Cuts and Graphs

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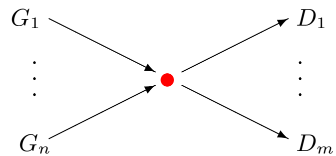
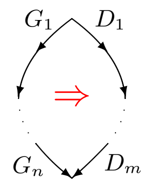
Plural sequents

Multple-conclusion sequents, classical sequents (introduced by Gentzen)

$$G_1, \ldots, G_n \vdash D_1, \ldots, D_m$$

$n, m \geq 0$.





Classical Lambek logic

V.M. Abrusci, *Phase semantics and sequent calculus for pure noncommutative classical linear propositional logic* (1991)

J. Lambek, *From categorial grammar to bilinear logic* (1993)

J. Hudelmaier and P. Schroeder-Heister, *Classical Lambek logic* (1995)

Cuts

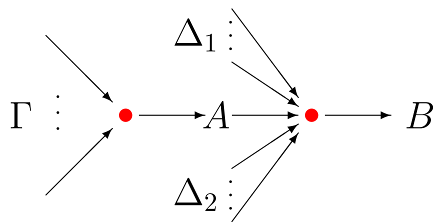
$$\frac{\Gamma \vdash \Theta, A \qquad A, \Delta \vdash \Lambda}{\Gamma, \Delta \vdash \Theta, \Lambda} \qquad \frac{\Gamma \vdash A, \Theta \qquad \Delta, A \vdash \Lambda}{\Delta, \Gamma \vdash \Lambda, \Theta}$$

$$\frac{\Gamma \vdash A \qquad \Delta_1, A, \Delta_2 \vdash \Lambda}{\Delta_1, \Gamma, \Delta_2 \vdash \Lambda} \qquad \frac{\Gamma \vdash \Theta_1, A, \Theta_2 \qquad A \vdash \Lambda}{\Gamma \vdash \Theta_1, \Lambda, \Theta_2}$$

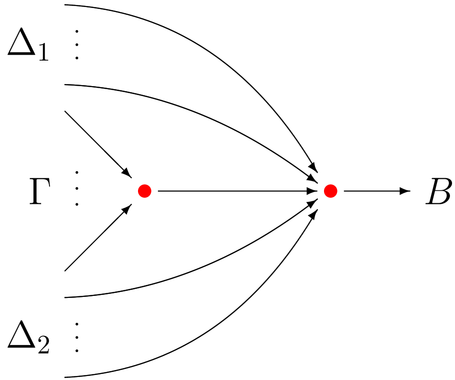
Combinatorial characterization of derivations in sequent systems given by these four cut rules and a set of *axiomatic sequents* of the form $\Gamma \vdash \Delta$.

Singular sequents

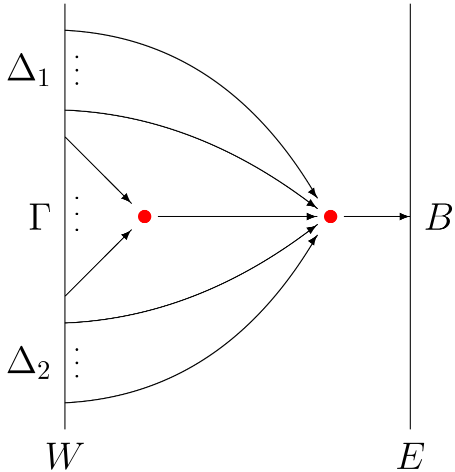
$$\frac{\Gamma \vdash A \qquad \Delta_1, A, \Delta_2 \vdash B}{\Delta_1, \Gamma, \Delta_2 \vdash B}$$



