The following sequent system is based on Gentzen's $\mathcal{LJ}$ (see [3]). The cut-elimination theorem that holds for $\mathcal{LJ}$ and $\mathcal{LK}$ is strong enough to deliver some very important properties of intuitionistic and classical logic. This gives to sequent systems an advantage over natural deduction systems.

We envisage a formal system whose alphabet extends the one introduced in Section 2 by the symbols $\vdash$ and ,. The language consists of sequents of the form $\Gamma \vdash A$, where Γ is a finite sequence of formulae, and A is a formula. The sequence Γ, which may be empty, is the *antecedent* of the sequent and consists of the *antecedent formulae*, while A is the *succedent* formula of the sequent. The axiom scheme is $A \vdash A$ and the inference figures are the following structural inference figures

$$\frac{\Gamma, A, B, \Delta \vdash D}{\Gamma, B, A, \Delta \vdash D}\ \text{interchange}$$

$$\frac{\Gamma \vdash D}{A, \Gamma \vdash D}\ \text{thinning} \qquad\qquad \frac{A, A, \Gamma \vdash D}{A, \Gamma \vdash D}\ \text{contraction}$$

$$\frac{\Gamma \vdash A \quad \Delta, A, \Theta \vdash D}{\Delta, \Gamma, \Theta \vdash D}\ \text{cut}$$

and the following operational inference figures

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash D}{\Gamma, A \to B, \Delta \vdash D}\ {\to}{\vdash} \qquad\qquad \frac{A, \Gamma \vdash D}{\Gamma \vdash A \to D}\ {\vdash}{\to}$$

This formal system is such that we derive sequents in it. However, this system may serve to define theorems of the corresponding fragment of logic. A formula A is a *theorem* of the implicational fragment of intuitionistic logic when the sequent $\vdash A$ is a theorem of this system.

Theorem 3.1. (Cut-Elimination Theorem) *Every derivation can be transformed into a derivation with the same endsequent and in which the cut inference figure does not occur.*

We prove here just a toy example of this theorem. The sequent system $\mathcal{IL}$, envisaged here, is in the same language as the previous one. This system corresponds to the implicational fragment of intuitionistic linear logic. Besides the axiomatic sequents $A \vdash A$, there are only two structural inference figures

$$\frac{\Gamma, A, B, \Delta \vdash D}{\Gamma, B, A, \Delta \vdash D}\ \text{interchange} \qquad\qquad \frac{\Gamma \vdash A \quad \Delta, A, \Theta \vdash D}{\Delta, \Gamma, \Theta \vdash D}\ \text{cut}$$

and two operational inference figures

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash D}{\Gamma, A \to B, \Delta \vdash D}\ {\to}{\vdash} \qquad\qquad \frac{A, \Gamma \vdash D}{\Gamma \vdash A \to D}\ {\vdash}{\to}$$

The formula $(p \to q) \to ((q \to r) \to (p \to r))$ is its theorem.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p \quad \dfrac{q \vdash q \quad r \vdash r}{q, q \to r \vdash r}}{p, p \to q, q \to r \vdash r}}{p \to q, q \to r \vdash p \to r}}{\dfrac{q \to r, p \to q \vdash (p \to r)}{p \to q \vdash (q \to r) \to (p \to r)}}}{\vdash (p \to q) \to ((q \to r) \to (p \to r))}$$

However, this system is rich enough to justify some basic categorial notions through the cut-elimination procedure.

By a formula in a derivation we always mean a particular occurrence of this formula as an antecedent or a succedent formula in this derivation. The sequent

$\Gamma \vdash A$ is the *left-premise* and $\Delta, A, \Theta \vdash D$ is the *right-premise* of the cut inference figure.

For every inference figure, every antecedent and succedent formula of the lower sequent, except $A \to B$ in $\to\vdash$ and $A \to D$ in $\vdash\to$, has the unique successor, an occurrence of the same formula, in the upper sequent. Let the *rank* of a formula A in a derivation be the number of formulae of that derivation that are related to A by the reflexive and transitive closure of the successor relation. For example, the red $p \to q$ has rank 4 in the above derivation. Let the formula A in the cut inference figure be called *cut formula*. Let the *degree of a cut* in a derivation be the number of occurrences of implication in the cut formula A. Let the *rank of a cut* in a derivation be the sum of the rank of A in the left premise and the rank of A in the right premise of this cut inference figure.

Proof of Theorem 3.1. It suffices to show that this theorem holds for a derivation whose last inference figure is cut and there is no other application of cut in the derivation. We proceed by induction on lexicographically ordered pairs (d, r), where d is the degree and r is the rank of the cut in such a derivation.

For the basis, when $(d, r) = (0, 2)$, the derivation is of the form

$$\frac{p \vdash p \quad p \vdash p}{p \vdash p} \tag{3.1}$$

and we transform it into the derivation consisting only of the axiomatic sequent $p \vdash p$.

When $d > 0$ and $r = 2$, the derivation is either of one of the following forms

$$\frac{A \vdash A \qquad \begin{array}{c}\mathcal{D}\\ \Delta, A, \Theta \vdash D\end{array}}{\Delta, A, \Theta \vdash D} \qquad\qquad \frac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A\end{array} \qquad A \vdash A}{\Gamma \vdash A} \tag{3.2}$$

which are transformed, respectively, into the following cut-free derivations

$$\begin{array}{c}\mathcal{D}\\ \Delta, A, \Theta \vdash D\end{array} \qquad\qquad \begin{array}{c}\mathcal{D}\\ \Gamma \vdash A\end{array}$$

or it is of the form

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ A_1, \Gamma \vdash A_2\end{array}}{\Gamma \vdash A_1 \to A_2} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Delta \vdash A_1\end{array} \quad \begin{array}{c}\mathcal{D}_3\\ A_2, \Theta \vdash D\end{array}}{\Delta, A_1 \to A_2, \Theta \vdash D}}{\Delta, \Gamma, \Theta \vdash D} \tag{3.3}$$

which is transformed into the following derivation.

$$\dfrac{\begin{array}{c}\mathcal{D}_2\\ \Delta \vdash A_1\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A_1, \Gamma \vdash A_2\end{array} \quad \begin{array}{c}\mathcal{D}_3\\ A_2, \Theta \vdash D\end{array}}{A_1, \Gamma, \Theta \vdash D}}{\Delta, \Gamma, \Theta \vdash D}$$

The two cuts in the new derivation are of the lower degree. Hence, by the induction hypothesis, we can first eliminate the upper cut to obtain a cut-free derivation of $A_1, \Gamma, \Theta \vdash D$. Then, again by the induction hypothesis, the lower cut may be eliminated.

When $r > 2$, either the derivation of the left premise of the cut ends with one of the following inference figures

$$\dfrac{\Gamma_1, B, C, \Gamma_2 \vdash A}{\Gamma_1, C, B, \Gamma_2 \vdash A} \qquad \dfrac{\Gamma_1 \vdash B \qquad C, \Gamma_2 \vdash A}{\Gamma_1, B \to C, \Gamma_2 \vdash A} \tag{3.4}$$

or the derivation of the right premise of the cut ends with one of the following inference figures.

$$\dfrac{\Delta_1, B, C, \Delta_2, A, \Theta \vdash D}{\Delta_1, C, B, \Delta_2, A, \Theta \vdash D} \qquad \dfrac{\Delta, A, \Theta_1, B, C, \Theta_2 \vdash D}{\Delta, A, \Theta_1, C, B, \Theta_2 \vdash D} \tag{3.5}$$

$$\dfrac{\Delta, B, A, \Theta \vdash D}{\Delta, A, B, \Theta \vdash D} \qquad \dfrac{\Delta, A, C, \Theta \vdash D}{\Delta, C, A, \Theta \vdash D}$$

$$\dfrac{\Delta_1, A, \Delta_2 \vdash B \qquad C, \Theta \vdash D}{\Delta_1, A, \Delta_2, B \to C, \Theta \vdash D} \qquad \dfrac{\Delta \vdash B \qquad C, \Theta_1, A, \Theta_2 \vdash D}{\Delta, B \to C, \Theta_1, A, \Theta_2 \vdash D}$$

$$\dfrac{D_1, \Delta, A, \Theta \vdash D_2}{\Delta, A, \Theta \vdash D_1 \to D_2}$$

It is obvious how to permute the cut with the above inference figures in order to decrease its rank by 1. For example, a derivation of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Delta, A, \Theta_1, B, C, \Theta_2 \vdash D\end{array}}{\Delta, A, \Theta_1, C, B, \Theta_2 \vdash D}}{\Delta, \Gamma, \Theta_1, C, B, \Theta_2 \vdash D}$$

is transformed into the following derivation.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\vdash A\end{array}\qquad\begin{array}{c}\mathcal{D}_2\\ \Delta,A,\Theta_1,B,C,\Theta_2\vdash D\end{array}}{\Delta,\Gamma,\Theta_1,B,C,\Theta_2\vdash D}}{\Delta,\Gamma,\Theta_1,C,B,\Theta_2\vdash D}$$

By the induction hypothesis the derivation ending with the lifted cut may be transformed into a cut-free derivation. □

4. From sequent systems to categories

In this section we use the strict variant of mixed language introduced in Section 2. The role of the propositional connective $\wedge$ is, roughly speaking, to amalgamate all the antecedent formulae of a sequent. Hence, an $\mathcal{IL}$ sequent $A_1,\dots,A_n\vdash B$ has now the form $A_1\wedge\ldots\wedge A_n\vdash B$, which is $\top\vdash B$ when $n=0$.