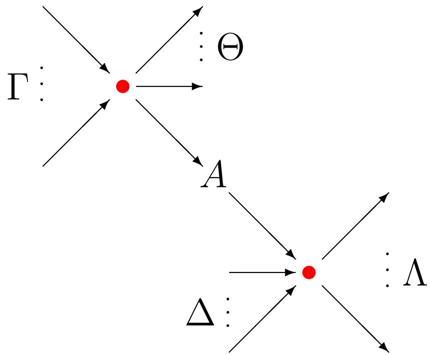
$$\frac{\Gamma \vdash A \qquad \Delta_1, A, \Delta_2 \vdash B}{\Delta_1, \Gamma, \Delta_2 \vdash B}$$



$$\frac{\Gamma \vdash A \qquad \Delta_1, A, \Delta_2 \vdash B}{\Delta_1, \Gamma, \Delta_2 \vdash B}$$



$$\frac{\Gamma \vdash \Theta, A \qquad A, \Delta \vdash \Lambda}{\Gamma, \Delta \vdash \Theta, \Lambda}$$



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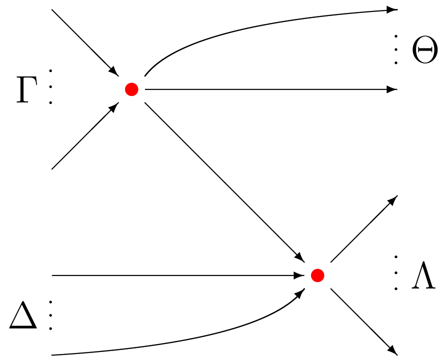
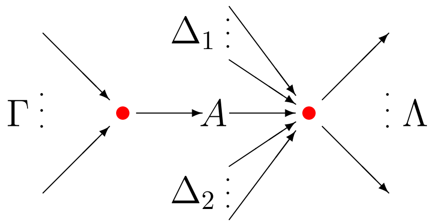
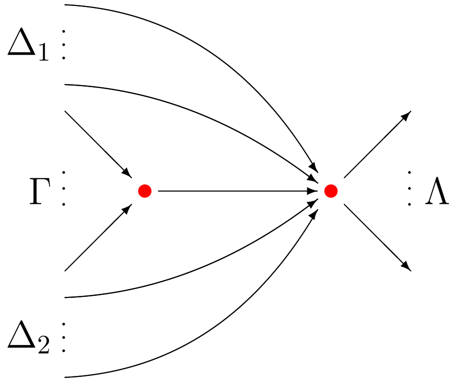


$$\frac{\Gamma \vdash A \qquad\qquad \Delta_1, A, \Delta_2 \vdash \Lambda}{\Delta_1, \Gamma, \Delta_2 \vdash \Lambda}$$



$$\frac{\Gamma \vdash A \qquad \Delta_1, A, \Delta_2 \vdash \Lambda}{\Delta_1, \Gamma, \Delta_2 \vdash \Lambda}$$



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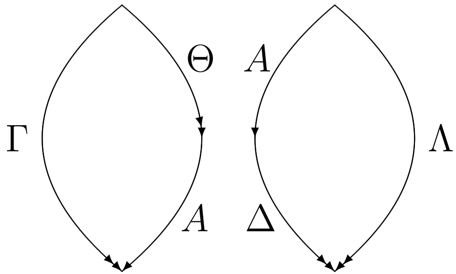

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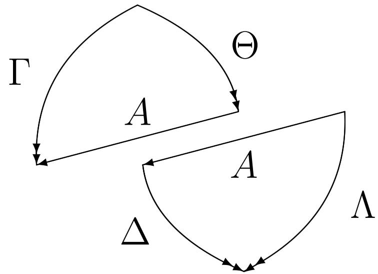

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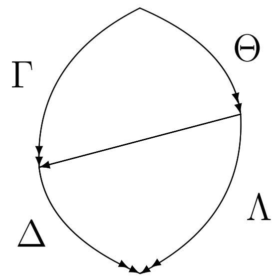
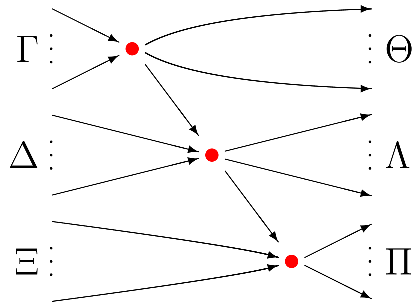
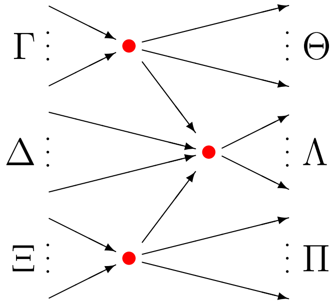


$$\dfrac{\dfrac{\Gamma \vdash \Theta, A \qquad A, \Delta \vdash \Lambda, B}{\Gamma, \Delta \vdash \Theta, \Lambda, B} \qquad B, \Xi \vdash \Pi}{\Gamma, \Delta, \Xi \vdash \Theta, \Lambda, \Pi} \qquad\qquad \dfrac{\Gamma \vdash \Theta, A \qquad \dfrac{A, \Delta \vdash \Lambda, B \qquad B, \Xi \vdash \Pi}{A, \Delta, \Xi \vdash \Lambda, \Pi}}{\Gamma, \Delta, \Xi \vdash \Theta, \Lambda, \Pi}$$



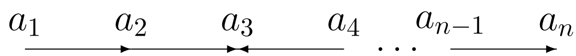
$$
\dfrac{\Xi \vdash B, \Pi \qquad \dfrac{\Gamma \vdash \Theta, A \qquad A, \Delta, B \vdash \Lambda}{\Gamma, \Delta, B \vdash \Theta, \Lambda}}{\Gamma, \Delta, \Xi \vdash \Theta, \Lambda, \Pi}
\qquad\qquad
\dfrac{\Gamma \vdash \Theta, A \qquad \dfrac{\Xi \vdash B, \Pi \qquad A, \Delta, B \vdash \Lambda}{A, \Delta, \Xi \vdash \Lambda, \Pi}}{\Gamma, \Delta, \Xi \vdash \Theta, \Lambda, \Pi}
$$



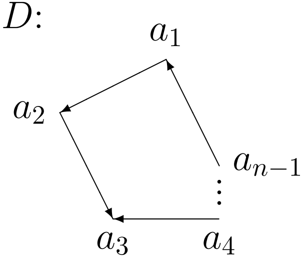
Oriented graphs, semipaths, semicycles

An *oriented graph* D is an irreflexive and antisymmetric binary relation on a finite nonempty set, called the set of *vertices* of D. The ordered pairs in D are its *edges*.

A *semipath* in D:



A *semicycle* in D:



$a_i \neq a_j$ when $i \neq j$. The vertex a_2 is an *inner vertex*.

Oriented graphs, semipaths, semicycles

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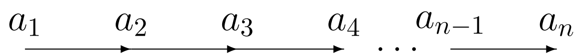
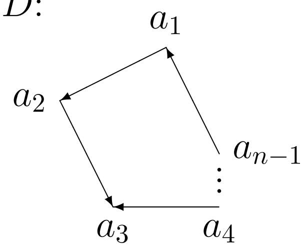
A *path* in D:



A *semicycle* in D:



$a_i \neq a_j$ when $i \neq j$. The vertex a_2 is an *inner vertex*.

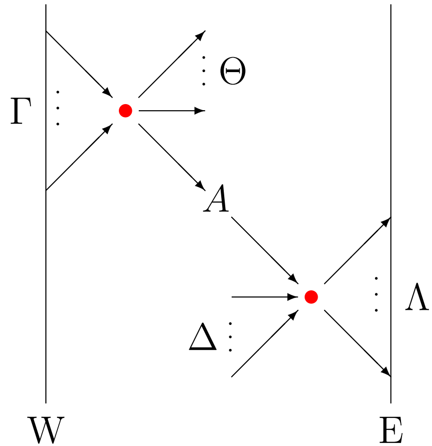
K-graphs

A K-*graph* is an oriented graph that corresponds to a derivation in a sequent system given by the four cut rules and a set of axiomatic sequents.

The following holds for K-graphs:
(1) every two vertices are connected by a semipath (it is weakly connected);
(2) it has no semicycles (it is asemicyclic)
(3) it has inner vertices and every non-inner vertex (*initial* or *terminal*) is tied to a single edge.

From now on we consider only oriented graphs with these three properties.