Based on the strict variant of mixed language, we build a new language whose members are equalities of the form $t=s$ where t and s are terms defined below. Some of these terms represent $\mathcal{IL}$ derivations. The alphabet introduced in Section 2 is extended by the symbols $\mathbf{1}$, $\mathbf{c}$, η, ε, $\circ$, $=$, :, $\vdash$ and ,.

The *terms* of this language are defined inductively as follows.

(1) If A and B are formulae, then

$$\mathbf{1}_A\colon A\vdash A,\qquad \mathbf{c}_{B,A}\colon B\wedge A\vdash A\wedge B,$$
$$\varepsilon_{A,B}\colon A\wedge(A\to B)\vdash B,\qquad \eta_{A,B}\colon B\vdash A\to(A\wedge B)$$

are terms called *primitive*.

(2) If $f\colon A\vdash B$ and $g\colon B\vdash C$ are terms, then $(g\circ f)\colon A\vdash C$ is a term.

(3) If $f_1\colon A_1\vdash B_1$ and $f_2\colon A_2\vdash B_2$ are terms, then $(f_1\wedge f_2)\colon A_1\wedge A_2\vdash B_1\wedge B_2$ is a term.

(4) If $f\colon B_1\vdash B_2$ is a term and A is a formula, then $(A\to f)\colon A\to B_1\vdash A\to B_2$ is a term.

(5) Nothing else is a term.

A *type* is a word of the form $A\vdash B$ where A and B are formulae. We say that $A\vdash B$ is the type of a term $f : A\vdash B$ and we say that this term has A as the *source* and B as the *target*. Sometimes, the subscripts determined by the context are omitted. Usually, we omit the type in writing a term and by term we mean just the part before the symbol ":". We omit the outermost parentheses in terms taking them for granted.

Every $\mathcal{IL}$ derivation of a sequent corresponds to a term whose type is that sequent. A derivation consisting solely of an axiomatic sequent $A\vdash A$ corresponds to $\mathbf{1}_A\colon A\vdash A$. A derivation ending with an inference figure corresponds to a term obtained from the terms corresponding to the derivations of the premises as follows:

$$\frac{f\colon G\wedge A\wedge B\wedge E\vdash D}{f\circ((\mathbf{1}_G\wedge \mathbf{c}_{B,A})\wedge \mathbf{1}_E)\colon G\wedge B\wedge A\wedge E\vdash D}\ \text{interchange}$$

$$\frac{f\colon G\vdash A \quad g\colon E\wedge A\wedge F\vdash D}{g\circ((\mathbf{1}_E\wedge f)\wedge \mathbf{1}_F)\colon E\wedge G\wedge F\vdash D}\ \text{cut}$$

$$\frac{f\colon G\vdash A \quad g\colon B\wedge E\vdash D}{g\circ((\varepsilon_{A,B}\wedge \mathbf{1}_E)\circ((f\wedge \mathbf{1}_{A\to B})\wedge \mathbf{1}_E))\colon G\wedge (A\to B)\wedge E\vdash D}\ \to\vdash$$

$$\frac{f\colon A\wedge G\vdash D}{(A\to f)\circ \eta_{A,G}\colon G\vdash A\to D}\ \vdash\to$$

For example, the derivation

$$\cfrac{\cfrac{\cfrac{\cfrac{p\vdash p \quad q\vdash q}{p\wedge(p\to q)\vdash q}}{(p\to q)\wedge p\vdash q}}{p\vdash (p\to q)\to q}}{\top\vdash p\to((p\to q)\to q)}$$

corresponds to the term

$$(p\to(((p\to q)\to(\mathbf{1}_q\circ(\varepsilon_{p,q}\circ(\mathbf{1}_p\wedge \mathbf{1}_{p\to q}))\circ \mathbf{c}_{p\to q,p}))\circ \eta_{p\to q,p}))\circ \eta_{p,\top}$$

whose type is $\top\vdash p\to((p\to q)\to q)$.

The *language* consists of words of the form $f=g$, where f and g are terms with the same type. Our goal is to define an equational system $\mathcal{E}$ in that language, whose theorems cover the cut-elimination procedure. This means that if a derivation corresponding to a term f is transformed by the cut-elimination procedure into a derivation corresponding to a term g, then $f=g$ is a theorem of $\mathcal{E}$.

The axiom schemata include the following

$$f=f,\quad f\wedge \mathbf{1}_\top = f = \mathbf{1}_\top\wedge f,\quad (f\wedge g)\wedge h = f\wedge(g\wedge h).$$

(The other axioms will appear through our analysis.) The inference figures are the following

$$\frac{f=g}{g=f}\qquad\qquad \frac{f=g\quad g=h}{f=h}$$

$$\frac{f_1\colon A\vdash B = f_2\colon A\vdash B \qquad g_1\colon B\vdash C = g_2\colon B\vdash C}{g_1\circ f_1 = g_2\circ f_2}$$

$$\frac{f_1=f_2\quad g_1=g_2}{f_1\wedge g_1 = f_2\wedge g_2}\qquad\qquad \frac{f_1=f_2}{A\to f_1 = A\to f_2.}$$

We start our analysis with two simple instances of Case (3.2) from the proof of Theorem 3.1, which cover also Case (3.1). A derivation of the form

$$\frac{\mathbf{1}_A \colon A \vdash A \quad g \colon A \vdash D}{g \circ \mathbf{1}_A \colon A \vdash D}$$

is transformed into

$$g \colon A \vdash D.$$

Also, a derivation of the form

$$\frac{f \colon G \vdash A \quad \mathbf{1}_A \colon A \vdash A}{\mathbf{1}_A \circ f \colon G \vdash A}$$

is transformed into

$$f \colon G \vdash A.$$

Hence, we add to $\mathcal{E}$ the following axiom schemata

$$g \circ \mathbf{1}_A = g \quad \text{and} \quad \mathbf{1}_A \circ f = f. \tag{4.1}$$

We always assume that both sides of our equalities are terms, which, for (4.1), means that g has A as the source and f has A as the target.

Consider the following instance of Case (3.4) in which the derivation

$$\frac{\dfrac{f \colon B \wedge C \vdash A}{f \circ \mathbf{c} \colon C \wedge B \vdash A} \qquad g \colon A \vdash D}{g \circ (f \circ \mathbf{c}) \colon C \wedge B \vdash D}$$

is transformed into

$$\dfrac{\dfrac{f \colon B \wedge C \vdash A \qquad g \colon A \vdash D}{g \circ f \colon B \wedge C \vdash D}}{(g \circ f) \circ \mathbf{c} \colon C \wedge B \vdash D}$$

We see that the axiom scheme

$$h \circ (g \circ f) = (h \circ g) \circ f \tag{4.2}$$

should be added to $\mathcal{E}$. Let $\equiv$ be the relation on the set of terms defined by

$$f \equiv g \quad \text{when} \quad f = g \text{ is a theorem of } \mathcal{E}.$$

This is an equivalence relation. Let $[f]$ be the equivalence class of a term f. We call $[f]$ an *arrow*. It is straightforward to check that the following definitions are correct. The *source* of $[f]$ is the source of f, the *target* of $[f]$ is the target of f, the *identity arrow* on A is $[\mathbf{1}_A]$ and, for a term $g \circ f$, the *composition* $[g] \circ [f]$ of $[g]$ with $[f]$ is $[g \circ f]$.

Categories

A *category* consists of two sets, O of *objects* and A of *arrows*, two functions

$$\text{source}, \text{target}\colon A \to O,$$

and two additional functions

$$\mathbf{1}\colon O \to A, \qquad \circ\colon A \times_O A \to A$$

called *identity* and *composition*, where

$$A \times_O A =_{df} \{(g, f) \mid g, f \in A \;\&\; \text{source}(g) = \text{target}(f)\}$$

is the set of all *composable pairs of arrows*. Moreover, for every $X, Y \in O$, and every $f, g, h \in A$ such that (h, g), (g, f) and $(\mathbf{1}_Y, f)$ are composable pairs of arrows, the following holds

$$\text{source}(\mathbf{1}_X) = X = \text{target}(\mathbf{1}_X),$$

$$\text{source}(g \circ f) = \text{source}(f), \quad \text{target}(g \circ f) = \text{target}(g),$$

and

$$g \circ \mathbf{1}_Y = g, \quad \mathbf{1}_Y \circ f = f, \quad h \circ (g \circ f) = (h \circ g) \circ f.$$

The set $O_{\mathbf{K}}$, which is the set of formulae, and the set $A_{\mathbf{K}} = \{[f] \mid f \text{ is a term}\}$, with the source and target function, identities and composition defined as above, make the *category* $\mathbf{K}$. In the sequel we denote an arrow $[f]$ just by f. We will delete occurrences of $\mathbf{1}$ in the immediate scope of $\circ$ and omit parentheses tied to $\circ$ in the immediate scope of $\circ$ without referring to (4.1) and (4.2).

Consider now a bit more complicated instance of Case (3.2) where

$$\frac{\mathbf{1}_A\colon A \vdash A \quad g\colon A \wedge B \vdash D}{g \circ (\mathbf{1}_A \wedge \mathbf{1}_B)\colon A \wedge B \vdash D}$$

is transformed into

$$g\colon A \wedge B \vdash D.$$

If we add the axiom scheme

$$\mathbf{1}_A \wedge \mathbf{1}_B = \mathbf{1}_{A \wedge B}, \tag{4.3}$$

to $\mathcal{E}$, then with the help of (4.1), we easily derive $g \circ (\mathbf{1}_A \wedge \mathbf{1}_B) = g$.

According to an instance of Case (3.5), the derivation

$$\dfrac{f\colon G \vdash A \qquad \dfrac{g\colon A \wedge B \wedge C \vdash D}{g \circ (\mathbf{1}_A \wedge \mathbf{c})\colon A \wedge C \wedge B \vdash D}}{g \circ (\mathbf{1}_A \wedge \mathbf{c}) \circ (f \wedge \mathbf{1}_{C \wedge B})\colon G \wedge C \wedge B \vdash D}$$

is transformed into

$$\dfrac{\dfrac{f\colon G \vdash A \quad g\colon A \wedge B \wedge C \vdash D}{g \circ (f \wedge \mathbf{1}_{B \wedge C})\colon G \wedge B \wedge C \vdash D}}{g \circ (f \wedge \mathbf{1}_{B \wedge C}) \circ (\mathbf{1}_G \wedge \mathbf{c})\colon G \wedge C \wedge B \vdash D}$$

If we add the axiom scheme

$$(4.4)\qquad (g_1 \wedge g_2)\circ(f_1 \wedge f_2) = (g_1 \circ f_1)\wedge(g_2 \circ f_2)$$

to $\mathcal{E}$, then we have

$$\begin{aligned}
g\circ(\mathbf{1}_A\wedge \mathbf{c})\circ(f\wedge \mathbf{1}_{C\wedge B}) &= g\circ((\mathbf{1}_A\circ f)\wedge(\mathbf{c}\circ \mathbf{1}_{C\wedge B})), && \text{by } (4.4)\\
&= g\circ((f\circ \mathbf{1}_G)\wedge(\mathbf{1}_{B\wedge C}\circ \mathbf{c})), && \text{by } (4.1)\\
&= g\circ(f\wedge \mathbf{1}_{B\wedge C})\circ(\mathbf{1}_G\wedge \mathbf{c}), && \text{by } (4.4)
\end{aligned}$$

Functors

Given two categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of two functions, both denoted by F, the *object function* $F\colon O_{\mathcal{C}} \to O_{\mathcal{D}}$ and the *arrow function* $F : A_{\mathcal{C}} \to A_{\mathcal{D}}$, such that for every $C \in O_{\mathcal{C}}$ and every composable pair (g, f) of arrows of $\mathcal{C}$

$$F\mathbf{1}_C = \mathbf{1}_{FC}, \quad F(g\circ f) = Fg\circ Ff.$$

The *identity functor* on a category consists of the identity function on objects and the identity function on arrows. The operation of *composition* of two functors consists of two compositions—composition of object functions and composition of arrow functions, hence this operation is associative.

Product of categories

The *product* $\mathcal{C}\times\mathcal{D}$, of categories $\mathcal{C}$ and $\mathcal{D}$ is the category whose objects make the cartesian product $O_{\mathcal{C}}\times O_{\mathcal{D}}$ and whose arrows make the cartesian product $A_{\mathcal{C}}\times A_{\mathcal{D}}$. The identity arrow on (C, D) is the pair $(\mathbf{1}_C, \mathbf{1}_D)$ and composition is defined componentwise.

If we define $\wedge(A, B)$ as $A\wedge B$ and $\wedge(f, g)$ as $f\wedge g$, then (4.3) and (4.4) guarantee that $\wedge$ is a functor from $\mathbf{K}\times\mathbf{K}$ to $\mathbf{K}$. (Note that the latter definition is correct since $f \equiv f'$ and $g \equiv g'$ implies $(f \wedge g) \equiv (f' \wedge g')$.)

According to another instance of Case (3.5), the derivation

$$\dfrac{f\colon G\vdash A \qquad \dfrac{g\colon B\wedge A\vdash D}{g\circ\mathbf{c}\colon A\wedge B\vdash D}}{g\circ\mathbf{c}\circ(f\wedge\mathbf{1}_B)\colon G\wedge B\vdash D}$$

is transformed into

$$\dfrac{\dfrac{f\colon G\vdash A \qquad g\colon B\wedge A\vdash D}{g\circ(\mathbf{1}_B\wedge f)\colon B\wedge G\vdash D}}{g\circ(\mathbf{1}_B\wedge f)\circ\mathbf{c}\colon G\wedge B\vdash D}$$

If we add the axiom scheme

$$\mathbf{c}_{A',B'} \circ (f \wedge g) = (g \wedge f) \circ \mathbf{c}_{A,B} \tag{4.5}$$

to $\mathcal{E}$, then with the help of (4.2) we easily derive $g \circ \mathbf{c} \circ (f \wedge \mathbf{1}_B) = g \circ (\mathbf{1}_B \wedge f) \circ \mathbf{c}$.

Natural transformations

Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\alpha : F \xrightarrow{\cdot} G$ is a function from $O_\mathcal{C}$ to $A_\mathcal{D}$, i.e., a family of arrows of $\mathcal{D}$ indexed by the objects of $\mathcal{C}$, such that for every $C \in O_\mathcal{C}$, source$(\alpha_C) = FC$ and target$(\alpha_C) = GC$, and for every $f : C \to C' \in A_\mathcal{C}$, the following diagram

$$\begin{tikzcd} FC \arrow[r, "\alpha_C"] \arrow[d, "Ff"'] & GC \arrow[d, "Gf"] \\ FC' \arrow[r, "\alpha_{C'}"'] & GC' \end{tikzcd}$$

commutes in $\mathcal{D}$. The arrows α_C for $C \in O_\mathcal{C}$, are the *components* of the natural transformation α.

Let $\mathbf{2}$ be the category with two objects, 0 and 1, and one nonidentity arrow $h : 0 \to 1$. Let $I_0, I_1 : \mathcal{C} \to \mathcal{C} \times \mathbf{2}$ be functors such that for every object C and every arrow f of $\mathcal{C}$, we have that $I_0(C) = (C, 0)$, $I_0(f) = (f, \mathbf{1}_0)$, and $I_1(C) = (C, 1)$, $I_1(f) = (f, \mathbf{1}_1)$. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. There is a bijection between the set of natural transformations $\alpha : F \xrightarrow{\cdot} G$, and the set of functors $A : \mathcal{C} \times \mathbf{2} \to \mathcal{D}$ such that $A \circ I_0 = F$ and $A \circ I_1 = G$. This bijection maps $\alpha : F \xrightarrow{\cdot} G$ to $A : \mathcal{C} \times \mathbf{2} \to \mathcal{D}$ defined by

$$A(C, 0) = FC, \quad A(C, 1) = GC, \quad A(f, \mathbf{1}_0) = Ff, \quad A(f, \mathbf{1}_1) = Gf,$$

and for $f : C \to C'$,

$$A(f, h) = Gf \circ \alpha_C = \alpha_{C'} \circ Ff.$$

Its inverse maps $A : \mathcal{C} \times \mathbf{2} \to \mathcal{D}$ to $\alpha : F \xrightarrow{\cdot} G$ defined by $\alpha_C = A(\mathbf{1}_C, h)$.

The scheme (4.5) says that the family indexed by the objects of $\mathbf{K} \times \mathbf{K}$

$$\mathbf{c} = \{\mathbf{c}_{A,B} \mid A, B \in Ob(\mathbf{K})\}$$

is a natural transformation from the above defined functor $\wedge : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$ to the functor $G : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$ defined so that

$$G(A, B) = B \wedge A \quad \text{and} \quad G(f, g) = g \wedge f.$$

Isomorphisms, natural isomorphisms

An arrow $f : A \to B$ of a category $\mathcal{C}$ is an *isomorphism* when there is an arrow $g : B \to A$ in $\mathcal{C}$, such that $g \circ f = \mathbf{1}_A$ and $f \circ g = \mathbf{1}_B$ in $\mathcal{C}$. A natural transformation is a *natural isomorphism* when all its components are isomorphisms.