Every K-graph can be drawn in the plane without crossings, so that every initial vertex belongs to the *West line* and every terminal vertex belongs to the *East line* and all the edges are directed from West to East.



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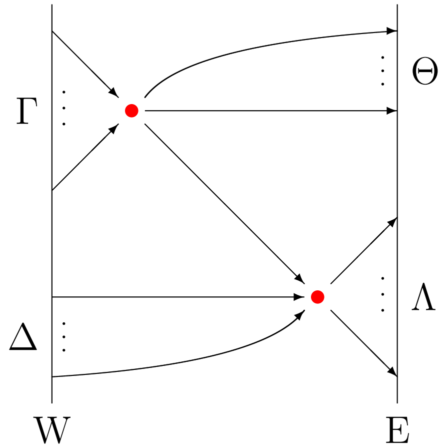
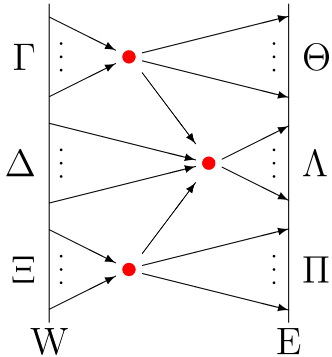


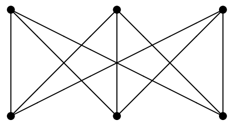
Geometric characterization of K-graphs

An oriented graph is a K-graph if and only if it can be drawn in the plane as above.

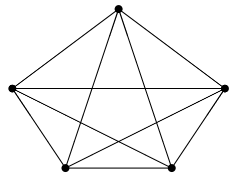


Kuratowski's planarity criterion

An ordinary (non-oriented) graph is planar if and only if it has no subgraph homeomorphic to $K_{3,3}$ and K_5.



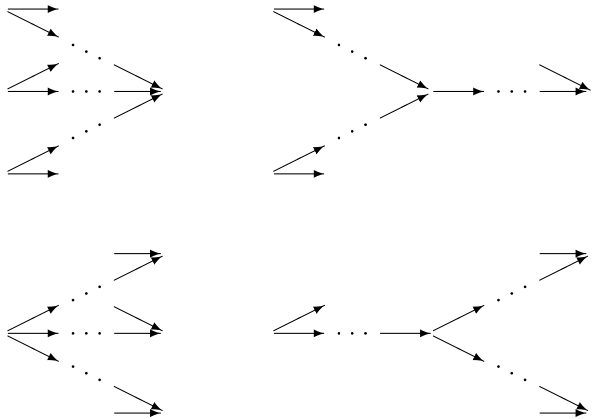
$K_{3,3}$



K_5

K-graph criterion

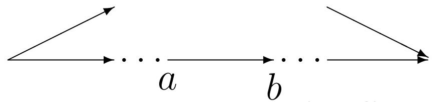
An oriented graph is a K-graph if and only if it has no subgraph of one of the following four (*obstacle*) forms:



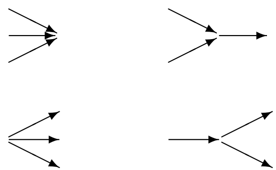
Transversal

A semipath is *proper* when it is not a path.

An edge (a, b) is *transversal* when there is a proper semipath starting with (a, b) and a proper semipath ending with (a, b).



A *bifurcation* is a triple of different edges that have a common vertex.



A bifurcation is *transversal* when all the three edges in it are transversal.

Propositions

1. If there is a transversal bifurcation, then there is an obstacle subgraph.

Assume there are no obstacle subgraphs.

2. The transversal edges make a semipath called the *transversal*.

3. Every connected component obtained by removing the transversal is either a tree (arborescence*) whose root belongs to the transversal, or its opposite is such.