The following two schemata say that $\mathbf{c}$ is a natural isomorphism, which satisfies a *coherence* condition.

$$\mathbf{c}_{B,A} \circ \mathbf{c}_{A,B} = \mathbf{1}_{A\wedge B} \tag{4.6}$$

$$\mathbf{c}_{A\wedge B,C} = (\mathbf{c}_{A,C} \wedge \mathbf{1}_B) \circ (\mathbf{1}_A \wedge \mathbf{c}_{B,C}) \tag{4.7}$$

According to an instance of Case (3.5), the derivation

$$\frac{f\colon G \vdash A \qquad \dfrac{g\colon D_1 \wedge A \vdash D_2}{(D_1 \to g)\circ \eta\colon A \vdash D_1 \to D_2}}{(D_1 \to g)\circ\eta\circ f\colon G \vdash D_1 \to D_2}$$

is transformed into

$$\frac{\dfrac{f\colon G \vdash A \quad g\colon D_1 \wedge A \vdash D_2}{g \circ (\mathbf{1}\wedge f)\colon D_1 \wedge G \vdash D_2}}{(D_1 \to (g\circ(\mathbf{1}\wedge f)))\circ\eta\colon G \vdash D_1 \to D_2}$$

If we add the axiom schemata

$$A \to (g\circ f) = (A \to g)\circ(A \to f) \tag{4.8}$$

$$\eta_{A,B'} \circ f = (A \to (\mathbf{1}_A \wedge f))\circ \eta_{A,B} \tag{4.9}$$

to $\mathcal{E}$, then we have

$$\begin{aligned} (D_1 \to g)\circ\eta\circ f &= (D_1 \to g)\circ(D_1 \to (\mathbf{1}\wedge f))\circ\eta && \text{by (4.9)}\\ &= (D_1 \to (g\circ(\mathbf{1}\wedge f)))\circ\eta && \text{by (4.8)} \end{aligned}$$

The scheme (4.8) together with the scheme

$$A \to \mathbf{1}_B = \mathbf{1}_{A\to B} \tag{4.10}$$

guarantee that for every A, we have the functor $A \to \_\colon \mathbf{K} \to \mathbf{K}$, mapping B to $A \to B$ and f to $A \to f$.

It is not difficult to see how to use (4.1), (4.3), (4.4), (4.10) and (4.8) in order to show that $G\colon \mathbf{K} \to \mathbf{K}$ defined by

$$G(B) = A \to (A \wedge B) \quad \text{and} \quad G(f) = A \to (\mathbf{1}_A \wedge f)$$

is indeed a functor. The scheme (4.9) says that for every $A \in Ob(\mathbf{K})$, the family

$$\eta_A = \{\eta_{A,B} \mid B \in Ob(\mathbf{K})\}$$

is a natural transformation from the identity functor on $\mathbf{K}$ to G.

According to an instance of Case (3.3), the derivation

$$\frac{\dfrac{f\colon A_1 \wedge G \vdash A_2}{(A_1 \to f)\circ\eta\colon G \vdash A_1 \to A_2} \qquad \dfrac{g\colon B \vdash A_1 \quad h\colon A_2 \vdash D}{h\circ\varepsilon\circ(g\wedge\mathbf{1})\colon B\wedge(A_1 \to A_2) \vdash D}}{h\circ\varepsilon\circ(g\wedge\mathbf{1})\circ(\mathbf{1}\wedge((A_1 \to f)\circ\eta))\colon B\wedge G \vdash D}$$

is transformed into

$$\dfrac{g\colon B\vdash A_1 \qquad \dfrac{f\colon A_1\wedge G\vdash A_2 \quad h\colon A_2\vdash D}{h\circ f\colon A_1\wedge G\vdash D}}{h\circ f\circ(g\wedge\mathbf{1})\colon B\wedge G\vdash D}$$

If we add the axiom schemata

$$\varepsilon_{A,B'}\circ(\mathbf{1}_A\wedge(A\rightarrow f)) = f\circ\varepsilon_{A,B} \tag{4.11}$$

$$\varepsilon_{A,A\wedge B}\circ(\mathbf{1}_A\wedge\eta_{A,B}) = \mathbf{1}_{A\wedge B} \tag{4.12}$$

to $\mathcal{E}$, then we have

$$\begin{aligned}
&h\circ\varepsilon\circ(g\wedge\mathbf{1})\circ(\mathbf{1}\wedge((A_1\rightarrow f)\circ\eta)) = \\
&\quad = h\circ\varepsilon\circ(g\wedge\mathbf{1})\circ((\mathbf{1}\circ\mathbf{1})\wedge((A_1\rightarrow f)\circ\eta)) && \text{by (4.1)}\\
&\quad = h\circ\varepsilon\circ(g\wedge\mathbf{1})\circ(\mathbf{1}\wedge(A_1\rightarrow f))\circ(\mathbf{1}\wedge\eta) && \text{by (4.4)}\\
&\quad = h\circ\varepsilon\circ(\mathbf{1}\wedge(A_1\rightarrow f))\circ(g\wedge\mathbf{1})\circ(\mathbf{1}\wedge\eta) && \text{by (4.1), (4.4)}\\
&\quad = h\circ f\circ\varepsilon\circ(g\wedge\mathbf{1})\circ(\mathbf{1}\wedge\eta) && \text{by (4.11)}\\
&\quad = h\circ f\circ\varepsilon\circ(\mathbf{1}\wedge\eta)\circ(g\wedge\mathbf{1}) && \text{by (4.1), (4.4)}\\
&\quad = h\circ f\circ(g\wedge\mathbf{1}) && \text{by (4.12)}
\end{aligned}$$

The scheme (4.11) says that for every $A\in Ob(\mathbf{K})$, the family

$$\varepsilon_A = \{\varepsilon_{A,B} \mid B\in Ob(\mathbf{K})\}$$

is a natural transformation from the functor $F\colon\mathbf{K}\rightarrow\mathbf{K}$ defined so that

$$F(B) = A\wedge(A\rightarrow B) \quad\text{and}\quad F(f) = \mathbf{1}_A\wedge(A\rightarrow f)$$

to the identity functor on $\mathbf{K}$.

Adjunction

Given two categories $\mathcal{C}$ and $\mathcal{D}$, an *adjunction* is given by two functors, $F:\mathcal{C}\rightarrow\mathcal{D}$ and $G:\mathcal{D}\rightarrow\mathcal{C}$, and two natural transformations, the *unit* $\eta\colon\mathbf{1}_{\mathcal{C}}\dot{\rightarrow} GF$ and the *counit* $\varepsilon\colon FG\dot{\rightarrow}\mathbf{1}_{\mathcal{D}}$, such that for every $C\in O_{\mathcal{C}}$ and every $D\in O_{\mathcal{D}}$

$$G\varepsilon_D\circ\eta_{GD} = \mathbf{1}_{GD}, \quad\text{and}\quad \varepsilon_{FC}\circ F\eta_C = \mathbf{1}_{FC}.$$

These two equalities are called *triangular identities*. The functor F is a *left adjoint* for the functor G, while G is a *right adjoint* for the functor F.

The schemata (4.9), (4.11), (4.12) together with

$$(A\rightarrow\varepsilon_{A,B})\circ\eta_{A,A\rightarrow B} = \mathbf{1}_{A\rightarrow B} \tag{4.13}$$

say that for every A, the functor $A\wedge\_\colon\mathbf{K}\rightarrow\mathbf{K}$, mapping B to $A\wedge B$ and f to $\mathbf{1}_A\wedge f$, is a left adjoint for the functor $A\rightarrow\_\colon\mathbf{K}\rightarrow\mathbf{K}$. The unit of this adjunction is the natural transformation η_A while the counit is the natural transformation ε_A. The schemata (4.12) and (4.13) are the triangular identities for this adjunction.

Symmetric monoidal closed categories

A *symmetric monoidal closed category* is a category $\mathcal{C}$ with a functor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$, an object I, and for every object A a functor $\_^A\colon \mathcal{C}\to\mathcal{C}$ such that the following holds:

1. There are three natural isomorphisms with components

$\alpha_{A,B,C}\colon A\otimes(B\otimes C)\to(A\otimes B)\otimes C, \lambda_A\colon I\otimes A\to A, \gamma_{A,B}\colon A\otimes B\to B\otimes A.$

2. The natural transformation γ is *self-inverse*, i.e., $\gamma_{B,A}\circ\gamma_{A,B}=\mathbf{1}_{A\otimes B}$.
3. For every object A, the functor $A\otimes\_$ is a left adjoint for $\_^A$.
4. The following diagrams commute (*coherence conditions*).

$$\begin{tikzcd}
A\otimes(B\otimes(C\otimes D)) \arrow[r, "\alpha"] \arrow[d, "\mathbf{1}\otimes\alpha"'] & (A\otimes B)\otimes(C\otimes D) \arrow[r, "\alpha"] & ((A\otimes B)\otimes C)\otimes D \\
A\otimes((B\otimes C)\otimes D) \arrow[r, "\alpha"'] & (A\otimes(B\otimes C))\otimes D \arrow[ur, "\alpha\otimes\mathbf{1}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(I\otimes A)\otimes B \arrow[dr, "\lambda\otimes\mathbf{1}"] & \\
I\otimes(A\otimes B) \arrow[u, "\alpha"] \arrow[r, "\lambda"'] & A\otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes(B\otimes C) \arrow[r, "\alpha"] \arrow[d, "\mathbf{1}\otimes\gamma"'] & (A\otimes B)\otimes C \arrow[r, "\gamma"] & C\otimes(A\otimes B) \arrow[d, "\alpha"] \\
A\otimes(C\otimes B) \arrow[r, "\alpha"'] & (A\otimes C)\otimes B \arrow[r, "\gamma\otimes\mathbf{1}"'] & (C\otimes A)\otimes B
\end{tikzcd}$$

A symmetric monoidal closed category is *symmetric strict monoidal closed* when all the components of α and λ are identities.