$*$ An *arborescence* is an oriented graph with a vertex u called the *root*, in which for any other vertex v, there is exactly one path from u to v.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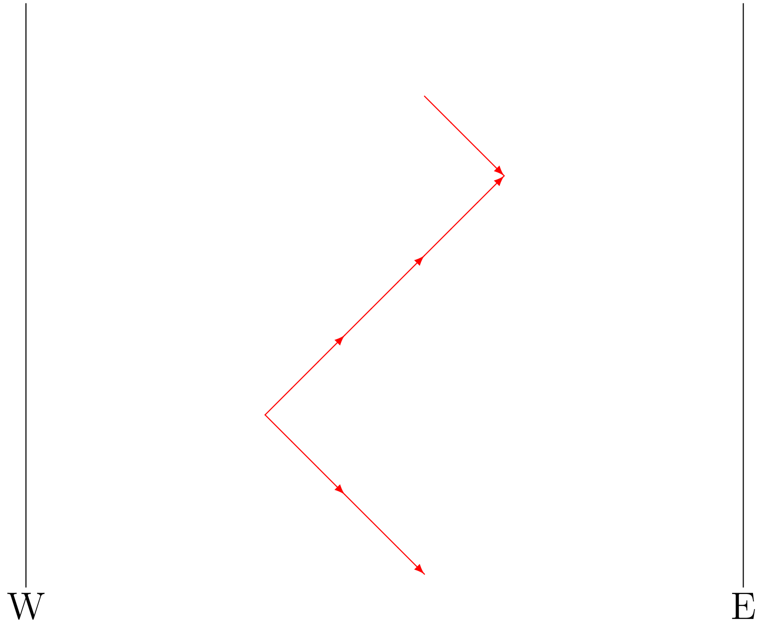

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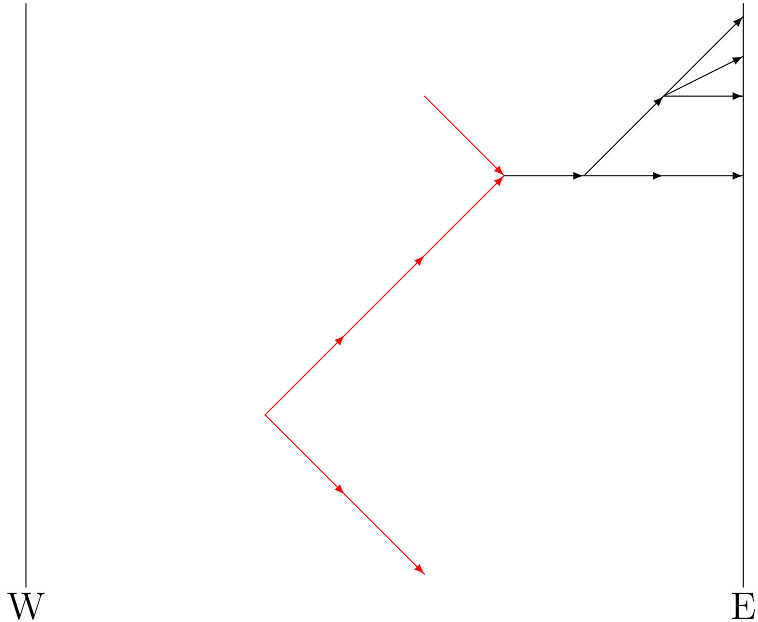
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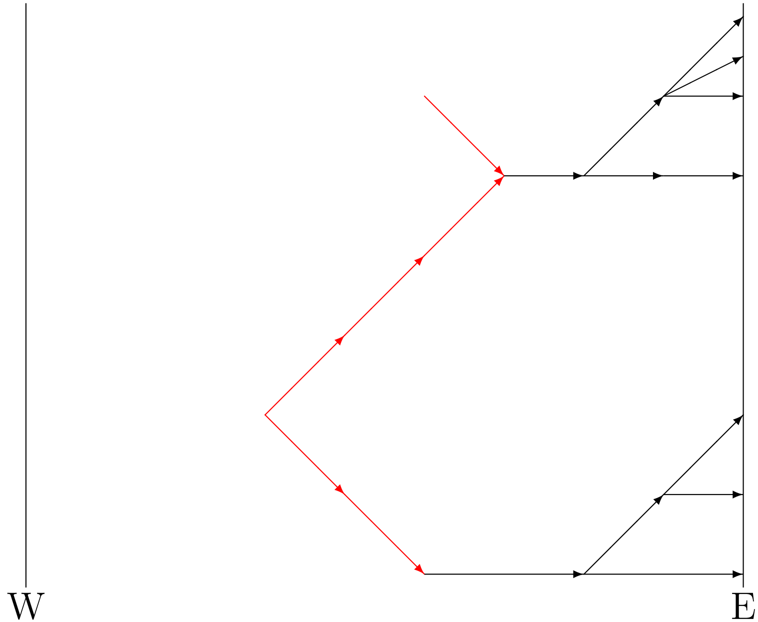
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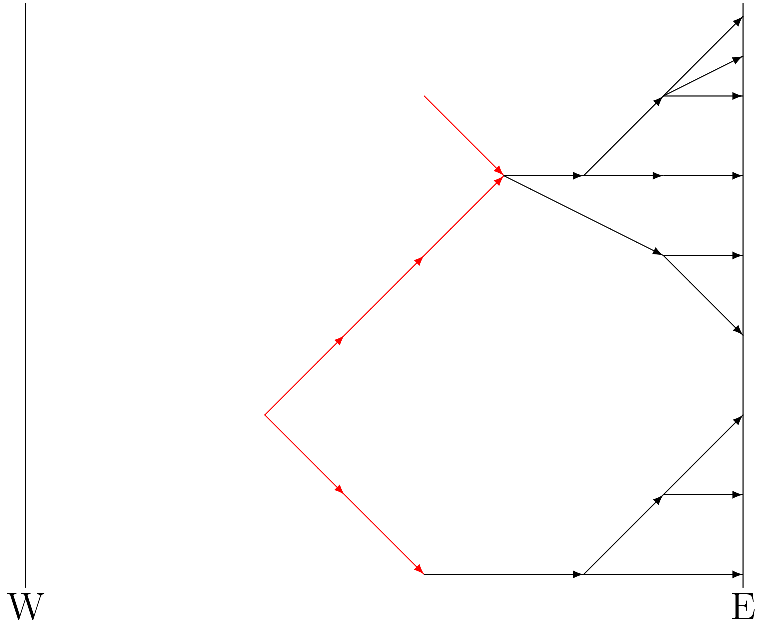