The category $\mathbf{K}$ whose objects are formulae and whose arrows are obtained via the equational system $\mathcal{E}$, axiomatized by

$$f=f,\quad f\wedge\mathbf{1}_\top=f=\mathbf{1}_\top\wedge f,\quad (f\wedge g)\wedge h=f\wedge(g\wedge h)$$

and (4.1)-(4.13), is a symmetric strict monoidal closed category in which $\otimes$ is $\wedge$, I is $\top$, $\_^A$ is $A\to\_$, and γ is $\mathbf{c}$. Moreover, it is a category of that kind freely generated by the set of propositional letters that belong to the alphabet. This means that every function from the set of propositional letters to the set of objects of a symmetric strict monoidal closed category $\mathcal{C}$ extends in a unique way to a functor from $\mathbf{K}$ to $\mathcal{C}$, which preserves the symmetric strict monoidal closed structure.

It is not difficult to show that $\mathcal{E}$ covers the cut-elimination procedure in the sense mentioned just before we started to define this equational system. We leave this as an exercise for the reader. Not all the schemata (4.1)-(4.13) are necessary for that. For example, there is no need for (4.7), (4.10) and (4.13). However, we keep these schemata not only for aesthetic or purely category-theoretic reasons but also for some other proof-theoretic needs. By using (4.7), the interchange of a formula with a sequence may be atomized, and if only atomic axiomatic sequents are allowed in

$\mathcal{IL}$, then (4.13) covers the expansion transforming $\mathbf{1}_{A\to B}$ into

$$\dfrac{\dfrac{\mathbf{1}_A : A \vdash A \quad \mathbf{1}_B : B \vdash B}{\varepsilon_{A,B} : A \wedge (A \to B) \vdash B}}{(A \to \varepsilon_{A,B}) \circ \eta_{A,A\to B} : A \to B \vdash A \to B}$$

For C and $f : A \vdash B$ from $\mathbf{K}$, we define $f \to C : B \to C \vdash A \to C$ as

$$(A \to (\varepsilon_{B,C} \circ (f \wedge \mathbf{1}_{B\to C}))) \circ \eta_{A,B\to C}.$$

From (4.3) and (4.13) it follows that

$$\mathbf{1}_A \to C = \mathbf{1}_{A\to C}. \tag{4.14}$$

Also, with the help of (4.9), (4.11), (4.4), (4.8) and (4.12) one can derive

$$(f \to C) \circ (g \to C) = (g \circ f) \to C. \tag{4.15}$$

Hence, for every object C of $\mathbf{K}$ we have the functor $\_ \to C : \mathbf{K}^{op} \to \mathbf{K}$, mapping A to $A \to C$ and f to $f \to C$.

Opposite category

Given a category $\mathcal{C}$, its *opposite category* $\mathcal{C}^{op}$ consists of the same set of objects and the same set of arrows. An arrow f of $\mathcal{C}$ envisaged as the arrow of $\mathcal{C}^{op}$ is denoted by f^{op}. The functions source and target switch the roles so that if $f : A \to B$ in $\mathcal{C}$, then $f^{op} : B \to A$ in $\mathcal{C}^{op}$. The identities are the same, while the composition $\circ$ in $\mathcal{C}^{op}$ is such that

$$f^{op} \circ g^{op} = (g \circ f)^{op}.$$

5. The classification of formulae

The main problem for a mathematician working in a particular category is to classify the objects of that category up to isomorphism. This could be a hard task, and in practise, for a lot of categories from the realm of algebra, geometry and topology, just some partial results are known. For a proof theorist ready to accept the program offered in the previous section, the classification of objects, i.e., formulae, is of similar importance. Isomorphic formulae may be considered to have the same proof-theoretical meaning.

Isomorphism $\cong$ between formulae is an equivalence relation which does not mean only that one formula is derivable from the other and vice versa. Since we are in a category, a formula A is *isomorphic* to a formula B when there are two derivations $f : A \vdash B$ and $g : B \vdash A$ such that the compositions $g \circ f$ and $f \circ g$ are equal respectively to the identity derivations $\mathbf{1}_A$ and $\mathbf{1}_B$.

In this section we explain how to classify the objects of a relaxed version of the category $\mathbf{K}$. Based on the non-strict variant of mixed language, following the lines of Section 4, the category $\mathbf{K}_{lax}$ is built out of syntax material. The set of objects of $\mathbf{K}_{lax}$ is the set of formulae of this language. The main difference is that $\wedge$ is not associative on objects and that $\top$ is not the unit for $\wedge$. However, we have the isomorphisms

$$A \wedge (B \wedge C) \cong (A \wedge B) \wedge C, \quad \top \wedge A \cong A \cong A \wedge \top$$

natural in A, B and C. Moreover, the three coherence conditions listed in Section 4 hold.

By an *invariant* of a category $\mathcal{C}$ we mean a function whose domain is the set of objects of $\mathcal{C}$ having the same (isomorphic) value on isomorphic objects. For example, the object function of a functor whose source is $\mathcal{C}$ is an invariant of $\mathcal{C}$. By a *complete* invariant of a category $\mathcal{C}$ we mean a family $\{F_i\}_{i\in\mathcal{I}}$ of functions whose domain is the set of objects of $\mathcal{C}$ such that for every A and B

$$A \cong B \text{ in } \mathcal{C} \quad \text{iff} \quad \text{for every } i \in \mathcal{I},\ F_iA = F_iB\ (F_iA \cong F_iB).$$

The arrows of $\mathbf{K}_{lax}$ do not go beyond derivations in classical logic, which means that if $f : A \vdash B$ is an arrow of $\mathbf{K}_{lax}$, then $A \to B$ is a tautology. Hence, if $A \cong B$, then $A \leftrightarrow B$ is a tautology, i.e., A and B correspond to the same Boolean function. This leads to a simple logical invariant mapping every formula to the corresponding Boolean function. By relying on this invariant we can show that $p \wedge q$ is not isomorphic to p in $\mathbf{K}_{lax}$, since $p \wedge q \leftrightarrow p$ is not a tautology. On the other hand, it leaves open the question whether $p \wedge p$ and p are isomorphic in $\mathbf{K}_{lax}$.

The category *Set*

Suppose there is a *universe* U satisfying:
(i) $x \in u \in U$ implies $x \in U$,
(ii) $u \in U$ and $v \in U$ imply $\{u, v\}, (u, v), u \times v \in U$,
(iii) $x \in U$ implies $\mathcal{P}(x) \in U$ and $\cup x \in U$,
(iv) $\omega \in U$, where $\omega = \{0, 1, 2, \ldots\}$ is the set of all finite ordinals,
(v) if $f : a \to b$ is a surjective function with $a \in U$ and $b \subset U$, then $b \in U$.
Every element of U is called a *small set*.

For a fixed universe U, the category *Set* has U as the set of objects and

$$\{(f, (u, v)) \mid f, u, v \in U \text{ and } f \subset u \times v \text{ is a function}\}$$

as the set of arrows. The source, target, identity and composition are defined as expected.

The category $\mathbf{K}_{lax}$ is a symmetric monoidal closed category freely generated by the set of propositional letters that belong to the alphabet. This means that every function from the set of propositional letters to the set of objects of a symmetric monoidal closed category $\mathcal{C}$ extends in a unique way to a functor, which preserves the symmetric monoidal closed structure, from $\mathbf{K}_{lax}$ to $\mathcal{C}$. The category *Set* is a symmetric monoidal closed category with $\otimes$ being the cartesian product and B^A being the set of all functions from A to B.

Consider the functor from $\mathbf{K}_{lax}$ to *Set* that extends a function mapping the letter p to a two element set X. This functor maps $p \wedge p$ to the four element set

$X \times X$ which is not isomorphic to X in *Set*. Hence, this functor is an invariant witnessing that $p \wedge p$ and p are not isomorphic in $\mathbf{K}_{lax}$.

These invariants give us just partial results concerning the classification of objects of $\mathbf{K}_{lax}$. In order to give the complete classification we introduce the following invariant. Let $\mathcal{S}$ be the equational system in which we write $\cong$ instead of $=$. The theorems of $\mathcal{S}$ are of the form $A \cong B$, where A and B are formulae of non-strict mixed language. The axiom schemata are,

$$\begin{array}{ll} C \cong C & \top \wedge C \cong C, \\ A \wedge (B \wedge C) \cong (A \wedge B) \wedge C, & A \wedge B \cong B \wedge A, \\ (A \wedge B) \to C \cong B \to (A \to C), & \top \to C \cong C, \end{array}$$

and we have the following inference figures

$$\frac{A \cong B}{B \cong A} \qquad\qquad \frac{A \cong B \quad B \cong C}{A \cong C}$$

$$\frac{A \cong B \quad C \cong D}{A \wedge C \cong B \wedge D} \qquad\qquad \frac{A \cong B \quad C \cong D}{B \to C \cong A \to D}.$$

The six axiom schemata are such that for arbitrary A, B and C, these isomorphisms hold in $\mathbf{K}_{lax}$. For the first four isomorphisms this is trivial, and the last two are left as an exercise for the reader. The first two inference figures are justified by the facts that the inverse of an isomorphism is an isomorphism and that the composition of isomorphisms is an isomorphism. The last two inference figures are justified by the functoriality of $\wedge$ and $\to$ (see (4.3), (4.4), (4.10), (4.8), (4.14), (4.15)). This suffices to conclude the following.

Lemma 5.1. *If $A \cong B$ is derivable in $\mathcal{S}$, then $A \cong B$ holds in $\mathbf{K}_{lax}$.*

The category *FinSet**, whose objects are the finite small sets with a selected base point and whose arrows are base-point-preserving functions, is a symmetric monoidal closed category. The functor $\otimes$ is defined on a pair of objects (X, Y) as

$$((X - \{*_X\}) \times (Y - \{*_Y\})) \cup \{*_{X \otimes Y}\}.$$

Any two element set with a base point may serve as the object I, i.e., a neutral (up to isomorphism) for $\otimes$. The functor $\_^X$ is defined so that Y^X is the set of base-point-preserving functions from X to Y, whose base point is the constant function, mapping every element of X to the base point of Y. The definition of $\otimes$ and $\_^X$ on arrows is then straightforward.

Let $\mathbf{N}^+$ be the set of positive natural numbers and let g be a function, called *valuation*, from the set of propositional letters to $\mathbf{N}^+$. Consider the following binary operations on $\mathbf{N}^+$

$$m \odot n =_{df} (m-1)(n-1)+1, \quad m^{\underline{n}} =_{df} m^{n-1}.$$

We define the function $\bar{g}$, the *extension* of g, that maps the objects of $\mathbf{K}_{lax}$ to $\mathbf{N}^+$ inductively as follows

$$\bar{g}(p) = g(p), \qquad\qquad \bar{g}(\top) = 2,$$
$$\bar{g}(A \wedge B) = \bar{g}(A) \odot \bar{g}(B), \qquad\qquad \bar{g}(A \to B) = \bar{g}(B)^{\underline{\bar{g}(A)}}.$$

For a valuation g, let $\dot{g}$ be a function from the set of propositional letters to the set of objects of *FinSet**, satisfying that the cardinality of $\dot{g}(p)$ is $g(p)$. By the universal property of $\mathbf{K}_{lax}$, there is a unique functor G from $\mathbf{K}_{lax}$ to *FinSet** that extends the function $\dot{g}$ and preserves the symmetric monoidal closed structure. Since every functor preserves isomorphisms, we have that $A \cong B$ in $\mathbf{K}_{lax}$ implies $GA \cong GB$ in *FinSet**.

Two objects of *FinSet** are isomorphic when they have the same cardinality. It is easy to verify that the cardinality of GA is $\bar{g}(A)$, for $\bar{g}$ being the extension of g. Hence, for every valuation g, its extension $\bar{g}$ is an invariant of $\boldsymbol{K}_{lax}$ in the sense that if $A \cong B$ in $\mathbf{K}_{lax}$, then $\bar{g}(A) = \bar{g}(B)$.

We write $\mathbf{N}^+ \models A \cong B$ when $\bar{g}(A) = \bar{g}(B)$ holds for every valuation g. By the preceding paragraph, we have the following result.

Lemma 5.2. *If $A \cong B$ holds in $\mathbf{K}_{lax}$, then $\mathbf{N}^+ \models A \cong B$.*

From Lemmata 5.1 and 5.2 we have that

$$\text{if } A \cong B \text{ is derivable in } \mathcal{S}, \text{ then } \mathbf{N}^+ \models A \cong B.$$

Whether the other direction of this implication holds is a question remained open since it is formulated in [1, Section 9]. The positive answer to this question, together with Lemma 5.1, guarantees that the family $\{\bar{g} \mid g \text{ is a valuation}\}$ is a complete invariant for $\mathcal{C}$. However, for the classification of formulae, we need just a restricted form of this implication, which is formulated below.

We say that a formula is *diversified* when every propositional letter occurs in it no more than once. The following result stems from [1, Arithmetical Completeness Theorem].

Lemma 5.3. *If $\mathbf{N}^+ \models A \cong B$, for A, B diversified, then $A \cong B$ is derivable in $\mathcal{S}$.*

From Lemmata 5.1, 5.2 and 5.3, for A and B diversified, we obtain the following triangle of implications.

$$\begin{tikzcd}
A \cong B \text{ is derivable in } \mathcal{S} \arrow[d, Rightarrow] & \mathbf{N}^+ \models A \cong B \arrow[l, Rightarrow] \\
A \cong B \text{ holds in } \mathbf{K}_{lax} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

With the help of these implications we obtain a classification of objects of $\mathbf{K}_{lax}$. For this we need the following notions. A formula A is an *instance* of a formula A' when for mutually distinct propositional letters $p_1, \ldots, p_n$ and not necessarily mutually distinct formulae $B_1, \ldots, B_n$, the formula A is the result of uniformly substituting the formula B_i for the letter p_i.

Similarly, a term $f : A \vdash B$ is an instance of a term $f' : A' \vdash B'$ when, for $p_1, \ldots, p_n$ and $B_1, \ldots, B_n$ as above, f is the result of uniformly substituting the formula B_i for the letter p_i in the indices of f'. For example, $\varepsilon_{p,p}\colon p\wedge(p\to p)\vdash p$ is an instance of $\varepsilon_{p,q}\colon p\wedge(p\to q)\vdash q$.

If $f: A\vdash B$ is an instance of $f': A'\vdash B'$, then A and B are instances, by the same substitution, of A' and B' respectively.

Remark 5.1. It is easy to show that if $A'\cong B'$ is derivable in $\mathcal{S}$ and A and B are instances, by the same substitution, of A' and B' respectively, then $A\cong B$ is derivable in $\mathcal{S}$.

The following result is taken over from [1, Diversification Lemma].

Lemma 5.4. *For every isomorphism $f\colon A\vdash B$ of $\mathbf{K}_{lax}$, there is an isomorphism $f'\colon A'\vdash B'$ of $\mathbf{K}_{lax}$, such that f is an instance of f', and A' and B' are diversified.*

As a corollary of Lemmata 5.1-5.4 we have the following.

Theorem 5.1. *$A\cong B$ is derivable in $\mathcal{S}$ iff $A\cong B$ holds in $\mathbf{K}_{lax}$.*

Proof. The direction from left to right is Lemma 5.1. For the other direction, suppose that $A\cong B$ holds in $\mathbf{K}_{lax}$. By relying on Lemma 5.4, there are diversified formulae A' and B' such that $A'\cong B'$ holds in $\mathbf{K}_{lax}$ and A and B are instances, by the same substitution, of A' and B' respectively. By Lemma 5.2, $\mathbf{N}^+\models A'\cong B'$, and by Lemma 5.3, $A'\cong B'$ is derivable in $\mathcal{S}$. By Remark 5.1, we conclude that $A\cong B$ is derivable in $\mathcal{S}$. □

The system $\mathcal{S}$ is decidable (see [1, Normal Form Lemma]), hence the relation $\cong$ is decidable. This completes the classification of objects of $\mathbf{K}_{lax}$.

6. Categories with products, coherence

Coherence results serve to describe the canonical arrows of categories of a particular kind. For example, if symmetric monoidal closed categories are concerned, then the canonical structure consists of all the arrows built out in terms of identities, α, λ, γ, the units and counits of the adjunctions, and operations of composition, $\otimes$ and $\_^A$. Usually, coherence provides a simple decision procedure for equality of canonical arrows. We refer to [2] as a source of various coherence results.

The best way to formulate and to understand a coherence theorem is to use a form of a standard logical completeness result. On the side of syntax, one has to built a category of a desired kind out of syntactical material (like our categories $\mathbf{K}$ and $\mathbf{K}_{lax}$). All the arrows of such a category are canonical. On the side of semantics, one has to find a manageable category of the same kind in which the equality of arrows is easily decidable. Usually, the arrows of such a category are some special finite relations, or some kind of diagrams representing them.

Full and faithful functors

A functor $F\colon\mathcal{C}\to\mathcal{D}$ is *full* when for every pair of objects C, C' of $\mathcal{C}$ and for every arrow $g\colon FC\to FC'$ of $\mathcal{D}$, there is an arrow $f\colon C\to C'$ of $\mathcal{C}$ with $g=Ff$. A functor $F\colon\mathcal{C}\to\mathcal{D}$ is *faithful* when for every pair $f,g\colon A\to B$ of arrows of $\mathcal{C}$, $F(f)=F(g)$ implies $f=g$.

The interpretation is now given by a functor F from the syntactical category to the manageable category. The existence of such F corresponds to a *soundness* result. The faithfulness of F corresponds to a *completeness* result.