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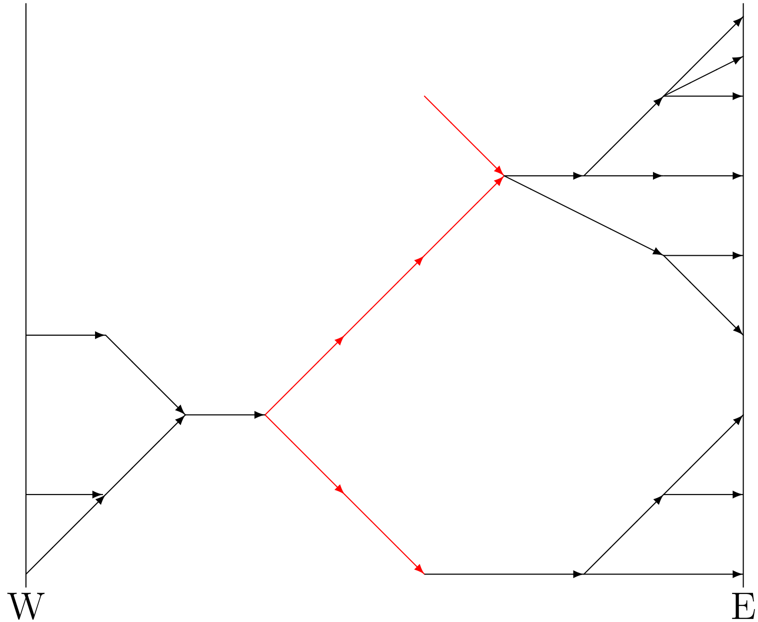
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The talk was based on: K. Došen and Z. Petrić, *Graphs of plural cuts*, Theoretical Computer Science 484 (2013) pp. 41-55 (available at: arXiv)