Products

A *product* of two objects A and B in a category $\mathcal{C}$ consists of an object $A \times B$ of $\mathcal{C}$ and a pair of arrows $\pi^1_{A,B} : A \times B \to A$, $\pi^2_{A,B} : A \times B \to B$ of $\mathcal{C}$, and it is characterized by the following *universal property*: for every pair of arrows $f : C \to A$ and $g : C \to B$ of $\mathcal{C}$, there is a unique arrow $h \colon C \to A \times B$ of $\mathcal{C}$ such that

$$\pi^1_{A,B} \circ h = f \quad \text{and} \quad \pi^2_{A,B} \circ h = g.$$

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi^1"] \arrow[r, "\pi^2"'] & B \end{tikzcd}$$

Categories with products

A category is a category with *binary products* when every pair of its objects has a product in this category. In another words, $\mathcal{C}$ is a category with binary products when for every pair (A, B) of its objects, there is an object $A \times B$ and a pair of arrows $\pi^1_{A,B} : A \times B \to A$ and $\pi^2_{A,B} : A \times B \to B$. Moreover, for every pair of arrows $f \colon C \to A$ and $g \colon C \to B$, there is an arrow $\langle f, g\rangle \colon C \to A \times B$ such that the following holds:

$$\pi^1_{A,B} \circ \langle f, g\rangle = f, \qquad \pi^2_{A,B} \circ \langle f, g\rangle = g, \tag{6.1}$$

$$\langle \pi^1_{A,B} \circ h, \pi^2_{A,B} \circ h\rangle = h. \tag{6.2}$$

It is easy to conclude that a category with binary products has n-ary products, which are defined analogously, for every $n \geqslant 1$ (cf. [5, III.5, Proposition 1]). We call these categories shortly *categories with products*.

Alternatively, this notion may be introduced as follows. The *diagonal functor* $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ for a category $\mathcal{C}$ is defined so that for every $C \in O_{\mathcal{C}}$ and every $f \in A_{\mathcal{C}}$, $\Delta(C) = (C, C)$ and $\Delta(f) = (f, f)$.
If there is a functor $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a right adjoint for Δ, then $\mathcal{C}$ is a category with products. The unit of this adjunction is the natural transformation from the identity functor on $\mathcal{C}$ to the composition $\times \circ \Delta$

$$w = \{w_A \colon A \to A \times A \mid A \in O_{\mathcal{C}}\},$$

while the counit is the natural transformation from the composition $\Delta \circ \times$ to the identity functor on $\mathcal{C} \times \mathcal{C}$

$$(\pi^1, \pi^2) = \{(\pi^1_{A,B} \colon A \times B \to A\ ,\ \pi^2_{A,B} \colon A \times B \to B) \mid A, B \in O_{\mathcal{C}}\}.$$

The equivalence of these two definitions is left as an exercise for the reader.

If categories with products are in question, then the canonical structure consists of all the arrows built out in terms of identities, projections, composition and the operation $\langle \_, \_ \rangle$ of *pairing*.

The syntactical category $\mathbf{L}$ is obtained in the same manner as we have obtained the category $\mathbf{K}$ in Section 4. The objects of $\mathbf{L}$ are the formulae of the conjunction language introduced in Section 2. Based on that language, we build a new language whose members are equalities of the form $t = s$ where t and s are terms defined below. The alphabet introduced in Section 2 is extended by the symbols $\mathbf{1}$, π^1, π^2, $\langle$, $\rangle$, $\circ$, $=$, $:$, $\vdash$ and ,. (Note that π^1, as well as π^2, is considered as one symbol.)

The *terms* of this language are defined inductively as follows.

(1) If A and B are formulae, then

$$\mathbf{1}_A \colon A \vdash A$$

$$\pi^1_{A,B} \colon A \wedge B \vdash A, \qquad \pi^2_{A,B} \colon A \wedge B \vdash B,$$

are terms called *primitive*.

(2) If $f \colon A \vdash B$ and $g \colon B \vdash C$ are terms, then $(g \circ f) \colon A \vdash C$ is a term.

(3) If $f \colon C \vdash A$ and $g \colon C \vdash B$ is a term, then $\langle f, g \rangle \colon C \vdash A \wedge B$ is a term.

(4) Nothing else is a term.

We define the type, the source and the target of a term, and use the same conventions, as in Section 4. The *language* consists of the words of the form $f = g$, where f and g are terms with the same type. We define an equational system $\mathcal{E}$ in this language in the same manner as in Section 4.

The axiomatic equalities are given by the following schemata

$$f = f, \quad f \circ \mathbf{1}_A = f = \mathbf{1}_B \circ f, \quad (h \circ g) \circ f = h \circ (g \circ f)$$

together with (6.1) and (6.2). The inference figures are

$$\frac{f = g}{g = f} \qquad \frac{f = g \quad g = h}{f = h}$$

$$\frac{f_1 \colon A \vdash B = f_2 \colon A \vdash B \qquad g_1 \colon B \vdash C = g_2 \colon B \vdash C}{g_1 \circ f_1 = g_2 \circ f_2}$$

$$\frac{f_1 \colon C \vdash A = f_2 \colon C \vdash A \qquad g_1 \colon C \vdash B = g_2 \colon C \vdash B}{\langle f_1, g_1 \rangle = \langle f_2, g_2 \rangle}$$

Let $\equiv$ be the relation on the set of terms defined by

$$f \equiv g \quad \text{when} \quad f = g \text{ is derivable in } \mathcal{E}.$$

Again, this is an equivalence relation. We define the arrows of $\mathbf{L}$, their sources and targets, composition, identities and pairing as for the category $\mathbf{K}$ in Section 4. This completes the definition of the syntactical category $\mathbf{L}$. This category corresponds to the conjunction fragment of both intuitionistic and classical logic. The arrows w_A (the components of the unit of the adjunction between Δ and $\times$) are tied to the structural rule of *contraction* and the arrows $\pi^1_{A,B}$ and $\pi^2_{A,B}$, called *projections*, are tied to the structural rule of *thinning*.

The category $\mathbf{L}$ is a category with products. Moreover, it is a category of that kind freely generated by the set of propositional letters that belong to the alphabet. This means that every function from the set of propositional letters to the set of objects of a category $\mathcal{C}$ with products, extends in a unique way to a functor, which preserves the product structure, from $\mathbf{L}$ to $\mathcal{C}$.

Since we have the scheme (4.2), we may omit the parentheses tied to a composition being in the immediate scope of another composition. Hence we write $f_n \circ \ldots \circ f_1$ and reconstruct the parentheses in an arbitrary way. The terms in *normal form* are defined inductively as follows. For a propositional letter p, a term of the form $\mathbf{1}_p \colon p \vdash p$ and the term of the form $f_n \circ \ldots \circ f_1 \colon A \vdash p$, where each f_i is either π^1 or π^2 is in normal form. If f and g are in normal form, then $\langle f, g\rangle$ is in normal form.

Theorem 6.1. (Normal-Form Theorem) *For every term f there exists a term f' in normal form such that $f \equiv f'$.*

Proof. We proceed by induction on the complexity of the target of f.

Let the target of f be a letter p. We may assume that f is of the form $f_n \circ \ldots \circ f_1$, where each f_i is not of the form $h \circ g$, and if $n > 1$, then each f_i is not of the form $\mathbf{1}_A$. If $n = 1$, then f is either of the form $\mathbf{1}_p$, or of the form $\pi^1_{p,A}$ or of the form $\pi^2_{A,p}$ and we are done.

If $n > 1$, then we start a new induction on the number of symbols $\langle$ in f. If $\langle$ does not occur in f, then we are done. Note that if $\langle$ occurs in f_i, then f_i is of the form $\langle g, h\rangle$ and if $\langle$ does not occur in f_i, then f_i is of the form π^1 or π^2. Since the target of f is p, we have that $\langle$ does not occur in f_n. For the greatest i such that $\langle$ occurs in f_i we have that f_{i+1} is of the form π^1 or π^2. In either case, by (6.1), f is equal to a term of the same form with one less occurrence of $\langle$ and we may apply the induction hypothesis. This concludes the basis of the first induction.

If the target of f is of the form $A \wedge B$, then by (6.2) we have that f is equal to $\langle \pi^1_{A,B} \circ f, \pi^2_{A,B} \circ f\rangle$ and we may apply the induction hypothesis to $\pi^1_{A,B} \circ f$ and $\pi^2_{A,B} \circ f$ whose targets are of lower complexity. $\square$

Subcategory, full subcategory

Given a category $\mathcal{C}$, its *subcategory* is a category whose set of objects is a subset of $O_{\mathcal{C}}$ and whose set of arrows is a subset of $A_{\mathcal{C}}$, while its source, target, identity and composition are just restrictions of the corresponding functions of $\mathcal{C}$. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is *full*, when for every pair of objects A_1 and A_2 of $\mathcal{A}$, if $f \colon A_1 \to A_2$ is an arrow of $\mathcal{C}$, then it is an arrow of $\mathcal{A}$, i.e., the embedding of $\mathcal{A}$ into $\mathcal{C}$ is a full functor.

Our manageable category is very simple. We start with its formal presentation and then skip to pictures as a more practical way to handle with arrows of this category. Let $\mathcal{O}$ be the category whose objects are the finite ordinals $0 = \emptyset$, $1 = \{0\}$

and in general $n = \{0, \ldots, n-1\}$, and whose arrows from n to m are the functions mapping the set n to the set m. This category is a full subcategory of the category *Set* of sets and functions.

Let $\mathcal{O}^{op}$ be the *opposite* category of the category $\mathcal{O}$. The arrows from n to m of this category are the functions whose domain is the set m and codomain is the set n. Since the category $\mathcal{O}$ is a category with coproducts (a notion dual to the notion of product) we have that $\mathcal{O}^{op}$ is a category with products.

Coproducts

A *coproduct* of two objects A and B in a category $\mathcal{C}$ consists of an object $A + B$ of $\mathcal{C}$ and a pair of arrows $\iota^1_{A,B} : A \to A + B$, $\iota^2_{A,B} : B \to A + B$ of $\mathcal{C}$, and it is characterized by the following *universal property*: for every pair of arrows $f : A \to C$ and $g : B \to C$ of $\mathcal{C}$, there is a unique arrow $h \colon A + B \to C$ of $\mathcal{C}$ such that

$$h \circ \iota^1_{A,B} = f \quad \text{and} \quad h \circ \iota^2_{A,B} = g.$$

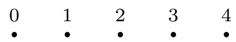
$$\begin{tikzcd} A \arrow[r, "\iota^1"] \arrow[dr, "f"'] & A + B \arrow[d, dotted, "h"] & B \arrow[l, "\iota^2"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

The coproducts in $\mathcal{O}$ are given by addition on objects and by putting "side by side" on arrows, i.e., the coproduct of $f : n \to m$ and $f' : n' \to m'$ is given by the function $g \colon n + n' \to m + m'$ such that

$$g(i) = \begin{cases} f(i), & \text{when} \quad 0 \leqslant i \leqslant n-1, \\ m + f'(i-n), & \text{when} \quad n \leqslant i \leqslant n + n' - 1. \end{cases}$$

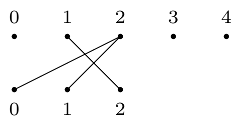
By inverting the arrows, this gives the products in $\mathcal{O}^{op}$. We proclaim $\mathcal{O}^{op}$ to be the manageable category.

The objects of $\mathcal{O}^{op}$ are presented so that n is presented by n vertices. For example, 5 is presented by the following picture.

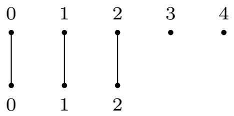


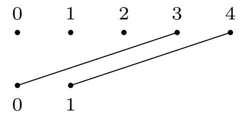
The object $n \times m$ is presented by the picture for $n + m$.

The arrows of $\mathcal{O}^{op}$ are presented by the standard pictures for finite functions. Note that the picture is such that the set of vertices at the bottom line is mapped to the set of vertices at the top line. For example, an arrow from 5 to 3 of $\mathcal{O}^{op}$ is presented by the following picture.

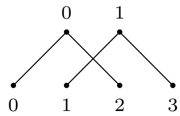


The arrows $\pi^1_{3,2} \colon 3 + 2 \vdash 3$ and $\pi^2_{3,2} \colon 3 + 2 \vdash 2$ are presented respectively by

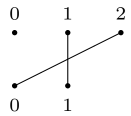


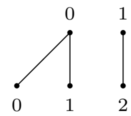


while $w_2 \colon 2 \vdash 2+2$ is presented by the following picture.

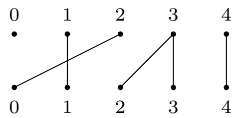


For $f \colon 3 \vdash 2$ and $g \colon 2 \vdash 3$ presented respectively by





the arrow $f \times g$ is presented by the following picture.

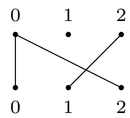


It is not difficult to show directly, without using the above categorial arguments involving dualities, that π^1, π^2 and the pairing defined as $\langle f, g\rangle =_{df} (f \times g) \circ w$, satisfy (6.1) and (6.2). Using the above pictures as arguments is very convincing. This is left as an exercise for the reader.

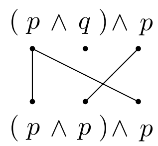
Let a constant function from the set of propositional letters to the set of objects of $\mathcal{O}^{op}$ be such that every letter is mapped to 1. Since $\mathbf{L}$ is a category freely generated by the set of propositional letters, this function extends in a unique way to a functor from $\mathbf{L}$ to $\mathcal{O}^{op}$. For example, the arrow f

$$\langle\langle \pi^1_{p,q} \circ \pi^1_{p\wedge q,p}, \pi^2_{p\wedge q,p}\rangle, \pi^1_{p,q} \circ \pi^1_{p\wedge q,p}\rangle \colon (p \wedge q) \wedge p \vdash (p \wedge p) \wedge p$$

of $\mathbf{L}$ is mapped by this functor to the following arrow of $\mathcal{O}^{op}$



which is usually drown as edges connecting letters in the source and the target of f.



Hence, we have a soundness result—to check that f and g are different in $\mathbf{L}$, it suffices to draw the corresponding pictures and find a difference.

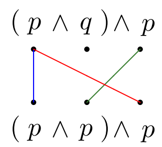
Theorem 6.2. (Soundness) *If f and g represent the same arrow of* $\mathbf{L}$*, then they have the same picture.*

For a completeness result, it is necessary to find a touching point of the syntax and semantics. This is why we represent the arrows of $\mathbf{L}$ by terms in normal form.

Let $f\colon A \vdash B$ be a term in normal form. The nesting of $\langle$ and $\rangle$ in f corresponds to the nesting of (and) in B and the subterms of f of the form $\mathbf{1}_p$ or $f_n \circ \ldots \circ f_1$, with A as the source and a letter as the target, are in one to one correspondence with the occurrences of letters in B.

It is not difficult to see that every such subterm corresponds to a unique edge in the picture, and vice versa, one can read such a subterm from an edge. This is the touching point for the syntax and semantics. For the term and the picture from the preceding example, we have the following correspondence.

$$\langle\langle \pi^1_{p,q} \circ \pi^1_{p\wedge q,p}, \pi^2_{p\wedge q,p}\rangle, \pi^1_{p,q} \circ \pi^1_{p\wedge q,p}\rangle \colon (p \wedge q) \wedge p \vdash (p \wedge p) \wedge p$$



A formalization of these observations is left as an exercise for the reader. Hence, if two terms, in normal form and with the same type, are not identical, then the corresponding pictures are different.

Theorem 6.3. (Completeness) *If f and g are two terms with the same picture, then f and g represent the same arrow of* $\mathbf{L}$*.*

Proof. Let f and g be two terms with the same type and having the same picture, i.e., mapped to the same arrow of $\mathcal{O}^{op}$. By the Normal-Form Theorem, there are f' and g' in normal form such that $f \equiv f'$ and $g \equiv g'$. By Theorem 6.2, f' and g' have the same picture and by the preceding paragraph, these terms are identical. Since $\equiv$ is an equivalence relation, we conclude that $f \equiv g$, and f and g represent the same arrow of $\mathbf{L}$. □

Hence, we have a completeness result—to check that f and g are equal in $\mathbf{L}$, it suffices to draw the corresponding pictures and find no difference.

7. Hints for the exercises

1. In order to prove that the isomorphisms $(A \wedge B) \to C \cong B \to (A \to C)$ and $\top \to C \cong C$ hold in $\mathbf{K}_{lax}$, let the components in $\mathbf{K}_{lax}$ of the natural isomorphism that replaces associativity be denoted by

$$\mathbf{a}_{A,B,C} : A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C, \quad \mathbf{a}^{-1}_{A,B,C} : (A \wedge B) \wedge C \vdash A \wedge (B \wedge C),$$

and let the components in $\mathbf{K}_{lax}$ of the natural isomorphism that replaces neutral conditions for $\top$ be denoted by

$$\mathbf{l}_C : \top \wedge C \vdash C, \quad \mathbf{l}_C^{-1} : C \vdash \top \wedge C.$$

Write down the naturality conditions for $\mathbf{a}$, $\mathbf{a}^{-1}$, $\mathbf{l}$ and $\mathbf{l}^{-1}$.

For $f : A \wedge B \vdash C$, denote by f^* the arrow

$$(A \rightarrow f) \circ \eta_{A,B} : B \vdash A \rightarrow C.$$

Show that for $g : D \vdash B$, the following equations hold:

(1) $f^* \circ g = (f \circ (\mathbf{1}_A \wedge g))^*$,
(2) $\varepsilon_{A,C} \circ (\mathbf{1}_A \wedge f^*) = f$,
(3) $\varepsilon^*_{A,B} = \mathbf{1}_{A \rightarrow B}$.

In terms of ε, $\mathbf{a}$, $\mathbf{a}^{-1}$ and $*$, define arrows of the type

$$(A \wedge B) \rightarrow C \vdash B \rightarrow (A \rightarrow C) \quad \text{and} \quad B \rightarrow (A \rightarrow C) \vdash (A \wedge B) \rightarrow C$$

and show that they are inverse to each other.

In terms of ε, $\mathbf{l}$, $\mathbf{l}^{-1}$ and $*$, define arrows of the type

$$\top \rightarrow C \vdash C \quad \text{and} \quad C \vdash \top \rightarrow C$$

and show that they are inverse to each other.

2. In order to show that the two definitions of a category with binary products are equivalent, let us first assume that $\mathcal{C}$ is a category of binary products according to the first definition. In terms of π^1, π^2 and pairing $\langle \_, \_ \rangle$ define the product $f \times g$ of two arrows and show that it has functorial properties. In the same terms define the components of natural transformations that may serve as the unit and counit of the adjunction between the diagonal functor and the product functor. Show that the unit and counit are natural and that the triangular equations hold.

On the other hand, if the second definition is assumed, then define the pairing $\langle \_, \_ \rangle$ in terms of the right adjoint $\times$ of the diagonal functor, and the unit of this adjunction. It remains to show that the equations (6.1) and (6.2) hold.

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